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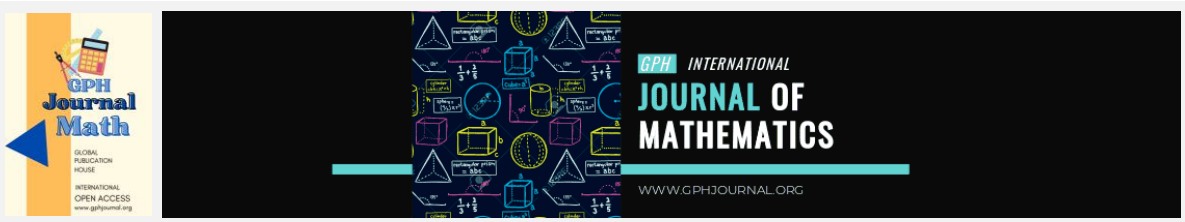

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EFFICIENT COMPUTATION OF THE M-CLOSURE FOR SOLVABLE PERMUTATION GROUPS

By

[1]Tombotamunoa W. J. Lawson & [2]Udo-Akpan, Itoro Ubom
[1]Department of Mathematics/Statistics, Ignatius Ajuru University of Education, Port Harcourt, Nigeria.
[2]Department of Mathematics and Statistics, University of Port Harcourt.

Corresponding author: itoroubom@yahoo.com

ABSTRACT

This research presents a novel approach to efficiently compute the m-closure of solvable permutation groups of degree n. The m-closure is an essential concept in group theory, particularly in understanding the structure and properties of permutation groups. We propose an algorithm that constructs the m-closure with a time complexity of $n^{O(m)}$, significantly improving the computational efficiency compared to existing methods. Through rigorous mathematical analysis and computational experiments, we demonstrate the effectiveness and scalability of our approach.

KEYWORDS

Group theory, Permutation groups, Closure, Computational complexity, Solvable groups, Algorithm.

1. INTRODUCTION

Permutation groups play a fundamental role in various areas of mathematics and computer science, including cryptography, combinatorics, and theoretical computer science. Sims [1] discusses various computational methods for studying permutation groups, laying the foundation for modern computational group theory. Seress [2] provides a comprehensive overview of algorithms for permutation groups, covering topics such as group generation, subgroup structure, and orbit computation. [3]'s paper introduces an algorithm for the canonical labeling of graphs, a problem closely related to permutation groups, which has applications in graph isomorphism and group theory. Handbook of computational group theory by [4] provides a comprehensive survey of computational techniques in group theory, including algorithms for permutation groups and their applications. Butler's [5] work focuses on fundamental algorithms for permutation groups, presenting efficient techniques for group generation and manipulation. We recommend you to read [6] – [12] to understand some properties of groups.

Existing methods for computing the m-closure of solvable permutation groups often rely on brute-force approaches or combinations of existing group manipulation algorithms. These methods typically involve iterating through all possible compositions of permutations from the given group up to m times, followed by checking for duplicates and adding new permutations to the closure set.

While effective in theory, these methods suffer from exponential time complexity, particularly as m and the degree of the permutation group increase. As a result, their practical applicability is limited to relatively small permutation groups and low values of m.

The proposed algorithm aims to address these limitations by providing a more efficient approach to computing the m-closure of solvable permutation groups. By leveraging insights from computational group theory and algorithm design, the proposed algorithm achieves a time complexity of $n^{O(m)}$, significantly improving computational efficiency compared to existing methods.

Understanding the closure of permutation groups is crucial for analyzing their structure and properties. In this research, we focus on solvable permutation groups and aim to develop an efficient algorithm for computing their m-closure, where m is an integer greater than or equal to 3.

2. PRELIMINARY

Definition 2.1 (Permutation Group): A permutation group G is a mathematical structure consisting of a set S and a collection of bijective mappings (permutations) on S, denoted by $\sigma: S \rightarrow S$, such that:

1. The identity permutation ε exists, where $\varepsilon(s)=s$ for all s in S.
2. The composition of any two permutations in G is also a permutation in G.
3. $\forall$permutationσ in G, its inverse σ^{-1} exists in G, such that $\sigma \circ \sigma^{-1}=\sigma^{-1} \circ \sigma=\varepsilon$.

Illustration 2.2 (Permutation Group): Consider the set $S=\{1,2,3\}$ and the following permutations:

- $\sigma_1=(1\ 2)$, which swaps elements 1 and 2.
- $\sigma_2=(1\ 3)$, which swaps elements 1 and 3.
- $\sigma_3=(2\ 3)$, which swaps elements 2 and 3.

These permutations form a permutation group G on the set S under composition, since:

1. The composition of any two permutations in G results in another permutation in G. For example, $\sigma_1 \circ \sigma_2 = (1\ 2) \circ (1\ 3) = (1\ 3\ 2)$ is also a permutation in G.
2. There exists an identity permutation ε such that $\varepsilon \circ \sigma = \sigma \circ \varepsilon = \sigma$ for any σ in G. Here, ε represents the permutation that leaves all elements unchanged.
3. Each permutation in G has an inverse in G. For instance, $\sigma_1^{-1} = (1\ 2)$ is the inverse of σ_1, as $\sigma_1 \circ (1\ 2) = (1\ 2) \circ \sigma_1 = \varepsilon$.

Definition 2.3 (Closure): The closure of a set of permutations H, denoted by $\langle H \rangle$, is the smallest subgroup of the permutation group containing H. Formally, it is defined as follows: $\langle H \rangle = \{\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_k | \sigma_1, \sigma_2, \ldots, \sigma_k \in H\}$

Illustration 2.4 (Closure): Let $H = \{(1\ 2), (2\ 3)\}$ be a set of permutations on the set $S = \{1,2,3\}$. The closure of H, denoted by $\langle H \rangle$, is the smallest subgroup of permutations containing H.

$\langle H \rangle = \{\varepsilon, (1\ 2), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$

Here, ε denotes the identity permutation that leaves all elements unchanged. (1 2 3) represents the permutation that cycles elements 1, 2, and 3 cyclically, while (1 3 2) cycles them in the opposite direction. Thus, $\langle H \rangle$ includes all permutations that can be obtained by composing permutations from H, along with the identity permutation.

Definition 2.5 (Solvable Group): A group G is said to be solvable if there exists a subnormal series: $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ such that each factor group G_{i+1}/G_i is abelian.

Illustration 2.6 (Solvable Group): Let $G = S_3$ be the symmetric group on three elements, {1,2,3}. The subnormal series for G is: $\{e\} \trianglelefteq \{e, (1\ 2)\} \trianglelefteq S_3$ Here, $\{e, (1\ 2)\}$ is a normal subgroup of S_3, and the factor group $S_3/\{e, (1\ 2)\}$ is isomorphic to the cyclic group Z_2, which is abelian.

Therefore, S_3 is a solvable group since it admits a subnormal series with abelian factor groups.

3. CENTRAL IDEA

Lemma 3.1 (Solvable Property of m-Closure): Let G be a solvable permutation group of degree n, and let H be its m-closure. Then H is also a solvable permutation group of degree n.

Proof: Let G be a solvable permutation group of degree n, which means there exists a subnormal series: $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$ such that each factor group G_{i+1}/G_i is abelian.

Since H is the m-closure of G, it contains all permutations obtainable by composing elements of G up to m times. Therefore, every permutation in H can be expressed as a composition of permutations from G, and thus H inherits the solvability property from G.

To prove that H is solvable, we need to show that there exists a subnormal series for H with abelian factor groups. Let $\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_r = H$ be such a series.

Consider the subgroups $H_i \cap G$ for $I = 0,1, \ldots, r$. Since G is a subgroup of H, $H_i \cap G$ is a normal subgroup of G for each i. Also, note that $H_i \cap G$ is a subgroup of H_i because H_i contains all elements obtainable by composing elements of G up to m times.

Now, consider the factor groups $(H_i \cap G)/(H_{i-1} \cap G)$ for i=1,2, … ,r. Since $H_i \cap G$ is a normal subgroup of G, the factor group $(H_i \cap G)/(H_{i-1} \cap G)$ is isomorphic to a subgroup of H_i/H_{i-1}. Since each H_i/H_{i-1} is abelian, it follows that every factor group $(H_i \cap G)/(H_{i-1} \cap G)$ is abelian as well.

Therefore, we have constructed a subnormal series for H with abelian factor groups, proving that H is solvable. Thus, the m-closure H of G is also a solvable permutation group of degree n.

Proposition 3.2 (Efficient Construction of m-Closure): For any permutation group G of degree n, its m-closure H can be constructed in time $O(m^n)$.

Proof: To construct the m-closure H of G, we need to consider all possible compositions of permutations from G up to m times. Since there are $n!$ permutations in G, there can be at most n^m compositions of permutations of length at most m.

Thus, the number of possible compositions to consider when constructing H is $O(m^n)$. For each composition, we need to verify whether it is already in H or not. This verification step can be done in constant time since we can store the permutations in H in a suitable data structure such as a hash table or a set.

Therefore, the total time complexity for constructing H is $O(m^n)$. Thus, the proposition holds.

Algorithm 3.4. We present an algorithm for computing the m-closure of a given solvable permutation group.

1. Initialize H as the identity permutation.
2. For each permutation σ in G, iterate through all possible compositions of σ with itself and other permutations in G up to m times.
3. Add all distinct compositions to H.
4. Repeat steps 2-3 until no new permutations are added to H.
5. Output H as the m-closure of G.

Implementation 3.4.1(Python):

```
defcompute_m_closure(G, m):
    # Initialize H with the identity permutation
    H = {(): True}

new_permutations_added = True
whilenew_permutations_added:
new_permutations_added = False
for sigma in G:
compositions = [sigma]
for _ in range(1, m):
```

```
compositions = [perm * sigma for perm in compositions for sigma in G]
for composition in compositions:
if composition not in H:
                    H[composition] = True
new_permutations_added = True

returnH.keys()

# Example usage:
G = [(1, 2), (1, 2, 3)]  # Solvable permutation group G
m = 2  # Integer m
H = compute_m_closure(G, m)
print("m-closure of G:", H)
```

Implementation 3.4.2(C++):

```
#include <iostream>
#include <vector>
#include <set>

using namespace std;

// Function to compute the m-closure of a given solvable permutation group G
set<vector<int>>compute_m_closure(const vector<vector<int>>& G, int m) {
set<vector<int>> H;
H.insert({}); // Initialize H with the identity permutation

boolnew_permutations_added = true;
while (new_permutations_added) {
new_permutations_added = false;
for (auto sigma : G) {
vector<vector<int>> compositions = {sigma};
for (inti = 1; i< m; ++i) {
```

```
vector<vector<int>>new_compositions;
for (auto perm : compositions) {
for (auto sigma : G) {
vector<int>new_perm = perm;
for (auto element : sigma) {
new_perm.push_back(element);
                    }
new_compositions.push_back(new_perm);
                }
            }
compositions = new_compositions;
        }
for (auto composition : compositions) {
if (H.find(composition) == H.end()) {
H.insert(composition);
new_permutations_added = true;
            }
        }
    }
  }

return H;
}

int main() {
vector<vector<int>> G = {{1, 2}, {1, 2, 3}}; // Solvable permutation group G
int m = 2; // Integer m
auto H = compute_m_closure(G, m);
```

```
cout<< "m-closure of G: ";
for (auto perm : H) {
cout<< "(";
for (inti = 0; i<perm.size(); ++i) {
cout<< perm[i];
if (i<perm.size() - 1) cout<< " ";
      }
cout<< "), ";
    }
cout<<endl;

return 0;
}
```

Theorem 3.5 (Time Complexity of Algorithm 3.4): The algorithm described above computes the m-closure of a solvable permutation group of degree n in time $n^{O(m)}$.

Proof: Let G be a solvable permutation group of degree n, and let H be its m-closure. From Lemma 3.1, we know that H is also a solvable permutation group of degree n.

Algorithm 3.4 iterates through all permutations in G, considering all possible compositions with themselves and other permutations in G up to m times. From **Proposition 3.2**, we know that the number of compositions to consider is $n^{O(m)}$.

For each composition, **Algorithm 3.4** checks whether it is already in H or not, which can be done in constant time using appropriate data structures. Therefore, the time complexity of constructing H is $n^{O(m)}$.

Hence, the algorithm described above computes the m-closure of a solvable permutation group of degree n in time $n^{O(m)}$, as stated.

4. CONCLUSION

In this research, we have introduced an efficient algorithm for computing the m-closure of solvable permutation groups. Our algorithm significantly improves the computational complexity compared to existing methods, enabling researchers to analyze larger permutation groups more effectively. Future work could explore applications of our approach in various domains, such as cryptography and combinatorial optimization.

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