ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Ji-Zhen Xu and Yong-Gao Chen

On subsets of asymptotic bases

Volume 362 (2024), p. 45-49

Online since: 2 February 2024

https://doi.org/10.5802/crmath.513

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE
1666

Comptes Rendus. Mathématique
2024, Vol. 362, p. 45-49
https://doi.org/10.5802/crmath.513

Research article / *Article de recherche*
Number theory / *Théorie des nombres*

On subsets of asymptotic bases

Ji-Zhen Xu [a, b] **and Yong-Gao Chen** [*, a]

[a] School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, People's Republic of China

[b] Nanjing Vocational College of Information Technology,Nanjing 210023, People's Republic of China

E-mails: 965165607@qq.com (J.-Z. Xu), ygchen@njnu.edu.cn (Y.-G. Chen)

Abstract. Let $h \ge 2$ be an integer. In this paper, we prove that if A is an asymptotic basis of order h and B is a nonempty subset of A, then either there exists a finite subset F of A such that $F \cup B$ is an asymptotic basis of order h, or for any $\varepsilon > 0$, there exists a finite subset F_ε of A such that $d_L(h(F_\varepsilon \cup B)) \ge h d_L(B) - \varepsilon$, where $d_L(X)$ denotes the lower asymptotic density of X and hX denotes the set of all $x_1 + \cdots + x_h$ with $x_i \in X$ $(1 \le i \le h)$. This generalizes a result of Nathanson and Sárközy.

2020 Mathematics Subject Classification. 11B13, 11B05, 11P99.

Funding. This work is supported by the National Natural Science Foundation of China, Grant No. 12171243.

Manuscript received 25 January 2023, accepted 13 May 2023.

1. Introduction

Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Let $h \ge 2$ be an integer. For $A \subseteq \mathbb{N}_0$, let

$$hA = \{a_1 + \cdots + a_h : a_1, \ldots, a_h \in A\}.$$

We define

$$d_L(A) = \liminf_{x \to +\infty} \frac{A(x)}{x},$$

where $A(x)$ is the number of positive integers in A which do not exceed x. Usually, $d_L(A)$ is called the *lower asymptotic density* of A. If

$$\lim_{x \to +\infty} \frac{A(x)}{x}$$

exists, then the limit value is called the *asymptotic density* of A and denote it by $d(A)$.

A set A is called an *asymptotic basis of order h* if hA contains all sufficiently large integers. An asymptotic basis A of order h is called *minimal* if no proper subset of A is an asymptotic basis of order h. The notation of minimal asymptotic bases was introduced by Stöhr [10] in 1955. In 1956, Härtter [4] proved that for each integer $h \ge 2$, there exist minimal asymptotic bases of order h. In 1988, Erdős and Nathanson [3] constructed a minimal asymptotic basis A with $d(A) = 1/h$. For related research, one may refer to Chen and Chen [1], Chen and Tang [2], Jańczak and Schoen [5], Nathanson [7, 8], Sun [11] and Tang and Lin [12].

Nathanson and Sárközy [9] proved the following results:

---

* Corresponding author.

Theorem A. *If A is an asymptotic basis of order h and B is a subset of A with $d_L(B) > 1/h$, then there exists a finite subset F of A such that $F \cup B$ is an asymptotic basis of order h.*

Theorem B. *If A is a minimal asymptotic basis of order h, then $d_L(A) \le 1/h$.*

In this paper, the following results are proved.

Theorem 1. *Let $h \ge 2$ be an integer. If A is an asymptotic basis of order h and B is a nonempty subset of A, then either there exists a finite subset F of A such that $F \cup B$ is an asymptotic basis of order h, or for any $\varepsilon > 0$, there exists a finite subset F_ε of A such that $d_L(h(F_\varepsilon \cup B)) \ge hd_L(B) - \varepsilon$.*

Remark. Theorem A is a corollary of Theorem 1. Let A be an asymptotic basis of order h and B_1 a subset of A with $d_L(B_1) > 1/h$. We take $\varepsilon = (hd_L(B_1) - 1)/2$. Then $hd_L(B_1) - \varepsilon = (hd_L(B_1) + 1)/2 > 1 \ge d_L(h(E \cup B_1))$ for any finite subset E of A. By Theorem 1, there exists a finite subset F of A such that $F \cup B_1$ is an asymptotic basis of order h.

Corollary 2. *Let $h \ge 2$ be an integer and let B be a nonempty set of nonnegative integers. Then either there exists a finite set F of nonnegative integers such that $F \cup B$ is an asymptotic basis of order h, or for any $\varepsilon > 0$, there exists a finite set F_ε of nonnegative integers such that $d_L(h(F_\varepsilon \cup B)) \ge hd_L(B) - \varepsilon$.*

Theorem 3. *Let $h \ge 2$ be an integer. If A is a minimal asymptotic basis of order h and B is a nonempty subset of A, then for any $\varepsilon > 0$, there exists a finite subset F_ε of A such that $d_L(h(F_\varepsilon \cup B)) \ge hd_L(B) - \varepsilon$.*

Theorem 4. *Let $h \ge 2$ be an integer. If A is a set of nonnegative integers with $d_L(A) > 0$, then there exists a subset B of A with $d_L(B) > 0$ such that $F \cup B$ is not an asymptotic basis of order h for any finite set F.*

2. Proofs

We will use a well known result of Kneser. If two sets X and Y of nonnegative integers are coincide from some point on, then we write $X \sim Y$. For any set X of nonnegative integers and any positive integer g, let $X^{(g)}$ be the set of all nonnegative integers n with $n \equiv x \pmod g$ for some $x \in X$.

In 1953, Kneser [6] proved the following profound result.

Lemma 5 (Kneser [6]). *Let $h \ge 2$ be an integer and X a nonempty set of nonnegative integers. Then either $d_L(hX) \ge hd_L(X)$ or there exists a positive integer g such that $hX \sim hX^{(g)}$ and*

$$d_L(hX) \ge hd_L(X) - \frac{h-1}{g}.$$

Proof of Theorem 1. If there exists a finite subset F of A such that $F \cup B$ is an asymptotic basis of order h, then we are done. Now we assume that for any finite subset F of A, $F \cup B$ is not an asymptotic basis of order h. Let $\varepsilon > 0$. For any positive integer g, let $A_g = \{a_{g,1}, \dots, a_{g,s_g}\}$ be a subset of A such that for every $a \in A$, there exists $1 \le i \le s_g$ with $a \equiv a_{g,i} \pmod g$. It is clear that $A_g^{(g)} = A^{(g)}$. Let

$$F_\varepsilon = \bigcup_{1 \le g < (h-1)/\varepsilon} A_g.$$

Then F_ε is finite. It is enough to prove that

$$d_L(h(F_\varepsilon \cup B)) \ge hd_L(B) - \varepsilon.$$

By Lemma 5, either

$$d_L(h(F_\varepsilon \cup B)) \ge hd_L(F_\varepsilon \cup B),$$

or there exists a positive integer g_1 such that $h(F_\varepsilon \cup B) \sim h((F_\varepsilon \cup B)^{(g_1)})$ and

$$d_L(h(F_\varepsilon \cup B)) \geq h d_L(F_\varepsilon \cup B) - \frac{h-1}{g_1}.$$

Since F_ε is finite, it follows that $d_L(F_\varepsilon \cup B) = d_L(B)$. Hence, either

$$d_L(h(F_\varepsilon \cup B)) \geq h d_L(B),$$

or there exists a positive integer g_1 such that $h(F_\varepsilon \cup B) \sim h((F_\varepsilon \cup B)^{(g_1)})$ and

$$d_L(h(F_\varepsilon \cup B)) \geq h d_L(B) - \frac{h-1}{g_1}.$$

If

$$d_L(h(F_\varepsilon \cup B)) \geq h d_L(B),$$

then we are done. Now we assume that there exists a positive integer g_1 such that $h(F_\varepsilon \cup B) \sim h((F_\varepsilon \cup B)^{(g_1)})$ and

$$d_L(h(F_\varepsilon \cup B)) \geq h d_L(B) - \frac{h-1}{g_1}.$$

If $(h-1)/g_1 \leq \varepsilon$, then we are done. Now we assume that $(h-1)/g_1 > \varepsilon$. We will derive a contradiction. By $(h-1)/g_1 > \varepsilon$, we have $g_1 < (h-1)/\varepsilon$. Thus,

$$A \subseteq A^{(g_1)} = A_{g_1}^{(g_1)} \subseteq F_\varepsilon^{(g_1)} \subseteq (F_\varepsilon \cup B)^{(g_1)}.$$

Hence

$$hA \subseteq h((F_\varepsilon \cup B)^{(g_1)}) \subseteq \mathbb{N}_0. \tag{1}$$

Since A is an asymptotic basis of order h, we have $hA \sim \mathbb{N}_0$. It follows from (1) that $h((F_\varepsilon \cup B)^{(g_1)}) \sim \mathbb{N}_0$. Noting that $h(F_\varepsilon \cup B) \sim h((F_\varepsilon \cup B)^{(g_1)})$, we have $h(F_\varepsilon \cup B) \sim \mathbb{N}_0$. This means that $F_\varepsilon \cup B$ is an asymptotic basis of order h, a contradiction.

This completes the proof of Theorem 1. □

Proof of Corollary 2. Since $\mathbb{N}_0$ is an asymptotic basis of order h and $B \subseteq \mathbb{N}_0$, Corollary 2 follows from Theorem 1 immediately. □

Proof of Theorem 3. Since A is a minimal asymptotic basis of order h, it follows that $hA \sim \mathbb{N}_0$. So $d_L(hA) = 1$. By Theorem B, $d_L(B) \leq d_L(A) \leq 1/h$. Thus, $h d_L(B) \leq 1$. Let $\varepsilon > 0$. If $A \setminus B$ is finite, then for $F_\varepsilon = A \setminus B$,

$$d_L(h(F_\varepsilon \cup B)) = d_L(hA) = 1 \geq h d_L(B) - \varepsilon.$$

Now we assume that $A \setminus B$ is infinite. Thus, for any finite subset F of A, we have $F \cup B \neq A$. Since A is a minimal asymptotic basis of order h, it follows that for any finite subset F of A, $F \cup B$ is not an asymptotic basis of order h. Now Theorem 3 follows from Theorem 1 immediately. □

Proof of Theorem 4. Let

$$B = \bigcup_{n=0}^{\infty} \left(\left((h+1)^{n^2+1}, (h+1)^{(n+1)^2} \right) \cap A \right).$$

For a sufficiently large x, let k be the integer with

$$(h+1)^{k^2} \leq x < (h+1)^{(k+1)^2}.$$

Let t be the integer with

$$(h+1)^{t-1} < \frac{h(h+1) + d_L(A)}{(h+1) d_L(A)} \leq (h+1)^t.$$

It is clear that $t \geq 1$. If $x \leq (h+1)^{k^2+t}$, then

$$\begin{aligned}\frac{B(x)}{x} &\geq \frac{B((h+1)^{k^2})}{x} \geq \frac{1}{(h+1)^t}\frac{B((h+1)^{k^2})}{(h+1)^{k^2}}\\ &> \frac{d_L(A)}{h(h+1)+d_L(A)}\frac{B((h+1)^{k^2})}{(h+1)^{k^2}}.\end{aligned}$$

Since

$$\begin{aligned}B((h+1)^{k^2}) &\geq A((h+1)^{k^2}) - \sum_{n=0}^{k-1}\Big((h+1)^{n^2+1}-(h+1)^{n^2}+1\Big)\\ &\geq A((h+1)^{k^2}) - k(h+1)^{(k-1)^2+1},\end{aligned}$$

it follows that

$$\frac{B((h+1)^{k^2})}{(h+1)^{k^2}} \geq \frac{A((h+1)^{k^2})}{(h+1)^{k^2}} + o(1) \geq d_L(A) + o(1).$$

Hence

$$\frac{B(x)}{x} \geq \frac{d_L(A)}{h(h+1)+d_L(A)}(d_L(A)+o(1)).$$

If $x > (h+1)^{k^2+t}$, then

$$\begin{aligned}B(x) &\geq A(x) - \sum_{n=0}^{k}\Big((h+1)^{n^2+1}-(h+1)^{n^2}+1\Big)\\ &= A(x) - h\sum_{n=0}^{k}(h+1)^{n^2} - k - 1\\ &\geq A(x) - h(h+1)^{k^2} - hk(h+1)^{(k-1)^2} - k - 1\\ &> A(x) - \frac{h}{(h+1)^t}x - \frac{hk}{(h+1)^{2k-1+t}}x - k - 1.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{B(x)}{x} &\geq \frac{A(x)}{x} - \frac{h}{(h+1)^t} - o(1)\\ &\geq d_L(A) - \frac{h(h+1)d_L(A)}{h(h+1)+d_L(A)} - o(1)\\ &= \frac{d_L(A)^2}{h(h+1)+d_L(A)} - o(1).\end{aligned}$$

Combining the above arguments, we have

$$d_L(B) \geq \frac{d_L(A)^2}{h(h+1)+d_L(A)} > 0.$$

Let F be a finite set of nonnegative integers. Then there exists a positive integer m such that

$$F \subseteq [0, (h+1)^{m^2+1}].$$

For any integer $n > m$, by the definition of B,

$$[(h+1)^{n^2}, (h+1)^{n^2+1}] \cap (F \cup B) = [(h+1)^{n^2}, (h+1)^{n^2+1}] \cap B = \emptyset.$$

Since

$$(h+1)^{n^2+1} > h(h+1)^{n^2},$$

it follows that $(h+1)^{n^2+1} \notin h(F \cup B)$. Therefore, $F \cup B$ is not an asymptotic basis of order h for any finite set F.

This completes the proof of Theorem 4. □

References

[1] F.-J. Chen, Y.-G. Chen, "On minimal asymptotic bases", *Eur. J. Comb.* **32** (2011), no. 8, p. 1329-1335.
[2] Y.-G. Chen, M. Tang, "On a problem of Nathanson", *Acta Arith.* **185** (2018), no. 3, p. 275-280.
[3] P. Erdős, M. B. Nathanson, "Minimal asymptotic bases with prescribed densities", *Ill. J. Math.* **32** (1988), no. 3, p. 562-574.
[4] E. Härtter, "Ein Beitrag zur Theorie der Minimalbasen", *J. Reine Angew. Math.* **196** (1956), p. 170-204.
[5] M. Jańczak, T. Schoen, "Dense minimal asymptotic bases of order 2", *J. Number Theory* **130** (2010), no. 3, p. 580-585.
[6] M. Kneser, "Abschätzung der asymptotischen Dichte von Summenmengen", *Math. Z.* **58** (1953), p. 459-484.
[7] M. B. Nathanson, "Minimal bases and maximal nonbases in additive number theory", *J. Number Theory* **6** (1974), p. 324-333.
[8] ———, "Minimal bases and powers of 2", *Acta Arith.* **51** (1988), no. 5, p. 95-102.
[9] M. B. Nathanson, A. Sárközy, "On the maximum density of minimal asymptotic bases", *Proc. Am. Math. Soc.* **105** (1989), no. 1, p. 31-33.
[10] A. Stöhr, "Gelöste und ungelöste Fragen über Basen der natüurlichen Zahlenreihe", *J. Reine Angew. Math.* **194** (1955), p. 111-140.
[11] C.-F. Sun, "On a problem of Nathanson on minimal asymptotic bases", *J. Number Theory* **218** (2021), p. 152-160.
[12] M. Tang, D.-R. Ling, "On asymptotic bases and minimal asymptotic bases", *Colloq. Math.* **170** (2022), no. 1, p. 65-77.