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The ordinal of dynamical degrees of birational maps of the projective plane

L'ordinal des degrés dynamiques des transformations birationelles du plan projectif

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Abstract. We show that the ordinal of the dynamical degrees of all birational maps of the complex projective plane is ω^ω.

Résumé. Nous démontrons que l'ordinal des degrés dynamiques de toutes les transformations birationnelles du plan projectif complexe est ω^ω.

Keywords. Dynamical degree, rational projective surfaces, ordinals.

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1. Introduction

Fix a smooth projective surface X over a field $\mathbf{k}$. Any birational self-map f of X can be analysed by considering the dynamical system its iterates provide. A measure of how chaotic such a system becomes is the dynamical degree of $f\colon X \dashrightarrow X$, which is given by

$$\lambda(f) = \lim_{n\to\infty} (D\cdot (f^n)^* D)^{1/n},$$

where $D \subseteq X$ is any ample divisor and $D\cdot C$ denotes the intersection form. If X is the projective plane, the definition agrees with $\lambda(f) = \lim_{n\to\infty}(\deg(f^n))^{1/n}$. Since the dynamical degree is invariant under conjugation, the set $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X)) \subseteq \mathbb{R}$ of all possible dynamical degrees of all elements of $\mathrm{Bir}_{\mathbf{k}}(X)$ coincides with $\Lambda(\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2_{\mathbf{k}}))$ whenever X is a rational projective surface. The dynamical degree has been considered in many different settings, often due to its close connection with entropy, see for example [1–3, 10, 12, 14, 16, 22].

Since any dynamical degree on a rational projective surface is an algebraic integer ≥ 1 (see [12, Theorem 5.1]), the set $\Lambda(\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2_{\mathbf{k}}))$ is at most countable. In fact, by [20, Theorem 1] the set of all dynamical degrees of all rational self-maps of all projective varieties is at most countable. By contrast, the theory of ordinal numbers allows to make a qualitative statement about accumulation points of $\Lambda(\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2_{\mathbf{k}}))$. A totally ordered set is called *well ordered* if every non-empty subset has a minimum. It is a nice exercise to prove that, assuming the axiom of choice, well orderedness is equivalent to any decreasing sequence becoming stationary; the statement can also be found in [8, Chapter III, §6.5, Proposition 6]. Recall that a well ordered set corresponds to a unique ordinal, that is, its order type, and that we have a total order on ordinals. The ordinals of finite cardinality are in bijection with the natural numbers, and we denote by ω the smallest infinite ordinal, which is the order type of the natural numbers with the standard order.

Blanc and Cantat showed in [5, Theorem 7.2] that the set of all dynamical degrees of all projective surfaces is well ordered with respect to the standard order on $\mathbb{R}$. Yet, by [5, Theorem B], the union of all $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X))$ where X is geometrically non-rational and where $\mathbf{k}$ is arbitrary is a discrete closed subset of $\mathbb{R}$ and thus its ordinal is at most ω. To fully classify the ordinals of $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X))$ for any smooth projective surface X, the following question remains: what is the ordinal of $\Lambda(\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2_{\mathbf{k}}))$? We show that it is ω^ω if $\mathbf{k} = \mathbb{C}$. More generally, we have:

Theorem 1. *For any field* $\mathbf{k}$ *and any projective geometrically rational surface* X *defined over* $\mathbf{k}$*, the order type of* $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X)) \subseteq \mathbb{R}$ *is bounded above by* ω^ω*. If, in addition,* $\mathbf{k}$ *contains the real algebraic numbers and* X *is rational over* $\mathbf{k}$*, then* $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X)) = \omega^\omega$*.*

The second statement of Theorem 1 can be interpreted as follows: whenever $\mathbf{k}$ contains the real algebraic numbers and X is rational over $\mathbf{k}$, there exists a sequence of birational transformations whose dynamical degrees are each limits of a sequence of dynamical degrees of other birational transformations. Now, infinitely many terms of these dynamical degrees are again each the limit of a sequence, most of whose members are again limits of sequences, and so on. One can prove the lower bound for $\Lambda(\mathrm{Bir}_{\mathbf{k}}(X))$ in two ways, either assuming $\mathbf{k} = \mathbb{C}$ and using the Weyl group as in Section 4 (see Theorem 20), or by explicitly constructing rational surface automorphisms with the desired dynamical degrees as in Section 5. The first strategy was pointed out to the author by Curtis T McMullen. Section 4 results in the following additional statement:

Theorem 2. *The Weyl spectrum* $\Lambda(W) = \bigcup_{n\geq 3}\Lambda(W(E_n))$ *has order type* ω^ω*.*

For the rest of the paper, we omit the field $\mathbf{k}$ from the notation, and write $\mathrm{Bir}(X)$ instead of $\mathrm{Bir}_{\mathbf{k}}(X)$, and $\mathbb{P}^2$ instead of $\mathbb{P}^2_{\mathbf{k}}$.

2. Bounding from above

Denote by $\mathrm{Bir}(\mathbb{P}^2)$ the set of birational maps $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, and by $\mathrm{Bir}_d(\mathbb{P}^2)$ all birational maps of $\mathbb{P}^2$ of degree d, meaning that the map f can be described by three homogeneous polynomials of degree d having no common divisor. Thus, $\mathrm{Bir}_d(\mathbb{P}^2)$ is a subset of some projective space. One can see that it is locally closed, and therefore, $\mathrm{Bir}_d(\mathbb{P}^2)$ can be seen as an algebraic variety and endowed with a Zariski topology, see also [6, Proposition 2.15]. For any subset $S \subseteq \mathrm{Bir}(\mathbb{P}^2)$, write $\Lambda(S)$ for the set of all dynamical degrees that elements of S can attain. As the degree is greater than or equal to 1, the set $\Lambda(S)$ is contained in $\mathbb{R}^{\geq 1}$.

Let X be a topological space and $I \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ a subset. A function $g\colon X \to I$ is called *lower semicontinuous* if for any $r \in I$, the set $L(r) = \left\{x \in X \mid g(x) \leq r\right\}$ is closed in X. The following result of Xie shows that the dynamical degree as a function $\lambda\colon \mathrm{Bir}_d(\mathbb{P}^2) \to \mathbb{R}^{\geq 1}$ is lower semicontinuous.

Theorem 3 ([21, Theorem 1.5]). *Let* $\mathbf{k}$ *be an algebraically closed field and* $d \geq 1$ *an integer. Then for any* $\lambda < d$*, the set* $U_\lambda = \{f \in \mathrm{Bir}_d(\mathbb{P}^2) \mid \lambda(f) > \lambda\}$ *is a Zariski dense open subset of* $\mathrm{Bir}_d(\mathbb{P}^2)$*.*

The proof of the next result uses the lower semicontinuity of the dynamical spectrum and the noetherianity of the Zariski topology. Such an argument was already used at the end of the proof of [5, Theorem 7.2] to prove that the set $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ is well ordered. Here we apply the same strategy and go one step further by describing the associated ordinals.

Theorem 4. *Let* $\mathbf{k}$ *be an algebraically closed field,* $d \geq 1$ *an integer and* $X \subseteq \mathrm{Bir}_d(\mathbb{P}^2)$ *a locally closed subset, with* n *denoting the number of irreducible components of* X*. Then* $\Lambda(X) \subseteq \mathbb{R}$ *is well ordered and of order type smaller than or equal to* $\omega^{\dim(X)} n + n$.

Proof. The set $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ is well ordered by [5, Theorem 7.2], and it contains $\Lambda(X)$. Thus, $\Lambda(X)$ must be well ordered, too. We fix $d \geq 1$ and prove the second claim by induction on the dimension of X.

If $\dim(X) = 0$, then any irreducible component of X is a point, and the claim follows. Suppose the claim is proved for any locally closed subvariety of $\mathrm{Bir}_d(\mathbb{P}^2)$ of dimension $c-1 \geq 0$, and consider $X \subseteq \mathrm{Bir}_d(\mathbb{P}^2)$ locally closed with $\dim(X) = c$. Decompose X into its irreducible components $X_1 \cup \ldots \cup X_n$. Pick X_i with $\dim(X_i) = \dim(X) = c$, as any irreducible component of dimension smaller than c has ordinal smaller than $\omega^{\dim(X)} + 1$ by induction.

Assume by contradiction that the order type of $\Lambda(X_i)$ is strictly greater than $\omega^c + 1$. Then there exist $v_1 < v_2$ in $\Lambda(X_i)$ such that $\{v \in \Lambda(X_i) \mid v < v_1\}$ is of order type ω^c. By lower semicontinuity (see Theorem 3), the set $L(v_1) = \{x \in \mathrm{Bir}_d(\mathbb{P}^2) \mid \lambda(x) \leq v_1\}$ is closed in $\mathrm{Bir}_d(\mathbb{P}^2)$. Thus $L(v_1) \cap X_i$ is closed in X_i. If the dimension of this set were strictly smaller than c, its image under the map $\lambda\colon \mathrm{Bir}_d(\mathbb{P}^2) \to \mathbb{R}^{\geq 1}$ in $\Lambda(X_i)$ would by induction be smaller than or equal to $\omega^{c-1} m + m$ for some m counting the irreducible components of $L(v_1) \cap X_i$. But $\omega^{c-1} m + m < \omega^c$, a contradiction. Thus, $L(v_1) \cap X_i = X_i$, which in turn implies that $v_2 \leq v_1$. This cannot be. Therefore, any irreducible component X_i of X satisfies $\Lambda(X_i) \leq \omega^c + 1$ and hence $\Lambda(X) \leq \omega^c n + n$. This finishes the induction and proves the theorem. □

Theorem 5. *Over any field* $\mathbf{k}$*, the ordinal of* $\Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$ *is less than or equal to* ω^{4d+6}*, and the ordinal of* $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ *is less than or equal to* ω^{ω}.

Proof. We may reduce to $\mathbf{k}$ being algebraically closed. If the result holds in that case, then we use that any birational map defined over some field is also defined over an algebraic closure, and deduce the claim.

By Theorem 4, we see that $\Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$ is of order type less than or equal to $\omega^{\dim(\mathrm{Bir}_d(\mathbb{P}^2))} = \omega^{4d+6}$, where $\dim(\mathrm{Bir}_d(\mathbb{P}^2)) = 4d+6$ by [4, Theorem 1]. This proves the first claim. As $\mathrm{Bir}(\mathbb{P}^2) = \bigcup_{d\geq 1} \mathrm{Bir}_d(\mathbb{P}^2)$, we find that $\Lambda(\mathrm{Bir}(\mathbb{P}^2)) = \bigcup_{d\geq 1} \Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$. According to [9, Theorem 1], the union of finitely many ordinals which are subsets of a common set is bounded above by the Hessenberg normal sum of the ordinals, here denoted by $\oplus$. Recall that the Hessenberg normal sum of two ordinals $\alpha = \sum_{k=0}^{t} \omega^k a_k$, $\beta = \sum_{k=0}^{t} \omega^k b_k$ expressed in Cantor normal form satisfies $\alpha \oplus \beta = \sum_{k=0}^{t} \omega^k (a_k + b_k)$. Consider $\bigcup_{1\leq d\leq e} \mathrm{Bir}_d(\mathbb{P}^2)$ with $e \geq 1$. Then, by writing $\Lambda(\mathrm{Bir}_d(\mathbb{P}^2)) = \sum_{i=0}^{4d+6} \omega^i a_{d,i}$ in the Cantor normal form, we find

$$\bigcup_{1\leq d\leq e} \Lambda(\mathrm{Bir}_d(\mathbb{P}^2)) \leq \bigcup_{1\leq d\leq e} \Lambda(\mathrm{Bir}_d(\mathbb{P}^2)) = \bigcup_{1\leq d\leq e} \sum_{i=0}^{4d+6} \omega^i a_{d,i} = \sum_{k=0}^{4e+6} \omega^k b_k,$$

where b_k are some sum of the $a_{d,i}$. But then $\bigcup_{1\leq d\leq e} \Lambda(\mathrm{Bir}_d(\mathbb{P}^2)) \leq \omega^{4e+7}$. Thus, $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ is of order type at most ω^{ω}. □

3. Arithmetic properties of auxiliary polynomials

To bound $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ from below, we first collect arithmetic facts about very specific, auxiliary polynomials and their largest real roots, all of which play a part in Sections 4 and 5. This current section may thus be skipped if one would first like to understand why the polynomials below and their largest positive roots are of importance.

Definition 6. *Fix $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$. Abbreviate $I_m = \{2, \dots, m\}$. Then, define*

$$p_{d,\underline{n}}(X) = (X^2 - (d-1)X - 1) \prod_{i=2}^{m} (X^{n_i} + 1) + X \sum_{i=2}^{m} \prod_{j \in I_m \setminus \{i\}} (X^{n_j} + 1).$$

Note that if $m = 1$, we have $p_{d,\emptyset}(X) = X^2 - (d-1)X - 1$, with roots $\frac{1}{2}(d - 1 \pm \sqrt{d^2 - 2d + 5})$.

The polynomial $p_{d,\underline{n}}(X)$ appears as a factor in the characteristic polynomial of a certain matrix: fix $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$, and consider the matrix

$$J_d^{\underline{n}} = \left(\begin{array}{c|c|c|c|c|c}
d & \begin{matrix} 0 & d-1 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & 1 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & 1 \end{matrix} & \cdots & \begin{matrix} 0 & \cdots & 0 & 1 \end{matrix} \\
\hline
\begin{matrix} -(d-1) \\ 0 \end{matrix} & \begin{matrix} 0 & -(d-2) \\ 1 & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 0 \end{matrix} \\
\hline
\begin{matrix} -1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 & -1 \\ & & & 0 \\ & \mathbb{1}_{n_2-1} & & \vdots \\ & & & 0 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\
\hline
\begin{matrix} -1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \mathbf{0} & \begin{matrix} 0 & \cdots & 0 & -1 \\ & & & 0 \\ & \mathbb{1}_{n_3-1} & & \vdots \\ & & & 0 \end{matrix} & \cdots & \mathbf{0} \\
\hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
\begin{matrix} -1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \mathbf{0} & \mathbf{0} & \cdots & \begin{matrix} 0 & \cdots & 0 & -1 \\ & & & 0 \\ & \mathbb{1}_{n_m-1} & & \vdots \\ & & & 0 \end{matrix}
\end{array}\right), \qquad (\diamondsuit)$$

with $\mathbb{1}_{n_k-1}$ the identity matrix of dimension $n_k - 1$. Here, $\mathbf{0}$ is short hand for the zero matrix, always of the respective suitable dimension.

Lemma 7. *Consider $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$. Then the characteristic polynomial* $\mathrm{char}_{d,\underline{n}}(X)$ *of $J_d^{\underline{n}}$ satisfies* $\mathrm{char}_{d,\underline{n}}(X) = (X-1)p_{d,\underline{n}}(X)$.

Proof. To calculate the characteristic polynomial of $J_d^{\underline{n}}$ consider first the last $n_m + 1$ rows of $X\mathbb{1} - J_d^{\underline{n}}$:

$$\left(\begin{array}{c|cc|c|c|c|c}
1 & 0 & 1 & & & & \begin{matrix} X & 0 & \cdots & 0 & 1 \\ -1 & X & & & 0 \\ & -1 & \ddots & & \vdots \\ & & \ddots & X & 0 \\ & & & -1 & X \end{matrix} \\
0 & 0 & 0 & \mathbf{0} & \cdots & \mathbf{0} & \\
\vdots & \vdots & \vdots & & & & \\
0 & 0 & 0 & & & &
\end{array}\right).$$

With row manipulations this can be reduced to

$$\left(\begin{array}{c|cc|c|c|c|ccccc} 1 & 0 & 1 & & & & 0 & 0 & \cdots & 0 & X^{n_m}+1 \\ 0 & 0 & 0 & & & & -1 & 0 & & & X^{n_m-1} \\ \vdots & \vdots & \vdots & \mathbf{0} & \cdots & \mathbf{0} & & -1 & \ddots & & \vdots \\ & & & & & & & & \ddots & 0 & X^2 \\ 0 & 0 & 0 & & & & & & & -1 & X \end{array}\right).$$

Repeat this for the other rows. Then, using Laplace expansion along the columns with the -1 entries, we find

$$\begin{aligned} \mathrm{char}_{d,\underline{n}}(X) &= \det\left(\begin{array}{ccc|cccc} X-d & 0 & -(d-1) & -1 & -1 & \cdots & -1 \\ d-1 & X & d-2 & 1 & 1 & \cdots & 1 \\ 0 & -1 & X & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & 1 & X^{n_2}+1 & & & \\ 1 & 0 & 1 & & X^{n_3}+1 & & \\ \vdots & \vdots & \vdots & & & \ddots & \\ 1 & 0 & 1 & & & & X^{n_m}+1 \end{array}\right) \\ &= \det\left(\begin{array}{ccc|cccc} X-d & 0 & -(d-1) & -1 & -1 & \cdots & -1 \\ X-1 & X & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & X & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & 1 & X^{n_2}+1 & & & \\ 1 & 0 & 1 & & X^{n_3}+1 & & \\ \vdots & \vdots & \vdots & & & \ddots & \\ 1 & 0 & 1 & & & & X^{n_m}+1 \end{array}\right) \end{aligned} \tag{1}$$

If we apply Laplace expansion along the fourth column, we find that $\mathrm{char}_{d,\underline{n}}(X)$ is equal to

$$\begin{aligned} &\det\left(\begin{array}{ccc|ccc} X-1 & X & -1 & 0 & \cdots & 0 \\ 0 & -1 & X & 0 & \cdots & 0 \\ \hline 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & X^{n_3}+1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ 1 & 0 & 1 & & & X^{n_m}+1 \end{array}\right) + (X^{n_2}+1)\det\left(\begin{array}{ccc|ccc} X-d & 0 & -(d-1) & -1 & \cdots & -1 \\ X-1 & X & -1 & 0 & \cdots & 0 \\ 0 & -1 & X & 0 & \cdots & 0 \\ \hline 1 & 0 & 1 & X^{n_3}+1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ 1 & 0 & 1 & & & X^{n_m}+1 \end{array}\right) \\ &\qquad = (X-1)X\prod_{i\in I_m\setminus\{2\}}(X^{n_i}+1) + (X^{n_2}+1)\det(A_2). \end{aligned}$$

Here, A_2 denotes the matrix obtained by deleting the fourth row and fourth column of the matrix in (1), which corresponds to the cofactor of the entry $X^{n_2}+1$. Denote by A_j the matrix obtained by also deleting the fourth row and fourth column of A_{j-1}. Then, with induction, we find

$$\begin{aligned} \det(A_j) &= (X-1)X\prod_{i>j}(X^{n_i}+1) + (X^{n_j}+1)\det(A_{j+1}) \\ &= (X-1)X\sum_{k>j}\prod_{i\neq k}(X^{n_i}+1) + \prod_{i>j}(X^{n_i}+1)\det(A_{2d-1}). \end{aligned}$$

Note that

$$\det(A_{2d-1}) = \det\begin{pmatrix} X-d & 0 & -(d-1) \\ X-1 & X & -1 \\ 0 & -1 & X \end{pmatrix} = X^3 - dX^2 + (d-2)X + 1 = (X-1)(X^2-(d-1)X-1).$$

Thus, $\mathrm{char}_{d,\underline{n}}(X) = (X-1)\big((X^2-(d-1)X-1)\prod_{i=2}^m(X^{n_i}+1) + X\sum_{i=2}^m\prod_{j\in I_m\setminus\{i\}}(X^{n_j}+1)\big) = (X-1)p_{d,\underline{n}}(X)$, as claimed. □

Using this correspondence between the polynomials $p_{d,\underline{n}}(X)$ and the characteristic polynomial of the matrices $J_d^{\underline{n}}$ in $(\diamondsuit)$, one can prove that there is at most one root of $p_{d,\underline{n}}(X)$ whose absolute value is larger than 1, and this root is real. To show this, we first prove the following lemma, for which we define $Q = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$. Also, we write A^T for the transpose of a matrix A.

Lemma 8. *For $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$, the matrix $J_d^{\underline{n}}$ satisfies $J_d^{\underline{n}} Q (J_d^{\underline{n}})^T = Q + H_{d,m}$, where*

$$H_{d,m} = \left(\begin{array}{cc|c} 2d-1-m & -(2d-1-m) & 0 \cdots 0 \\ -(2d-1-m) & 2d-1-m & 0 \cdots 0 \\ \hline 0 & 0 & \\ \vdots & \vdots & \mathbf{0} \\ 0 & 0 & \end{array}\right).$$

The matrix $H_{d,m}$ is positive semidefinite.

Proof. The first part of the claim follows by a direct calculation using Q and $J_d^{\underline{n}}$ given in $(\diamondsuit)$. The characteristic polynomial of $H_{d,m}$ is equal to the product of $X - 2(2d-1-m)$ and some power of X. Since $m \leq 2d-1$, all eigenvalues of $H_{d,m}$ are non-negative. Thus, $H_{d,m}$ is positive semidefinite. □

Equipped with this lemma, we can prove that there is at most one root of $p_{d,\underline{n}}(X)$ whose absolute value is greater than 1, and this must be a real root. The idea of the proof comes directly from the proof of [12, Theorem 5.1, Assertion (1)]. Yet for the purpose of this section, we are solely concerned with matrices and their characteristic polynomials, and thus do not need to apply [12, Theorem 5.1] in its full generality.

Proposition 9. *Fix $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$. Then $p_{d,\underline{n}}(X)$ has at most one root whose modulus is strictly larger than 1, and this root is real.*

Proof. By Lemma 7, $(X-1)p_{d,\underline{n}}(X) = \mathrm{char}_{d,\underline{n}}(X)$, where $\mathrm{char}_{d,\underline{n}}(X)$ is the characteristic polynomial of the matrix $J_d^{\underline{n}}$ defined in $(\diamondsuit)$. Thus, a root of $p_{d,\underline{n}}(X)$ corresponds to an eigenvalue of $J_d^{\underline{n}}$. Suppose by contradiction that we have two eigenvalues $\mu_1, \mu_2 \in \mathbb{C}$ of $J_d^{\underline{n}}$ with $|\mu_1|, |\mu_2| > 1$, and denote by v_1, v_2 some corresponding eigenvectors. We claim that on the subspace $\mathbb{C}v_1 + \mathbb{C}v_2$, the hermitian form corresponding to Q is positive semidefinite. This is equivalent to proving that $v_i^T Q \overline{v_i} \geq 0$ for $i = 1,2$ and $(v_1^T Q \overline{v_1})(v_2^T Q \overline{v_2}) - |v_1^T Q \overline{v_2}|^2 \geq 0$.

Abbreviate $H = H_{d,m}$ the matrix from Lemma 8. Note that $\mu_i \overline{\mu_j} v_i^T Q \overline{v_j} = v_i^T J_d^{\underline{n}} Q (J_d^{\underline{n}})^T \overline{v_j} \overset{8}{=} v_i^T Q \overline{v_j} + v_i^T H \overline{v_j}$ for $i,j = 1,2$, and thus $v_i^T Q \overline{v_i} = \frac{v_i^T H \overline{v_i}}{|\mu_i|^2 - 1}$ and $v_1^T Q \overline{v_2} = \frac{v_1^T H \overline{v_2}}{\mu_1 \overline{\mu_2} - 1}$. Since H is positive semidefinite by Lemma 8, we directly see that $v_i^T Q \overline{v_i} \geq 0$ holds. Furthermore,

$$(v_1^T Q \overline{v_1})(v_2^T Q \overline{v_2}) - |v_1^T Q \overline{v_2}|^2 = \frac{1}{(|\mu_1|^2-1)(|\mu_2|^2-1)} (v_1^T H \overline{v_1})(v_2^T H \overline{v_2}) - \frac{1}{|\mu_1 \overline{\mu_2} - 1|^2} |v_1^T H \overline{v_2}|^2.$$

Since $(|\mu_1|^2 - 1)(|\mu_2|^2 - 1) \leq |\mu_1 \overline{\mu_2} - 1|^2$ by expanding $|\mu_1 - \mu_2|^2 \geq 0$, we deduce

$$(v_1^T Q \overline{v_1})(v_2^T Q \overline{v_2}) - |v_1^T Q \overline{v_2}|^2 \geq \frac{1}{(|\mu_1|^2-1)(|\mu_2|^2-1)} ((v_1^T H \overline{v_1})(v_2^T H \overline{v_2}) - |v_1^T H \overline{v_2}|^2) \geq 0,$$

where the last inequality is due to H being positive semidefinite by Lemma 8. Thus, Q is positive semidefinite on $\mathbb{C}v_1 + \mathbb{C}v_2$. But the signature of Q is $(1, \dim(Q) - 1)$, so there cannot be a subspace of dimension 2 on which Q is positive semidefinite. Therefore, $\mathbb{C}v_1 = \mathbb{C}v_2$ and $\mu_1 = \mu_2$. As $\overline{\mu_1}$ is also an eigenvalue of $J_d^{\underline{n}}$, we additionally deduce $\mu_1 = \overline{\mu_1}$. The root of largest absolute value must thus be real. This finishes the proof of the proposition. □

Definition 10. *For $d \geq 1$, $1 \leq m \leq 2d-1$ and $\underline{n}$ we denote by $\lambda_{d,\underline{n}} \in \mathbb{R}$ the root of largest absolute value of $p_{d,\underline{n}}(X)$.*

We are interested in describing $\lambda_{d,\underline{n}}$; by the following proposition, it must be larger than 2 under mild assumptions.

Proposition 11. *Fix $d \geq 4$, $1 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$ with $n_i \geq 2$ for all $2 \leq i \leq m$. Then $\lambda_{d,\underline{n}}$ is strictly larger than 2.*

Proof. The claim for $m = 1$ follows by observing that the largest root λ of $p_{d,\emptyset}(X) = X^2 - (d-1)X - 1$ satisfies $\lambda - \frac{1}{\lambda} = d-1$ and thus $2 < d-1 < \lambda$. Hence, assume $m \geq 2$. The polynomial $p_{d,\underline{n}}(X)$ is monic. Thus, if at $X = 2$, its value is negative, there must be a zero of $p_{d,\underline{n}}(X)$ which is larger than 2. The value $p_{d,\underline{n}}(2)$ is strictly negative if and only if

$$2\sum_{i=2}^{m} \prod_{j \in I_m \setminus \{i\}} (2^{n_j}+1) < (2d-5)\prod_{i=2}^{m}(2^{n_i}+1).$$

Note that $4 < 2^{n_i}+1$ since $n_i \geq 2$. Therefore:

$$\begin{aligned} 2\sum_{i=2}^{m} \prod_{j \in I_m \setminus \{i\}} (2^{n_j}+1) &< 2\frac{1}{4}\sum_{i=2}^{m}\prod_{j=2}^{m}(2^{n_j}+1) = \frac{1}{2}(m-1)\prod_{i=2}^{m}(2^{n_i}+1) \\ &\leq \frac{1}{2}(2d-2)\prod_{i=2}^{m}(2^{n_i}+1) \leq (2d-5)\prod_{i=2}^{m}(2^{n_i}+1), \end{aligned}$$

where in the last step we used $d \geq 4$. This proves the claim. □

Whenever we increase one of the n_i defining some $p_{d,\underline{n}}(X)$, the largest root also increases strictly.

Proposition 12. *Fix $d \geq 4$, $2 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_m)$ with $n_i \geq 2$. For $2 \leq k \leq m$, denote $\underline{n}_k = (n_2, \dots, n_{k-1}, n_k+1, n_{k+1}, \dots, n_m)$. Then $\lambda_{d,\underline{n}} < \lambda_{d,\underline{n}_k}$.*

Proof. The polynomials $p_{d,\underline{n}}(X)$ and $p_{d,\underline{n}_k}(X)$ are related by

$$(X^{n_k}+1)p_{d,\underline{n}_k}(X) = (X^{n_k+1}+1)p_{d,\underline{n}}(X) - (X-1)X^{n_k+1}\prod_{j \in I_m \setminus \{k\}}(X^{n_j}+1).$$

By Proposition 11, we know that $\lambda_{d,\underline{n}} > 2$. Since $p_{d,\underline{n}}(\lambda_{d,\underline{n}}) = 0$, we deduce $p_{d,\underline{n}_k}(\lambda_{d,\underline{n}}) < 0$. The polynomial $p_{d,\underline{n}_k}(X)$ is monic, which implies that it must have a zero which is strictly larger than $\lambda_{d,\underline{n}}$. Hence, $\lambda_{d,\underline{n}_k} > \lambda_{d,\underline{n}}$, which implies the claim. □

As a last property of the auxiliary polynomials, we prove that if we fix $n_2, \dots, n_m \geq 2$ and consider all $\lambda_{d,\underline{n}'}$ where $\underline{n}' = (n_2, \dots, n_m, n_{m+1})$ with $n_{m+1} \geq 2$, then the sequence $(\lambda_{d,\underline{n}'})_{n_{m+1}}$ has $\lambda_{d,\underline{n}}$ as its limit.

Lemma 13. *Fix $d \geq 4$, $2 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_{m-1})$, and $\underline{n}' = (n_2, \dots, n_{m-1}, n_m)$, with all $n_i \geq 2$. Then $\lambda_{d,\underline{n}'} < \lambda_{d,\underline{n}}$.*

Proof. The two polynomials $p_{d,\underline{n}}(X)$ and $p_{d,\underline{n}'}(X)$ are related as $p_{d,\underline{n}'}(X) = (X^{n_m}+1)p_{d,\underline{n}}(X) + X\prod_{j \in I_{m-1}}(X^{n_j}+1)$. By Proposition 11, we find $\lambda_{d,\underline{n}'} > 2$. Therefore, plugging in $\lambda_{d,\underline{n}'}$, and using that it is a root of $p_{d,\underline{n}'}(X)$, we find

$$p_{d,\underline{n}}(\lambda_{d,\underline{n}'}) = -\lambda_{d,\underline{n}'}\prod_{j \in I_{m-1}}(\lambda_{d,\underline{n}'}^{n_j}+1) < 0$$

As $p_{d,\underline{n}}(X)$ is monic, this proves $\lambda_{d,\underline{n}'} < \lambda_{d,\underline{n}}$. □

Proposition 14. *Fix $d \geq 4$, $2 \leq m \leq 2d-1$ and $\underline{n} = (n_2, \dots, n_{m-1})$ with $n_i \geq 2$. Denote $\underline{n}_{n_m} = (n_2, \dots, n_{m-1}, n_m)$. Then the limit of the sequence $(\lambda_{d,\underline{n}_{n_m}})_{n_m}$ exists and is equal to $\lambda_{d,\underline{n}}$.*

Proof. Fix some $\underline{n} = (n_2, \ldots, n_{m-1})$ and $2 < \lambda < \lambda_{d,\underline{n}}$. Note that the polynomials satisfy

$$p_{d,\underline{n}_{n_m}}(X) = (X^{n_m} + 1) p_{d,\underline{n}}(X) + X \prod_{j \in I_{m-1}} (X^{n_j} + 1).$$

Note that since by Proposition 9, $\lambda_{d,\underline{n}}$ is the only real root of $p_{d,\underline{n}}(X)$ which is larger than 1, on $(2, \lambda_{d,\underline{n}})$, the polynomial $p_{d,\underline{n}}(X)$ is negative. Thus, there exists $n_m(\lambda) \geq 2$ such that for all $n_m \geq n_m(\lambda)$, we have

$$p_{d,\underline{n}_{n_m}}(\lambda) = (\lambda^{n_m} + 1) p_{d,\underline{n}}(\lambda) + \lambda \prod_{j \in I_{m-1}} (\lambda^{n_j} + 1) < 0.$$

This implies that for $n_m \geq n_m(\lambda)$, we have $\lambda < \lambda_{d,\underline{n}_{n_m}} < \lambda_{d,\underline{n}}$, where the last inequality is due to Lemma 13. This proves the claim. □

To finish this section, we prove that we can construct a well ordered set out of the $\lambda_{d,\underline{n}}$ such that we can precisely describe the ordinal of that set.

Proposition 15. *Consider for $d \geq 4$ and $m \leq 2d - 1$ the inductively defined sets*

$$\Lambda_{d,1} = \{\tfrac{1}{2}(d - 1 + \sqrt{d^2 - 2d + 5})\},$$
$$\Lambda_{d,2} = \{\lambda_{d,(n_2)} \mid 2 \leq n_2 \text{ and } d - 1 < \lambda_{d,(n_2)}\},$$
$$\Lambda_{d,m} = \left\{ \lambda_{d,(n_2,\ldots,n_m)} \,\middle|\, \begin{array}{r} 2 \leq n_2 < \ldots < n_m \text{ and } \lambda_{d,(n_2,\ldots,n_{m-1}-1)} \in \Lambda_{d,m-1} \text{ and } \lambda_{d,(n_2,\ldots,n_{m-1})} \in \Lambda_{d,m-1} \\ \text{and } \lambda_{d,(n_2,\ldots,n_{m-1}-1)} < \lambda_{d,(n_2,\ldots,n_m)} < \lambda_{d,(n_2,\ldots,n_{m-1})} \end{array} \right\}.$$

The following holds:

1. *The well ordering on the sets $\Lambda_{d,m}$ with respect to the lexicographic ordering on $(n_2, \ldots, n_m)$ agrees with the (well) ordering inherited from the one on the real line.*
2. *The ordinal of $\Lambda_{d,m}$ with respect to this ordering is ω^{m-1}.*
3. *The disjoint union $\sqcup_{d\geq 4} \Lambda_{d,2d-1}$ has ordinal equal to ω^ω.*

Proof. We prove Claims 1 and 2 simultaneously by induction. For $m = 1$ it is clear. For $m = 2$, the elements of $\Lambda_{d,2}$ indexed by n_2 form by Proposition 12 an ascending chain of elements. Thus $\Lambda_{d,2}$ is well ordered with ordinal ω. Now assume we know for some $m \in \{2, \ldots, 2d-2\}$ that the well ordering on $\Lambda_{d,m}$ given by the lexicographic ordering of the indexing set is the same as the one inherited from the real line and that $\Lambda_{d,m}$ has ordinal ω^{m-1} with respect to this ordering.

Define $J_m = \{(n_2, \ldots, n_m) \in \mathbb{Z}^{m-1} \mid 2 \leq n_2 < \ldots < n_m$ and $\lambda_{d,(n_2,\ldots,n_m-1)} \in \Lambda_{d,m}$ and $\lambda_{d,(n_2,\ldots,n_m)} \in \Lambda_{d,m}\}$ and for each $(n_2, \ldots, n_m) \in J_m$, define the open interval $I_{(n_2,\ldots,n_m)} = (\lambda_{d,(n_2,\ldots,n_m-1)}, \lambda_{d,(n_2,\ldots,n_m)})$. These intervals serve the purpose of making the sequences in $\Lambda_{d,m}$ disjoint. The deduction on the ordinal will then be immediate.

By Proposition 14, for each $(n_2, \ldots, n_m) \in J_m$, there exists $n_{m+1}(\lambda_{d,(n_2,\ldots,n_m-1)}) > n_m$ such that for $n_{m+1} \geq n_{m+1}(\lambda_{d,(n_2,\ldots,n_m-1)})$, the values $\lambda_{d,(n_2,\ldots,n_{m+1})}$ form a sequence with limit $\lambda_{d,(n_2,\ldots,n_m)}$. By Proposition 12, this sequence is strictly increasing. Furthermore, we can choose this sequence to lie entirely within the interval $I_{(n_2,\ldots,n_m)}$. Thus, the sequence within $I_{(n_2,\ldots,n_m)}$ is well ordered. Moreover, we can write $\Lambda_{d,m+1}$ as a union of such sequences. Since the intervals $I_{(n_2,\ldots,n_m)}$ are disjoint, so are the sequences making up $\Lambda_{d,m+1}$. Therefore, $\Lambda_{d,m+1}$ is well ordered in regard to the lexicographic ordering on its indexing set, too, and this ordering agrees again with the ordering on $\mathbb{R}$. This proves 1. Each sequence has ordinal ω, and by induction, $\Lambda_{d,m}$ has ordinal ω^{m-1}; this implies that $\Lambda_{d,m+1}$ has ordinal ω^m, as claimed in 2.

As for the last claim, note that for a fixed $d \geq 4$, the $\Lambda_{d,m}$ with $m \in \{1, \ldots, 2d-1\}$ are disjoint by construction and by Lemma 13. For $4 \leq d < d'$, note that all elements of $\Lambda_{d',m}$ for all $m \in \{1, \ldots, 2d' - 1\}$ are larger than $d' - 1$ by construction. Furthermore, all elements of $\Lambda_{d,m}$ with $m \in \{2, \ldots, 2d-1\}$ are smaller than $\frac{1}{2}(d - 1 + \sqrt{d^2 - 2d + 5})$, which is smaller than $d \leq d' - 1$ for $d \geq 4$. Therefore, in particular, the $\Lambda_{d,2d-1}$ are disjoint, and their union has ordinal ω^ω. This proves 3, and hence the proposition. □

Remark 16. Note that the sets $\Lambda_{d,m}$ do not contain any accumulation points in the open interval topology on $\mathbb{R}$. In fact, for any $m \in \{2,\dots,2d-1\}$, the accumulation points of the set $\Lambda_{d,m}$ lie in $\Lambda_{d,m-1}$ by Proposition 14, and the closure of $\Lambda_{d,m}$ is equal to $\bigsqcup_{1\le i\le m} \Lambda_{d,i}$.

4. Bounding from below using the Weyl group

We introduce the Weyl group in a brief way; for a more comprehensive introduction to Coxeter groups see [15] and [7], and for an introduction to the Weyl group as it relates to this setting see [13] and [16]. The *Weyl group* is defined as the Coxeter group $W_n = W(E_n)$ given by the Coxeter-Dynkin diagram E_n on n vertices as in Figure 1.

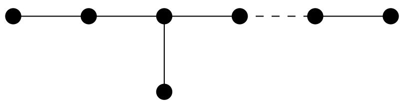

Figure 1. The graph E_n on n vertices.

Alternatively, fix $n \ge 3$. Write $\mathbb{Z}^{1,n}$ for the lattice $\mathbb{Z}^{n+1}$ with inner product $x\cdot x = x_0^2 - x_1^2 - \cdots - x_n^2$ and standard basis $(e_0, e_1, \dots, e_n)$. Then the Weyl group W_n is the subgroup of $O(\mathbb{Z}^{1,n})$ generated by the reflections $x \mapsto x + (x\cdot\alpha_i)\alpha_i$, where $\alpha_0 = e_0 - e_1 - e_2 - e_3$ and $\alpha_i = e_i - e_{i+1}$ for $i = 1,\dots,n-1$. There exists an isometry

$$\mathbb{Z}^{1,n} \longrightarrow \operatorname{Pic}(X), \quad e_0 \longmapsto L, \quad e_i \longmapsto \mathscr{E}_i, \quad 1 \le i \le n$$

for a rational surface X obtained as the blow-up of $\mathbb{P}^2$ in n points, where L is the pullback of a general line and $\mathscr{E}_i$ are the exceptional curves. Via this isometry, W_n is the subgroup of $\operatorname{Aut}(\operatorname{Pic}(X)) \cong \mathrm{GL}_{n+1}(\mathbb{Z})$ generated by the permutation matrices with the upper left entry fixed as 1, and by the matrix

$$A_0 = \left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & \\ -1 & 0 & -1 & -1 & \\ -1 & -1 & 0 & -1 & \\ -1 & -1 & -1 & 0 & \\ \hline & & & & \mathbb{1} \end{array}\right).$$

Even more, Nagata [17] proved that the image of $\operatorname{Aut}(X)$ under the homomorphism $\operatorname{Aut}(X) \to \operatorname{Aut}(\operatorname{Pic}(X))$ lies in W_n. This homomorphism is injective whenever out of the n points blown up we can pick four points where no three are collinear. Note that W_n must not necessarily be equal to the subgroup consisting of elements preserving the intersection form and the anticanonical divisor: take for example the reflection determined by $3e_0 - \sum_{i=1}^{10} e_i + e_{11}$.

The matrices with $m = 2d-1$ introduced in ($\diamondsuit$) are all elements of a Weyl group.

Lemma 17. *For any $d \ge 2$ and $\underline{n} = (n_2,\dots,n_{2d-1})$ with $n_i \in \mathbb{Z}^{\ge 1}$ the matrix $J_d^{\underline{n}}$ lies in W_n with $n = \sum_{i=2}^{2d-1} n_i + 2$.*

Proof. Fix some $\underline{n} = (n_2,\dots,n_{2d-1})$ and $J_d^{\underline{n}}$ as in ($\diamondsuit$). Any permutation matrix with fixed upper left entry lies in W_n. Consider P_1 such a permutation matrix which by right multiplication permutes columns 3, n_2+3 and n_2+n_3+3 of any matrix into second, third and fourth position, respectively. Similarly, there exists a permutation matrix P_1' in W_n which by multiplication from the left permutes rows 4 and n_2+4 of a matrix into third and fourth position. Multiplying from the right and the left, we can bring $J^{\frac{n}{d}}$ into the following form:

$$P_1' J^{\frac{n}{d}} P_1 = \left(\begin{array}{c|c|c} \begin{array}{cccc} d & d-1 & 1 & 1 \\ -(d-1) & -(d-2) & -1 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{array} & * & \mathbf{0} \\ \hline * & * & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbb{1} \end{array}\right).$$

Then, multiplying by A_0 (defined above) from the right, we obtain

$$P_1' J^{\frac{n}{d}} P_1 A_0 = \left(\begin{array}{c|c|c} \begin{array}{cccc} d & d-1 & 1 & 1 \\ -(d-1) & -(d-2) & -1 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{array} & * & \mathbf{0} \\ \hline * & * & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbb{1} \end{array}\right) \left(\begin{array}{c|c} \begin{array}{cccc} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{array} & \\ \hline & \mathbb{1} \end{array}\right) = \left(\begin{array}{c|c|c} \begin{array}{cccc} d-1 & d-2 & 0 & 0 \\ -(d-2) & -(d-3) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} & * & \mathbf{0} \\ \hline * & * & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbb{1} \end{array}\right).$$

The $2d-2$ columns and rows indicated by $*$ in $P_1' J^{\frac{n}{d}} P_1 A_0$ above are of the form $e_0 - e_i - e_j$ or $(e_0 - e_i - e_j)^T$ for some $i \neq j$. Thus, there exist permutation matrices $P_2, P_2' \in W_n$, both with 1 in the upper left entry, such that

$$P_2' P_1' J^{\frac{n}{d}} P_1 A_0 P_2 = \left(\begin{array}{c|c|c} \begin{array}{cccc} d-1 & d-2 & 1 & 1 \\ -(d-2) & -(d-3) & -1 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{array} & * & \mathbf{0} \\ \hline * & * & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbb{1} \end{array}\right).$$

We can therefore inductively repeat the process of multiplying by A_0 from the right and permuting the rows and columns, and obtain that $P_{d-2}' \cdots P_1' J^{\frac{n}{d}} P_1 A_0 \cdots P_{d-2} A_0$ is the identity matrix. Thus, $J^{\frac{n}{d}}$ is indeed an element of W_n. This proves the lemma. □

Define the *Weyl spectrum* $\Lambda(W)$ as the set of all spectral radii of all elements of all W_n with $n \geq 3$. By the following result of Uehara, rephrased to fit the framework of this article, the Weyl spectrum agrees with all possible dynamical degrees of rational surface automorphisms.

Theorem 18 ([19, Theorem 1.1]). *If the ground field is $\mathbb{C}$, then the Weyl spectrum $\Lambda(W)$ agrees with*

$$\Lambda\big(\{f \in \mathrm{Bir}(\mathbb{P}^2) \,\big|\, \exists\, \pi\colon X \to \mathbb{P}^2 \colon \pi^{-1} f \pi \in \mathrm{Aut}(X)\}\big).$$

With Uehara's result, we can infer the lower bound ω^ω on the ordinal of $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ by showing it for $\Lambda(W)$.

Proposition 19.

(1) *The Weyl spectrum $\Lambda(W)$ is a well ordered subset of $\mathbb{R}$ and its ordinal is greater than or equal to ω^ω.*

(2) *The ordinal of $\Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$ is greater than or equal to ω^{2d-2}.*

Proof. The Weyl spectrum $\Lambda(W)$ is a subset of $\Lambda(\mathrm{Bir}(\mathbb{P}^2)) \subseteq \mathbb{R}$, which is well ordered by [5, Theorem 7.2]. Therefore, $\Lambda(W)$ is also well ordered. By Lemma 17, all the matrices $J^{\frac{n}{d}}$ in ($\diamondsuit$) with $m = 2d-1$ belong to the respective Weyl group W_n with $n = \sum_{i=2}^{2d-1} n_i + 2$. Therefore, by Lemma 7, the disjoint sets $\Lambda_{d,2d-1}$ defined in Proposition 15 lie in $\Lambda(W)$ and thus $\bigsqcup_{d\geq 4} \Lambda_{d,2d-1} \subseteq \Lambda(W)$. By Proposition 15 3, we deduce that the ordinal of $\Lambda(W)$ is at least ω^ω, which proves (1). Claim (2) follows from observing that $\Lambda_{d,2d-1} \subseteq \Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$. This proves the proposition. □

Theorem 20. *If the base field is $\mathbb{C}$, then both the ordinal of $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ and the ordinal of the Weyl spectrum $\Lambda(W)$ are equal to ω^ω.*

Proof. Since by Theorem 18, we have the inclusion $\Lambda(W) \subseteq \Lambda(\mathrm{Bir}(\mathbb{P}^2))$. Thus, by Proposition 19 1, the ordinal of $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ is also greater than or equal to ω^ω. This together with Theorem 5 proves the claims. □

5. Bounding from below using explicit realisations

In this section, we prove that the ordinal of $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ can be bounded from below by ω^ω without appealing to Theorem 18. We construct suitable birational maps having the elements of $\Lambda_{d,2d-1}$ defined in Proposition 15 as dynamical degrees. To this end, we show a general result (Proposition 23): Suppose we are given a matrix, an eigenvalue and a corresponding eigenvector, satisfying some geometrically inspired conditions. Then, there exist points on a cuspidal cubic in $\mathbb{P}^2$ such that their blow-up admits an automorphism having precisely this matrix encoding its action on the Picard group. In the terminology of [19], we then prove the realisability of the orbit data explicitly. The proof we provide is more direct, and easier; it also shows that our realisations are Jonquières maps. We refer to [11, 16, 18, 19] for similar constructions.

Note that in this section, for the constructions to work, the field of definition needs to contain the field of real algebraic integers; this implies further that $\mathbf{k}$ is of characteristic 0.

5.1. *Existence of maps with base points on a cuspidal cubic*

Consider a birational map $f \in \mathrm{Bir}(\mathbb{P}^2)$. Assume that there exist points $p_1, \dots, p_n \in \mathbb{P}^2$ such that f lifts to an automorphism of the blow-up X of $\mathbb{P}^2$ in the points $p_1, \dots, p_n$. Call this automorphism f as well. The automorphism f induces an automorphism of the Picard group via pushforward, i.e. $f_*\colon [D] \mapsto [f(D)]$. This automorphism can be represented by an integer-valued matrix in the ordered basis $([L], [E_1], \dots, [E_n])$, where E_i denotes the exceptional curve above p_i. Denote this matrix by F. We know that the spectral radius $\rho(F)$ is equal to the dynamical degree $\lambda(f)$; in addition, since f preserves the canonical divisor, $(3, -1, \dots, -1)^T$ is an eigenvector of F with eigenvalue 1. Setting

$$Q = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}, \tag{$\clubsuit$}$$

the fact that f preserves the intersection form translates to $FQF^T = Q$, where Q is the matrix given in ($\clubsuit$) and F^T denotes the transpose of F.

If, conversely, we want to find a birational map of $\mathbb{P}^2$ with corresponding matrix F, we can prove a partial converse by choosing appropriate points on a cuspidal cubic curve. Consider the singular cuspidal cubic $C = V(XY^2 - Z^3)$ in $\mathbb{P}^2$ with unique singular point $[1:0:0]$. Set $C^* = C \setminus \{[1:0:0]\}$ the set of smooth points. The isomorphism $\mathbb{A}^1 \xrightarrow{\sim} C^*$, $t \mapsto [t^3 : 1 : t]$ allows to parametrise the smooth points of C with $\mathbb{A}^1$ and to endow C^* with a group structure, for which $t_1 + t_2 + t_3 = 0$ if and only if the points t_1, t_2, t_3 on C^* are collinear.

Lemma 21. *If for $3d \geq 3$ not necessarily distinct points $t_i \in C^*$, there exists a homogeneous polynomial P of degree d whose restriction to C is not zero and such that for $p_i = [t_i^3 : 1 : t_i]$ we have $\mathrm{div}_C(P) = p_1 + \cdots + p_{3d}$, then $\sum_{i=1}^{3d} t_i = 0$.*

Proof. This is a well known consequence of the group law on the Picard group of C^*. We provide a direct proof for the sake of completeness. Suppose we have $3d$ points $p_i = [t_i^3 : 1 : t_i]$ with $t_i \in C^*$ such that $\mathrm{div}_C(P) = p_1 + \cdots + p_{3d}$ for a homogeneous polynomial $P \in \mathbb{C}[X, Y, Z]$ of degree d not a multiple of the defining equation of C. As $P(t^3, 1, t) \in \mathbb{C}[t]$ is a polynomial of degree at most $3d$ and $\mathrm{div}_C(P) = p_1 + \cdots + p_{3d}$, we must have $P(t^3, 1, t) = a\prod_{i=1}^{3d}(t - t_i)$ for some $a \in \mathbb{C}^*$. The coefficient of t^{3d-1} has to be zero. Otherwise, we could find a term $X^iY^jZ^k$ of P with $3i + k = 3d - 1$ and $i + k \leq d$. This, however, would imply $2k \leq 1$, leading to the desired contradiction. Therefore, the coefficient of t^{3d-1} is precisely equal to $-a\sum_{i=1}^{3d} t_i = 0$, which finishes the proof. □

Lemma 22. *Consider any matrix $F \in \mathrm{GL}_{n+1}(\mathbb{Z})$ satisfying $FQF^T = Q$, where Q is defined as in (♣), and $F^T(3,1,\ldots,1)^T = (3,1,\ldots,1)^T$. Fix $a \in \mathbb{C}\setminus\{1\}$ and $b, v_1,\ldots,v_n \in \mathbb{C}$. If the vector $(b,v_1,\ldots,v_n)^T$ is an eigenvector of F^T with respect to a, then we have the following system of equations:*

$$3b = F_{21}\frac{1}{a-1}(3v_1-b)+\cdots+F_{n+1,1}\frac{1}{a-1}(3v_n-b), \tag{♡}$$

$$b = F_{2,i+1}\frac{1}{a-1}(3v_1-b)+\cdots+(F_{i+1,i+1}-a)\frac{1}{a-1}(3v_i-b)+\cdots+F_{n+1,i+1}\frac{1}{a-1}(3v_n-b). \tag{♠}$$

Proof. Assume that the vector $(b,v_1,\ldots,v_n)^T$ is an eigenvector of F^T with respect to the eigenvalue a. Then $(b,v_1,\ldots,v_n)^T$ satisfies

$$(F_{11}-a)b+F_{21}v_1+\cdots+F_{n+1,1}v_n = 0,$$
$$F_{1,i+1}b+F_{2,i+1}v_1+\cdots+(F_{i+1,i+1}-a)v_i+\cdots+F_{n+1,i+1}v_n = 0, \quad 1\le i\le n.$$

By multiplying each equation by 3, and subtracting and adding $\sum_{j=2}^{n+1}F_{j1}b$ from the first equation and $\sum_{j=2}^{n+1}F_{j,i+1}b$ from the remaining n equations, the equations imply

$$3(F_{11}-a)b+F_{21}(3v_1-b)+\cdots+F_{n+1,1}(3v_n-b)+\sum_{j=2}^{n+1}F_{j1}b = 0,$$

$$\left(3F_{1,i+1}+\sum_{j=2}^{n+1}F_{j,i+1}-a\right)b+F_{2,i+1}(3v_1-b)+\cdots+(F_{i+1,i+1}-a)(3v_i-b)+\cdots+F_{n+1,i+1}(3v_n-b) = 0,$$

where $1\le i\le n$. By assumption, since $F^T(3,1,\ldots,1)^T=(3,1,\ldots,1)^T$, we have $3F_{11}+\sum_{j=2}^{n+1}F_{j1}=3$ and $3F_{1,i+1}+\sum_{j=2}^{n+1}F_{j,i+1}=1$, and thus find

$$3(1-a)b+F_{21}(3v_1-b)+\cdots+F_{n+1,1}(3v_n-b) = 0,$$
$$(1-a)b+F_{2,i+1}(3v_1-b)+\cdots+(F_{i+1,i+1}-a)(3v_i-b)+\cdots+F_{n+1,i+1}(3v_n-b) = 0,$$

which after dividing by $(a-1)$ implies (♡) and (♠). □

Proposition 23. *Let $C\subseteq\mathbb{P}^2$ be a singular cuspidal cubic given by $C=V(XY^2-Z^3)$, fix $n\ge 10$ and let $p_1,\ldots,p_n\in C^*$ be distinct points. Call $\pi\colon X\to\mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ in the points $p_1,\ldots,p_n$. Denote by $E(p_i)=\pi^{-1}(p_i)$ the exceptional curves and by L the pullback of a general line in $\mathbb{P}^2$. Consider $F\in\mathrm{Aut}(\mathrm{Pic}(X))\setminus\{\mathrm{id}\}$, viewing it as a matrix in the basis $([L],[E(p_1)],\ldots,[E(p_n)])$ with entries F_{ij}. Suppose that F sends the anticanonical divisor $-K_X$ to itself and that it preserves the intersection form. Furthermore, assume that for each $i\in\{1,\ldots,n\}$ there exists an irreducible curve C_i of X in the class of $F([E(p_i)])$. Fix an eigenvalue $a\in\mathbb{C}\setminus\{1\}$ of F and consider an eigenvector $(b,v_1,\ldots,v_n)^T$ of F^T. If $p_i=\frac{1}{a-1}(3v_i-b)$, then there exists an automorphism $f\in\mathrm{Aut}(X)$ such that the induced action on $\mathrm{Pic}(X)$ is equal to F, and the restriction of f to the strict transform $\widetilde{C}$ of C induces an automorphism of $C^*\cong\mathbb{A}^1$ given by $z\mapsto az+b$.*

Proof. Note that $E(p_i)\cdot E(p_j)=-\delta_{ij}$, which implies $F([E(p_i)])\cdot F([E(p_j)])=-\delta_{ij}$, since F preserves the intersection form. By assumption, there are irreducible curves C_i lying in $F([E(p_i)])$, for which we therefore find that they are disjoint (-1)-curves. Hence there exists a contraction $\eta\colon X\to\mathbb{P}^2$ of the curves C_i. As we blow up only points on C^*, the strict transform $\widetilde{C}$ of C lies in $-K_X$. As F preserves the intersection form and since $F(-K_X)=-K_X$ by assumption, we find $C_i\cdot\widetilde{C}=[C_i]\cdot(-K_X)=F([E(p_i)])\cdot F(-K_X)=[E(p_i)]\cdot(-K_X)=1$, and thus any C_i meets $\widetilde{C}$ transversally in one point. From this, we deduce that the birational morphism η maps $\widetilde{C}$ to an irreducible cubic curve $\eta(\widetilde{C})$. Additionally, the $\eta(C_i)$ cannot be singular points of $\eta(\widetilde{C})$, since the cusp has multiplicity 2 but $\widetilde{C}\cdot C_i=1$. Since none of the points blown up by η equals the singular point and since an irreducible cubic curve can have at most one double point, $\eta(\widetilde{C})$ is again cuspidal. We can therefore choose an automorphism $\beta\in\mathrm{Aut}(\mathbb{P}^2)$ which maps $\eta(\widetilde{C})$ to C, and by replacing η with $\beta\circ\eta$ we may assume that $\eta(\widetilde{C})=C$. Consequently, the birational map $\pi\circ\eta^{-1}$ restricts to an automorphism of C, say given by $z \mapsto a'z + b'$. By assumption, there exists an eigenvalue $a \neq 1$ of F. Any automorphism of C given by $z \mapsto \widehat{a}z$, where $\widehat{a} \in \mathbb{C}^*$, extends to some automorphism $\gamma_{\widehat{a}} \in \mathrm{Aut}(\mathbb{P}^2)$. Hence, by replacing η by $\gamma_{aa'^{-1}}^{-1} \circ \eta$, we may assume that $\pi \circ \eta^{-1}$ restricts to the automorphism $C^* \to C^*, z \mapsto az + b'$. Denote $\varphi_F = \pi \circ \eta^{-1}$.

We first prove $b' = b$. For this, consider the pullback $\eta^{-1}(L_{\mathbb{P}^2})$ of a line $L_{\mathbb{P}^2}$ not passing through any of the $\eta(C_i)$ via η, and not being tangent to C. Since $\eta^{-1}(L_{\mathbb{P}^2})$ is disjoint from all the C_i, the Picard group of X admits the basis $([\eta^{-1}(L_{\mathbb{P}^2})], [C_1], \dots, [C_n])$. Writing $F([L])$ in that basis, we see that $F([L]) = d[\eta^{-1}(L_{\mathbb{P}^2})]$ for some $d \in \mathbb{Z}^{\geq 1}$, since $F([L]) \cdot [C_i] = F([L]) \cdot F([E(p_i)]) = [L] \cdot [E(p_i)] = 0$. Then, since $F([L])$ has self-intersection 1, we deduce $d = 1$ and thus $\eta^{-1}(L_{\mathbb{P}^2}) \in F([L])$. Call r_1, r_2, r_3 the three points of intersection of $L_{\mathbb{P}^2}$ and C. As $L_{\mathbb{P}^2}$ and C intersect in three points, the curve $\eta^{-1}(L_{\mathbb{P}^2})$ has three points of intersection with $\widetilde{C}$. The images of r_1, r_2, r_3 under $\varphi_F|_{C^*}$ equal the images of these three points $\eta^{-1}(r_1), \eta^{-1}(r_2), \eta^{-1}(r_3)$ under π. Since $F([L]) = F_{11}[L] + F_{21}[E(p_1)] + \dots + F_{n+1,1}[E(p_n)]$, the curve $\pi(\eta^{-1}(L_{\mathbb{P}^2}))$ is a curve of degree F_{11} passing through p_i with multiplicity $-F_{i+1,1}$ and through $\varphi_F|_{C^*}(r_1), \varphi_F|_{C^*}(r_2), \varphi_F|_{C^*}(r_3)$ with multiplicity 1 each. Thus, by the above, on $C^* \cong \mathbb{A}^1$ we have the following equality:

$$3b' \overset{\text{Lem. 21}}{=} a(r_1 + r_2 + r_3) + 3b' = \varphi_F|_{C^*}(r_1) + \varphi_F|_{C^*}(r_2) + \varphi_F|_{C^*}(r_3) \overset{\text{Lem. 21}}{=} F_{21}p_1 + \dots + F_{n+1,1}p_n.$$

By Lemma 22 (♡), we know that $F_{21}p_1 + \dots + F_{n+1,1}p_n = 3b$, and thus find $b' = b$.

We analyse how $\varphi_F|_{C^*}$ maps the points $\eta(C_i)$. The curve C_i is an element of $F([E(p_i)]) = F_{1,i+1}[L] + \sum_{j=1}^n F_{j+1,i+1}[E(p_j)]$; this corresponds to the $(i+1)$-th column of F. By the argument laid out in the first paragraph of the proof, there is exactly one point lying in the intersection of C_i and $\widetilde{C}$, say $\widetilde{p_i}$, and $\pi(\widetilde{p_i}) = \varphi_F|_{C^*}(\eta(C_i))$. The point $\pi(\widetilde{p_i})$ lies on $\pi(C_i)$ and the curve $\pi(C_i)$ is a curve of degree $F_{1,i+1}$ with multiplicity $-F_{j+1,i+1}$ at p_j and multiplicity 1 at $\pi(\widetilde{p_i})$. Considering the divisor on C given by $\pi(C_i)$, we thus find

$$a\eta(C_i) + b = \varphi_F|_{C^*}(p_i) = \pi(\widetilde{p_i}) \overset{\text{Lem. 21}}{=} F_{2,i+1}p_1 + \dots + F_{n+1,i+1}p_n.$$

By Lemma 22 (♠), this must be equal to $ap_i + b$, which implies $\eta(C_i) = p_i$. Thus, φ_F lifts to an automorphism on X, whose induced action on $\mathrm{Pic}(X)$ is equal to F and whose restriction to $\widetilde{C}$ is the same automorphism as $\varphi_F|_{C^*}: z \mapsto az + b$. □

Remark 24. A similar result holds for an arbitrary irreducible cubic, but one has to take a bit more care, see for instance Diller [11] for the degree 2 case. However, only very few dynamical degrees outside the unit circle arise on a non-cuspidal cubic, see [11, Theorem 2].

5.2. *Existence of Jonquières maps*

We take a similar set-up as in the previous section, describing the points blown up more precisely. Consider a birational map $f \in \mathrm{Bir}(\mathbb{P}^2)$. Write $\mathrm{Base}(f) = \{p_1, \dots, p_m\}$ for the base points of f and $\mathrm{Base}(f^{-1}) = \{q_1, \dots, q_m\}$ for the base points of its inverse. Label them so that the multiplicities satisfy $\mathrm{mult}_{p_1}(f^{-1}(L)) \geq \ldots \geq \mathrm{mult}_{p_m}(f^{-1}(L))$ and $\mathrm{mult}_{q_1}(f(L)) \geq \ldots \geq \mathrm{mult}_{q_m}(f(L))$. Fix this ordering of the points. Assume that the points of $\mathrm{Base}(f^{-1})$, and $\mathrm{Base}(f)$, respectively, are proper (often called not infinitely near), which by definition means that none of the base points lie on an exceptional curve above some other base point. Suppose that for $1 \leq i \leq m$, we have $n_i \in \mathbb{Z}^{\geq 0}$ such that $f^{n_i - 1}(q_i) = p_i$. Take the minimal n_i's satisfying these conditions. Then the points $f^s(q_t)$ for $1 \leq t \leq m$ and $0 \leq s \leq n_t - 1$ are pairwise distinct. Call the tuple $(n_1, \dots, n_m)$ the *orbit data of* f, the set $O(q_t) = \{f^s(q_t) \mid 0 \leq s \leq n_t - 1\}$ the *orbit of* q_t and write $O(f) = \bigcup_{1 \leq t \leq m} O(q_t)$. Moreover, denote $q_i^j = f^j(q_i)$, where $q_i^0 = q_i$ and $q_i^{n_i - 1} = p_i$; fix once and for all the ordering of the points q_i^j of $O(f)$ induced by the lexicographic ordering on (i, j). Note that if we blow up $O(f)$, then f lifts to an automorphism on the blow-up X, and the ordering on $O(f)$ results in an ordered basis

$([L],([E(q_i^j)]))$ of the Picard group $\mathrm{Pic}(X)$, where L is the pullback of a general line in $\mathbb{P}^2$ and $E(q_i^j)$ denotes the exceptional curve above q_i^j.

Recall that a *Jonquières map* of degree d is a map $j_d \in \mathrm{Bir}(\mathbb{P}^2)$ with base point p_1 of multiplicity $d-1$ and base points $p_2,\dots,p_{2d-1}$ of multiplicity 1. Call the base points of the inverse $q_1,\dots,q_{2d-1}$, where q_1 is the unique one of multiplicity $d-1$ and all others have multiplicity 1. Write $j_d^{\underline{n}}$ for a Jonquières map with orbit data $\underline{n}=(n_1,\dots,n_{2d-1})$ and $\lambda_{d,\underline{n}}$ for its dynamical degree. Also, since it will make computations easier, we set $n_1=2$ and abbreviate $\underline{n}=(n_2,\dots,n_{2d-1})$. In the basis of $\mathrm{Pic}(X)$ described above, the matrix corresponding to $j_d^{\underline{n}}$ takes precisely the form of $J_d^{\underline{n}}$ defined in ($\diamondsuit$) for $m=2d-1$. We call such a matrix $J_d^{\underline{n}}$ a *Jonquières matrix.* A priori, we do not know if these Jonquières maps $j_d^{\underline{n}}$ can be realised. Using Section 5.1, we have an ansatz for which points on the smooth locus C^* of the cuspidal cubic we are looking for. Thus, we calculate an eigenvector of $(J_d^{\underline{n}})^T$ with respect to the eigenvalue $\lambda_{d,\underline{n}}$ realising the spectral radius in the following lemma:

Lemma 25. *Fix $\underline{n}$ with $n_1=2$ and $n_i\geq 2$ for all $i\in\{2,\dots,2d-1\}$. Abbreviate $\lambda=\lambda_{d,\underline{n}}$. Then the vector $(\lambda+1,1,\lambda,\frac{\lambda}{\lambda^{n_2}+1},\frac{\lambda^2}{\lambda^{n_2}+1},\dots,\frac{\lambda^{n_2}}{\lambda^{n_2}+1},\dots,\frac{\lambda^{n_{2d-1}}}{\lambda^{n_{2d-1}}+1})^T$ is an eigenvector of $(J_d^{\underline{n}})^T$ with respect to the eigenvalue λ.*

Proof. Since λ is strictly larger than 2 by Proposition 11, the vector is well-defined. We check directly that $(\lambda+1,1,\lambda,\frac{\lambda}{\lambda^{n_2}+1},\frac{\lambda^2}{\lambda^{n_2}+1},\dots,\frac{\lambda^{n_2}}{\lambda^{n_2}+1},\dots,\frac{\lambda^{n_{2d-1}}}{\lambda^{n_{2d-1}}+1})^T$ is an eigenvector of $(J_d^{\underline{n}})^T$ with respect to the eigenvalue λ. Indeed, for all the entries where $(J_d^{\underline{n}})^T$ only permutes the entries of the eigenvector, this follows by construction of the vector. The only equations we need to check are

$$\begin{aligned} d(\lambda+1)-(d-1)-\sum_{i=2}^{2d-1}\frac{\lambda}{\lambda^{n_i}+1}&=\lambda(\lambda+1),\\ (d-1)(\lambda+1)-(d-2)-\sum_{i=2}^{2d-1}\frac{\lambda}{\lambda^{n_i}+1}&=\lambda^2,\\ \lambda+1-1-\frac{\lambda}{\lambda^{n_i}+1}&=\lambda\frac{\lambda^{n_i}}{\lambda^{n_i}+1},\quad i\in\{2,\dots,2d-1\}. \end{aligned}$$

Multiplying the two upper equations by $\prod_{i=2}^{2d-1}(\lambda^{n_i}+1)$, we find the expression of the characteristic polynomial, which evaluates to 0 at λ, and the last equation also holds, which can be seen after multiplying both sides by $\lambda^{n_i}+1$. $\square$

The following technical lemma will be used several times in the proposition thereafter.

Lemma 26. *Consider $d\geq 2$, $(n_1,n_2,\dots,n_{2d-1})$ with $n_1=2$, $n_i\in\mathbb{Z}^{\geq 2}$ all pairwise distinct, and $\lambda>2$. Define $q_1^0=3-(\lambda+1)$, $q_1^1=3\lambda-(\lambda+1)$ and $q_i^j=3\frac{\lambda^{j+1}}{\lambda^{n_i}+1}-(\lambda+1)$ for $i\geq 2$. Then for any $i_1,i_2,i_3\in\{1,\dots,2d-1\}$ and any $j_k\in\{0,\dots,n_{i_k}-1\}$ with $k=1,2,3$, we have $q_{i_1}^{j_1}+q_{i_2}^{j_2}+q_{i_3}^{j_3}\neq 0$.*

Proof. We prove the claim by contradiction. We first prove it for when $i_1,i_2,i_3\neq 1$. Suppose that there exist $i_1,i_2,i_3\in\{2,\dots,2d-1\}$ and any $j_k\in\{0,\dots,n_{i_k}-1\}$ with $k=1,2,3$ such that $q_{i_1}^{j_1}+q_{i_2}^{j_2}+q_{i_3}^{j_3}=0$. This implies that

$$\lambda+1=\frac{\lambda^{j_1+1}}{\lambda^{n_{i_1}}+1}+\frac{\lambda^{j_2+1}}{\lambda^{n_{i_2}}+1}+\frac{\lambda^{j_3+1}}{\lambda^{n_{i_3}}+1}.$$

Rewrite it as $(\lambda+1)(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1)(\lambda^{n_{i_3}}+1)=\lambda^{j_1+1}(\lambda^{n_{i_2}}+1)(\lambda^{n_{i_3}}+1)+\lambda^{j_2+1}(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_3}}+1)+\lambda^{j_3+1}(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1)$, where after observing $\lambda^{j_k+1}<\lambda^{n_{i_k}}+1$, we deduce

$$(\lambda+1)(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1)(\lambda^{n_{i_3}}+1)<3(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1)(\lambda^{n_{i_3}}+1).$$

But this would imply $\lambda<2$, a contradiction.

The cases where $i_k = 1$ for all $k = 1,2,3$ are done similarly: If $3q_1^0 = 0$, we would find $\lambda = 2$, and if $3q_1^1 = 0$, then $2\lambda = 1$, both times in contradiction to $\lambda > 2$. Moreover, $2q_1^0 + q_1^1 = 3 \neq 0$ and $q_1^0 + 2q_1^1 = 3\lambda \neq 0$.

We consider now the cases where — up to labelling the indices — $i_1 = i_2 = 1$. Denote $i := i_3$ and $j := j_3$. If $2q_1^0 + q_i^j = 0$, then $3(\lambda+1) = 6 + 3\frac{\lambda^{j+1}}{\lambda^{n_i}+1}$, which is equivalent to $(\lambda-1)(\lambda^{n_i}+1) = \lambda^{j+1}$. Using once more that $\lambda^{j+1} < \lambda^{n_i}+1$, we obtain $\lambda < 2$, in contradiction to $\lambda > 2$. If $2q_1^1 + q_i^j = 0$, a similar calculation leads to $1 - \lambda = \frac{\lambda^{j+1}}{\lambda^{n_i}+1}$, which cannot be, as the left side is negative and the right side is positive. If $q_1^0 + q_1^1 + q_i^j = 0$, a direct calculation results in $\frac{\lambda^{j+1}}{\lambda^{n_i}+1} = 0$, again a contradiction.

The cases where only one of the i_k equals 1 remain, say $i_3 = 1$. If $q_1^0 + q_{i_1}^{j_1} + q_{i_2}^{j_2} = 0$, then $3(\lambda+1) = 3 + 3\frac{\lambda^{j_1+1}}{\lambda^{n_{i_1}}+1} + 3\frac{\lambda^{j_2+1}}{\lambda^{n_{i_2}}+1}$. But this results, using again $\lambda^{j_k+1} < \lambda^{n_{i_k}}+1$, in $\lambda < 2$, a contradiction. If $q_1^1 + q_{i_1}^{j_1} + q_{i_2}^{j_2} = 0$, we obtain $(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1) = \lambda^{j_1+1}(\lambda^{n_{i_2}}+1) + \lambda^{j_2+1}(\lambda^{n_{i_1}}+1)$. If both $j_1 < n_{i_1} - 1$ and $j_2 < n_{i_2} - 1$, then $2\lambda^{j_k+1} < \lambda^{n_{i_k}}+1$ for both $k = 1,2$. Therefore, we find $(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1) < \frac{1}{2}(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1) + \frac{1}{2}(\lambda^{n_{i_1}}+1)(\lambda^{n_{i_2}}+1)$, a contradiction. Assume now without loss of generality that $j_2 = n_{i_2}+1$. This implies $\lambda^{n_{i_1}}+1 = \lambda^{j_1+1}(\lambda^{n_{i_2}}+1)$. Hence, $n_{i_1} > j_1 + n_{i_2}$ and $n_{i_1} < j_1 + n_{i_2} + 2$. This implies $n_{i_1} = j_1 + n_{i_2} + 1$. We thus find $\lambda^{j_1+1} = 1$, a contradiction. This proves the lemma. □

With the choice on the points of Lemma 25 and a weak assumption on the orbits, we can ensure that the Jonquières map $j_d^{\underline{n}}$ exists. Parts of the proof follow the same arguments as in the proof of [5, Proposition 5.12].

Proposition 27. *Let $d \geq 4$ and denote by $C \subseteq \mathbb{P}^2$ the cuspidal cubic $C = V(XY^2 - Z^3)$. Fix the orbit data $\underline{n} = (n_2, \ldots, n_{2d-1})$ with $n_i \geq 2$ for each $i \in \{2, \ldots, 2d-1\}$ and all the n_i pairwise distinct. Denote by $\lambda \in \mathbb{R}$ the eigenvalue $\lambda_{d,\underline{n}}$ realising the spectral radius of the Jonquières matrix $J_d^{\underline{n}}$ and set*

$$\begin{aligned} q_1^0 = q_1 &= \frac{1}{\lambda-1}(2-\lambda), \\ q_1^1 = p_1 &= \frac{1}{\lambda-1}(2\lambda-1), \\ q_i^0 = q_i &= \frac{1}{\lambda-1}\Big(3\frac{\lambda}{\lambda^{n_i}+1} - (\lambda+1)\Big), \\ q_i^j &= \frac{1}{\lambda-1}\Big(3\frac{\lambda^{j+1}}{\lambda^{n_i}+1} - (\lambda+1)\Big), \quad 2 \leq i \leq 2d-1, 1 \leq j \leq n_i - 1, \end{aligned}$$

where the points are identified with their corresponding point on $C^* \cong \mathbb{A}^1$. Then there exists a Jonquières map $j_d^{\underline{n}} \in \mathrm{Bir}_d(\mathbb{P}^2)$ with base points $p_1, p_2 = q_2^{n_2-1}, \ldots, p_{2d-1} = q_{2d-1}^{n_{2d-1}-1}$, orbits $O(q_i) = \{q_i^j \mid 0 \leq j \leq n_i - 1\}$ and dynamical degree λ.

Proof. To prove the proposition, we show that there exists an automorphism realising the Jonquières matrix $J_d^{\underline{n}}$ of (♢) on the blow-up $\pi\colon X \to \mathbb{P}^2$ in the points q_i^j by applying Proposition 23. By Lemma 17, any $J_d^{\underline{n}}$ sends $-K_X$ to itself and preserves the intersection form. Thus, to apply Proposition 23, we need to prove the following two claims: that the points q_i^j are all distinct, and that in each class of $J_d^{\underline{n}}([E(q_i^j)])$ there exists an irreducible curve, where $E(q_i^j) = \pi^{-1}(q_i^j)$ denotes the exceptional curve above q_i^j.

We start with proving that the points are distinct. Assume $q_1 = p_1$. This is equivalent to $\lambda = 1$, a contradiction to $\lambda > 2$ by Proposition 11. Now, take any $i \neq 1$ and assume $q_1 = q_i^j$. This is equivalent to $\lambda^{n_i} + 1 = \lambda^{j+1}$, a contradiction to $j \leq n_i - 1$ and $\lambda > 2$. Similarly, we find that $p_1 = q_i^j$ is equivalent to $\lambda^{n_i} + 1 = \lambda^j$, again a contradiction. Then, for a fixed $i \neq 1$, we can also immediately see that $q_i^k \neq q_i^\ell$ whenever $k \neq \ell$. Assume therefore that $i, j \neq 1$ and that we have $0 \leq k \leq n_i - 1$ and $0 \leq \ell \leq n_j - 1$ such that $q_i^k = q_j^\ell$. Assume without loss of generality that $n_i > n_j$. This is equivalent to $\lambda^k(\lambda^{n_j}+1)=\lambda^\ell(\lambda^{n_i}+1)$. Since $n_i>n_j$ and $\lambda>2$, the equation yields $k>\ell$ and is equivalent to $\lambda^m(\lambda^{n_j}+1)=\lambda^{n_i}+1$, where $m=k-\ell$. As $\lambda>2$ it must follow that $\lambda^{m+n_j}<\lambda^m(\lambda^{n_j}+1)-1=\lambda^{n_i}$, and thus $m+n_j<n_i$. Hence $\lambda^m(\lambda^{n_j}+1)<2\lambda^{m+n_j}<\lambda^{m+n_j+1}\le\lambda^{n_i}<\lambda^{n_i}+1$, a contradiction. Therefore, none of the points blown up by π agree.

Now that we know that the point q_i^j are distinct, we turn to proving that there exists an irreducible curve in X contained in $J_d^n([E(q_i^j)])$, for each $q_i^j\in O(f)$. Note that using ($\diamondsuit$), the only $J_d^n([E(q_i^j)])$ which are not equal to the class of an exceptional divisor are:

$$J_d^n([E(q_1^1)])=J_d^n([E(p_1)])=(d-1)[L]-(d-2)[E(q_1)]-\sum_{i=2}^{2d-1}[E(q_i)],$$
$$J_d^n([E(q_i^{n_i-1})])=J_d^n(E(p_i))=[L]-[E(q_1)]-[E(q_i)],\quad i\in\{2,\dots,2d-1\}.$$

As $J_d^n([E(p_i)])$ with $i>1$ contains the strict transform of a line through q_1 and q_i, we are left with securing the existence of an irreducible degree $d-1$ curve through q_1 with multiplicity $d-2$ and through $q_2,\dots,q_{2d-1}$ with multiplicity 1 each.

Start with the blow-up $\tau\colon X_1\to\mathbb{P}^2$ of $\mathbb{P}^2$ in q_1 to the first Hirzebruch surface $X_1=\mathbb{F}_1$. Blow up the point on X_1 corresponding to q_2 and blow down the strict transform of the line through q_1 and q_2, resulting in a birational map $\tau_2\colon X_1\dashrightarrow X_2$, where X_2 is either $\mathbb{F}_0$ or $\mathbb{F}_2$. By proving that for all $2\le i<j\le 2d-1$, the points q_i and q_j cannot lie on a line through q_1, we can repeat this procedure. In this way, we construct a sequence of birational maps $X_1\overset{\tau_2}{\dashrightarrow}X_2\overset{\tau_3}{\dashrightarrow}\cdots\overset{\tau_{2d-1}}{\dashrightarrow}X_{2d-1}$, where at each step, τ_i equals the blow-up of the point corresponding to q_i followed by the blow-down of the curve we obtain from pushing forward the strict transform of the line through q_1 and q_i by the map $\tau_{i-1}\cdots\tau_2$; moreover, X_i is equal to some Hirzebruch surface $\mathbb{F}_s$ for some $s\ge 0$ with $s\equiv i \mod 2$. Thus assume by contradiction that there exist $2\le i<j\le 2d-1$ such that the points q_1, q_i and q_j lie on a line. By Lemma 21, this implies $q_1+q_i+q_j=0$, which cannot be by Lemma 26. Therefore, a sequence of birational maps $\tau_{2d-1}\cdots\tau_2$ as described above does indeed exist.

Now, X_{2d-1} is equal to some $\mathbb{F}_s$ with s odd. We prove $s=1$. Assume there exists a section of $\mathbb{F}_s\to\mathbb{P}^1$ of self-intersection ≤ -3. It must come from a curve in $\mathbb{P}^2$ of degree k passing through q_1 with multiplicity $k-1$ and through ℓ points out of $q_2,\dots,q_{2d-1}$ with multiplicity 1. The self-intersection of its strict transform in X_1 is $k^2-(k-1)^2=2k-1$. Whenever one of the ℓ points through which it passes is blown up, the self-intersection decreases by one; yet for every point it does not pass through, the self-intersection increases by one, since we contract a (-1)-curve passing through that point. Thus, the irreducible curve of self-intersection ≤ -3 has self-intersection $2k-1-\ell+2d-2-\ell=2(d+k-\ell)-3\le -3$. Therefore, $d+k\le\ell$. But, as all the q_i lie on the smooth locus of the cubic curve C, with Bézout, we find for any curve of $\mathbb{P}^2$ of degree k passing through q_1 with multiplicity $k-1$ and ℓ other points with multiplicity 1 that $3k-(k-1)-\ell\ge 0$, implying $2k+1\ge\ell\ge d+k$ and thus $k\ge d-1$ and $\ell\ge 2d-1$. But $\ell\le 2d-2$, a contradiction. Hence, $X_{2d-1}=\mathbb{F}_1$, and we can contract the (-1)-curve to obtain a birational transformation $f\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ as in the following diagram, where π decomposes into the blow-up π_1 of the points $q_1,\dots,q_{2d-1}$, and π_2 the blow-up of the remaining q_i^j:

$$\begin{tikzcd}
 & & X \arrow[dl, "\pi_2"'] \arrow[dd, "\pi"] \\
 & X' \arrow[dl] \arrow[dr, "\pi_1"] & \\
\mathbb{P}^2 & & \mathbb{P}^2. \arrow[ll, dashed, "f"']
\end{tikzcd}$$

The irreducible curve of self-intersection -1 on X_{2d-1} must by the same reasoning as above come from an irreducible curve on $\mathbb{P}^2$ of degree $d-1$ passing through q_1 with multiplicity $d-2$ and through all the other $q_2,\dots,q_{2d-1}$ with multiplicity 1, which lifts to an irreducible curve on X' linearly equivalent to $(d-1)L-(d-2)E(q_1)-\sum_{i=2}^{2d-1}E(q_i)$. Furthermore, any line through q_1 and q_i lifts to an irreducible curve in X'.

To verify that the lines through q_1 and q_i and the curve of degree $d-1$ described above indeed lift to X as desired, we need to prove that none of the other points q_i^j, $j>0$, which are blown up by π_2, lie on one of these lines through q_1 and q_k, or on the curve of degree $d-1$ passing through q_1 with multiplicity $d-2$ and through $q_2,\dots,q_{2d-1}$ with multiplicity 1. By combining Lemma 21 and Lemma 26, no three can lie on a line.

Thus, assume by contradiction that some q_i^j lies on a degree $d-1$ curve through $q_1,\dots,q_{2d-1}$ with all points having multiplicity 1 except q_1 with multiplicity $d-2$. By Lemma 21, this implies $q_i^j=-(d-2)q_1-\sum_{k=2}^{2d-1}q_k$. By replacing the q_i with their explicit values as defined in the statement and using $\lambda\sum_{k=2}^{2d-1}\prod_{\ell\neq k}(\lambda^{n_\ell}+1)=-(\lambda^2-(d-1)\lambda-1)\prod_{k=2}^{2d-1}(\lambda^{n_k}+1)$, we find

$$
\begin{aligned}
(\lambda-1)((d-2)q_1+\sum_{k=2}^{2d-1}q_k) &= (d-2)(2-\lambda)+\sum_{k=2}^{2d-1}\left(3\frac{\lambda}{\lambda^{n_k}+1}-(\lambda+1)\right)\\
&= (d-2)(2-\lambda)+\frac{3}{\prod_{k=2}^{2d-1}(\lambda^{n_k}+1)}\lambda\sum_{k=2}^{2d-1}\prod_{\ell\neq k}(\lambda^{n_\ell}+1)-(2d-2)(\lambda+1)\\
&= (d-2)(2-\lambda)-3(\lambda^2-(d-1)\lambda-1)-(2d-2)(\lambda+1)\\
&= -3\lambda^2+(\lambda+1).
\end{aligned}
$$

Therefore, $q_i^j=-(d-2)q_1-\sum_{k=2}^{2d-1}q_k$ would imply $\lambda^2+1=0$ if $q_i^j=q_1$, $\lambda^2+\lambda=0$ if $q_i^j=p_1$ and $\lambda^2(\lambda^{n_i}+1)+\lambda^j=0$ otherwise. But for any of these equations, the left side is positive, since $\lambda>2$, leading to the desired contradiction.

By the above arguments, there exist irreducible curves lying in $J_d^n([E(q_i^j)])$ on the blow-up $\pi\colon X\to\mathbb{P}^2$ in the points q_i^j. Thus, applying Proposition 23, there exists a Jonquières map $j_d^n\in\mathrm{Bir}_d(\mathbb{P}^2)$ with base points $p_1,\dots,p_{2d-1}\in C^*$ and orbits $O(q_i)=\{q_i^j\,|\,0\le j\le n_i-1\}$ which lifts to an automorphism on X such that the induced action on $\mathrm{Pic}(X)$ is equal to the Jonquières matrix J_d^n. As the spectral radius of J_d^n is equal to λ by assumption, j_d^n has dynamical degree λ. This finishes the proof. □

By Proposition 23, the sets $\Lambda_{d,2d-1}$ of Proposition 15 lie in $\Lambda(\mathrm{Bir}_d(\mathbb{P}^2))$. This implies Theorem 1 whenever the ground field $\mathbf{k}$ contains the real algebraic numbers.

Remark 28. As noted in Remark 16, the accumulation points of $\Lambda_{d,2d-1}$ lie in $\Lambda_{d,2d-2}$, which are not Salem numbers any more, but Pisot numbers. *Salem numbers* are algebraic integers strictly larger than 1 whose other Galois conjugates lie in the closure of the unit disk, with at least one of the conjugates on the boundary, and a *Pisot number* is an algebraic integer strictly larger than 1 whose other Galois conjugates lie in the open unit disk (see for example [5, 1.1.2]). One can see that the elements of $\Lambda_{d,2d-1}$ are Salem numbers and the elements of $\Lambda_{d,m}$ with $m<2d-1$ are Pisot numbers by noting that $m=2d-1$ if and only if $H_{d,m}$ from Lemma 8 is the zero matrix.

The main difference between Salem and Pisot numbers is that any birational map which can be realised as an automorphism on some blow-up must be Salem, and any birational map which cannot is Pisot, except for quadratic reciprocal integers, which can occur in both cases. Yet by [5, Theorem D], the set $\Lambda(\mathrm{Bir}(\mathbb{P}^2))$ is closed as soon as $\mathbf{k}$ is uncountable and algebraically closed (for example $\mathbf{k}=\mathbb{C}$), and thus the accumulation points $\Lambda_{d,2d-2}$ and in fact all elements of the closure $\overline{\Lambda_{d,2d-1}}=\bigsqcup_{1\le m\le 2d-1}\Lambda_{d,m}$ are also realised as dynamical degrees of birational maps.

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