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Existence of a weak solution to a regularized moving boundary fluid-structure interaction problem with poroelastic media

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Abstract. We study the existence of a weak solution to a regularized, moving boundary, fluid-structure interaction problem with multi-layered poroelastic media consisting of a reticular plate located at the interface between the free flow of an incompressible, viscous fluid modeled by the 2D Navier–Stokes equations, and a poroelastic medium modeled by the 2D Biot equations. The existence result holds for both the elastic and viscoelastic Biot model. The free fluid flow and the poroelastic medium are coupled via the moving interface (the reticular plate) through the kinematic and dynamic coupling conditions. The reticular plate is "transparent" to fluid flow. The nonlinear coupling over the moving interface presents a major difficulty since both the fluid domain *and* the poroelastic medium domain are functions of time, and the finite energy spaces do not provide sufficient regularity for the corresponding weak formulation to be well-defined. This is why in this manuscript we consider a regularized problem by employing convolution with a smooth kernel only where needed. The resulting problem is still very challenging due to the nonlinear coupling and the motion of the fluid and Biot domains. We provide a constructive existence proof for this regularized fluid-poroelastic structure interaction problem. This regularized problem is consistent with the original, nonregularized problem in the sense that the weak solutions constructed here, converge, as the regularization parameter tends to zero, to a classical solution of the original, nonregularized problem when such a classical solution exists, assuming viscoelasticity in the Biot poroelastic matrix [1]. Furthermore, the existence result presented in this manuscript, is a crucial stepping stone for the singular limit problem in which the thickness of the reticular plate tends to zero. Namely, in [1] we will show that, in the case of a Biot poroviscoelastic matrix, the *moving boundary problem* obtained in the singular limit as the thickness of the plate tends to zero, has a weak solution.

Keywords. moving boundary problem, fluid-poroelastic structure interaction, well-posedness, nonlinear coupling, Biot equations, Navier–Stokes equations.

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1. Introduction and literature review

We are interested in weak solutions to a fluid-structure interaction problem between the flow of an incompressible, viscous free fluid and a poroelastic medium, coupled over a *moving interface*. The free fluid flow is modeled by the 2D Navier–Stokes equations for an incompressible, viscous fluid, and the poroelastic medium is modeled by the 2D Biot equations, which are coupled to a reticular plate serving as the interface between the free fluid flow and the Biot poro(visco)elastic medium. See Figure 1.

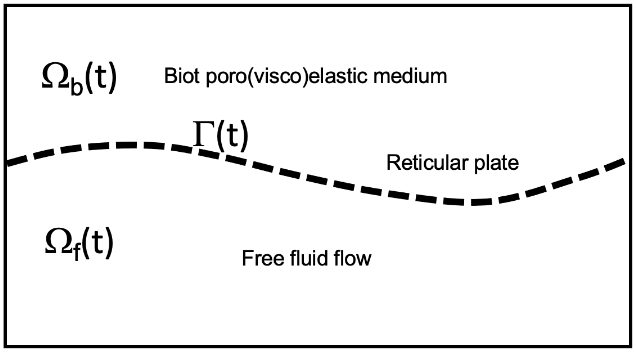

Figure 1. *A sketch of the fluid-poroelastic structure interaction domain.*

A reticular plate is a lattice-type structure characterized by two properties: periodicity and small thickness, where periodicity refers to periodic cells (holes) distributed in all directions [2]. Reticular plates, shells or membranes are models for reticular tissue, which is a connective tissue made up of a network of supportive fibers that provide a framework for soft organs. The reticular plate is transparent to fluid flow. It only provides mass (inertia) and elastic energy to the interface. The free fluid flow and the structure consisting of the Biot poro(visco)elastic medium with a reticular plate at the interface, are fully coupled across the *moving (plate) interface*, giving rise to a *nonlinearly coupled* fluid-poroelastic structure interaction (FPSI) problem.

We focus on the finite energy weak solutions analogous to the Leray–Hopf class for the Navier–Stokes equations. Such solutions have been widely studied in the context of fluid-structure interaction (FSI) with purely elastic structures (no poroelasticity) when the structure is lower dimensional, i.e. described by the plate/shell type of models, see e.g. [3–5] and the references therein. However, in the case of FSI problems with bulk elasticity (i.e. when the dimension of the fluid domain is the same as the dimension of the structure domain) the existence of weak solutions is still open. The main issue is that the resulting energy inequality does not provide sufficient regularity of the interface to define the moving domain and the corresponding traces, as well as the integrals over the interface and over the moving structure domain in the weak formulation. To the best of our knowledge, there are currently only two approaches to circumvent this issue. The first one is to associate mass and elastic energy to the interface (elastic interface with mass), as in [6, 7]. The elastic interface with mass regularizes the problem, as was proved in [8], and makes it possible to define and construct weak solutions. The second one is to add viscoelasticity to the structure so that the resulting nonlinear second order viscoelastic model has energy that is coercive in $W^{1,p}$, $p > 3$, see [9]. Our approach to the study of FPSI in this manuscript is closer to the former.

In moving boundary problems with *poroelasticity*, the issues are further exacerbated by the fact that the Biot equations are defined on moving domains $\Omega_b(t)$ in Eulerian coordinates, and the corresponding weak formulation has terms that are not well defined. Consequently, in the analysis of moving boundary FPSI problems it is not enough to regularize the interface, rather one needs more regularity of the poroelastic matrix displacement in the entire domain. To deal with this difficulty, we define *weak solutions to a regularized form of the given moving boundary FPSI problem* by using a suitably constructed convolution in spatial variables. We regularize only the "problematic" terms, i.e. the terms that are not well defined in the finite energy regularity class. The resulting regularized problem is still a nonlinear moving boundary problem of FPSI type and *very challenging*. To justify our regularization procedure and the corresponding definition of weak solutions to the regularized FPSI problem, in [1] we prove that weak solutions to the regularized FPSI problem indeed converge to the classical (smooth) solution of the original FPSI problem when such a classical solution exists, as the regularization parameter tends to zero. This is a weak-classical consistency result. Namely, we prove that if a smooth solution exists for the FPSI problem without regularization, then there exists a time $T > 0$ such that the sequence of weak solutions to the regularized FPSI problem constructed in this manuscript, converges to the classical solution on $[0, T]$ as the regularization parameter converges to 0 [1].

In this manuscript we provide the *existence of a weak solution to the regularized FPSI problem* for both the viscoelastic and elastic Biot models. While the result of this manuscript holds for the more difficult elastic case, we include viscoelasticity since the weak-classical consistency result from [1] is proved for the viscoelastic case.

1.1. *Outline of the manuscript*

This manuscript is organized as follows. In Section 2 we provide the statement of the original, nonregularized FPSI problem defined on moving domains, together with the coupling conditions. Here, we introduce the Lagrangian map which maps the Biot problem from the reference domain onto the moving, physical Eulerian domain. In Section 3 we further introduce the Lagrangian mapping for the plate problem, and the Arbitrary Lagrangian-Eulerian (ALE) mapping for the fluid domain, so that we can state the weak formulation for the nonregularized FPSI problem. In this section we remark which term in the weak formulation is not well-defined due the motion of the Biot domain $\Omega_b(t)$ and the lack of regularity of finite energy solutions. In Section 4 we regularize the FPSI problem using a convolution with the convolution parameter δ, and specify weak formulations for the regularized FPSI problem both on fixed and moving domains. An energy equality satisfied by the regularized solution is also presented. Existence of a weak solution to the regularized FPSI problem is presented in the rest of the paper. The main result and the construction of approximate weak solutions is given in Section 5. Approximate solutions are constructed by semi-discretizing the coupled problem in time and splitting the reticular plate problem from the fluid-Biot subproblem. In Section 6 we provide uniform energy bounds for the approximate sequences. Since the problem is highly nonlinear, to show that approximate solutions converge to a weak solution of the continuous regularized FPSI problem, we need some strong convergence results to pass to the limit in the weak formulation. We outline the compactness results ensuring the desired strong convergence of approximate subsequences in Section 7. In Section 8 we pass to the limit in the weak formulation. This is, however, far from straight-forward since the test functions in the weak formulation depend on the moving fluid domain. We will explain in Section 8 how to address this difficulty.

Conclusions and upcoming related work are discussed in Section 9.

While all the details necessary to follow the results presented here are provided in this manuscript, further detailed calculations will be provided in the associated manuscript [1].

1.2. *A brief literature review*

By now, there is extensive work on fluid-structure interaction involving incompressible, Newtonian fluids interacting with elastic structures. The models first considered in the literature are linearly coupled fluid-structure interaction models [10–12], which pose the fluid equations on a fixed reference fluid domain as a linearization that approximates real-life dynamics well when structure displacements are small. Nonlinearly coupled models, defined on time-dependent moving fluid domains, have been extensively studied by many authors [3–5, 13–38]. The main difficulty in the analysis of these problems is geometric nonlinearities arising from the moving boundary.

Many elastic materials, such as biological tissues and sediments, are not impermeable and can admit fluid flow through their pores. The study of poroelastic media was initiated by Biot modeling soil consolidation [39, 40]. Since then, a number of authors have contributed to the study of poroelastic media, both in terms of modeling and well-posedness analysis [35, 41–71].

More recently, the mathematical studies involving porous/poroelastic media have focused on *coupled* problems including viscous, incompressible fluids. First, the coupled problems involving Stokes/Navier–Stokes viscous fluid flow equations and the *Darcy equation* were studied in, e.g., [72–74] in the steady-state case, and in [75–77] in the time-dependent case. Very recently, studies focusing on the well-posedness of fully coupled fluid-poroelastic structure interaction systems, involving an incompressible fluid coupled to a structure that is *poroelastic, modeled by the Biot equations*, were considered in [78–80], where existence of (mild and strong) solutions was studied. In 2021, existence of a weak solution to a fluid-structure interaction model involving a multi-layered poroelastic medium and the time-dependent flow of a Stokes fluid was proved in [81] assuming both linear and nonlinear permeability in the Biot model. The proof uses a Lie operator splitting scheme, also known as the Marchuk–Yanenko scheme, which has been used by Roland Glowinski and others in numerical computations, see [82] and the references therein. A version of this operator splitting scheme is also used in the current paper as a skeleton for the construction of approximate solutions. We emphasize that the existing work on fluid-poroelastic structure interaction is solely for linearly coupled models, in which the fluid equations are posed on a fixed reference fluid domain and the Biot equations describing poroelasticity are posed on a fixed reference poroelastic structure domain. There has been no prior work that has considered the case of nonlinearly coupled fluid-poroelastic structure interaction, in which the poroelastic structure and fluid are both posed on moving domains that depend on the poroelastic structure displacement. The goal of the current manuscript is to provide a first step in the development of a well-posedness theory for nonlinearly coupled (moving boundary) fluid-poroelastic structure interaction problems by constructing new tools for dealing with the equations of poroelasticity defined on time-dependent and a priori unknown domains.

2. Statement of the problem

We consider a 2D benchmark, moving boundary FPSI problem defined on a simple rectangular domain. See Figure 1. This benchmark problem already embodies significant mathematical difficulties associated with the well-posedness analysis of moving domain FPSI problems. We will use the "hat" notation to denote the reference domain:

$$\widehat{\Omega} = \widehat{\Omega}_b \cup \widehat{\Omega}_f \cup \widehat{\Gamma},$$

where $\widehat{\Omega}_f = (0, L) \times (-R, 0)$, $\widehat{\Omega}_b = (0, L) \times (0, R)$, and $\widehat{\Gamma} = (0, L) \times \{0\}$ denote the fluid reference domain, the Biot reference domain and the interface reference configuration, respectively. Because we are considering a problem with nonlinear coupling, these domains will evolve in time, giving rise to time-dependent $\Omega(t) = \Omega_b(t) \cup \Omega_f(t) \cup \Gamma(t)$.

2.1. *The Biot poroviscoelastic structure subproblem*

The Biot system consists of the elastodynamics equation, which in this work will be defined on the Lagrangian domain $\widehat{\Omega}_b$, and the fluid equation, which in this work will be defined on the Eulerian domain $\Omega_b(t)$. The elastodynamics equation will be given in terms of the displacement of the Biot poroelastic matrix from the reference configuration $\widehat{\boldsymbol{\eta}} : [0,T] \times \widehat{\Omega}_b \to \mathbb{R}^2$ and fluid pore pressure $\widehat{p} : \widehat{\Omega}_b \to \mathbb{R}$. To specify the fluid equation given in terms of the fluid pore pressure in Eulerian formulation, we introduce the Lagrangian map by

$$\widehat{\boldsymbol{\Phi}}_b^{\eta}(t,\cdot) = \text{Id} + \widehat{\boldsymbol{\eta}}(t,\cdot) : \widehat{\Omega}_b \to \Omega_b(t), \tag{1}$$

with $(\boldsymbol{\Phi}_b^{\eta})^{-1}(t,\cdot) : \Omega_b(t) \to \widehat{\Omega}_b$ denoting its inverse. The Biot equations are then given by [83, 84]:

$$\rho_b \partial_{tt} \widehat{\boldsymbol{\eta}} = \widehat{\nabla} \cdot \widehat{S}_b(\widehat{\nabla}\widehat{\boldsymbol{\eta}}, \widehat{p}), \quad \text{in } \widehat{\Omega}_b, \tag{2}$$

$$\frac{c_0}{\left[\det\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)\right] \circ \left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1}} \frac{D}{Dt} p + \alpha \nabla \cdot \frac{D}{Dt} \boldsymbol{\eta} - \nabla \cdot (\kappa \nabla p) = 0, \quad \text{in } \Omega_b(t), \tag{3}$$

where $\frac{D}{Dt} = \frac{d}{dt} + (\partial_t \boldsymbol{\eta}(t,.) \circ (\boldsymbol{\Phi}_b^{\eta})^{-1}(t,\cdot)) \cdot \nabla)$ is the material derivative. To recover the filtration fluid velocity $\boldsymbol{q}$, Darcy's law is used:

$$\boldsymbol{q} = -\kappa \nabla p \quad \text{on } \Omega_b(t), \tag{4}$$

where κ is a positive permeability constant.

In this work, we will consider both the viscoelastic and the purely elastic consitutive models for the Biot poroelastic matrix with the Piola–Kirchhoff stress tensor for the viscoelastic case given by

$$\widehat{S}_b\left(\nabla\boldsymbol{\eta}, p\right) = 2\mu_e \widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\eta}}) + \lambda_e(\widehat{\nabla} \cdot \widehat{\boldsymbol{\eta}})\boldsymbol{I} + 2\mu_v \widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\eta}}_t) + \lambda_v(\widehat{\nabla} \cdot \widehat{\boldsymbol{\eta}}_t)\boldsymbol{I} - \alpha \det\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right) \widehat{p} \left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)^{-t}, \tag{5}$$

where superscript t denotes matrix transposition and $A^{-t} = (A^{-1})^t$. The *purely elastic case* will have the coefficients λ_v and μ_v equal to zero. Here, $\boldsymbol{D}$ denotes the symmetrized gradient, μ_e and λ_e are the Lamé parameters related to the elastic stress, μ_v and λ_v are the corresponding parameters related to the viscoelastic stress, and $\widehat{\boldsymbol{\Phi}}_b^{\eta}$ is the Lagrangian map defined above. In equation (3) the Biot material displacement $\boldsymbol{\eta}$ and the pore pressure p are defined on the physical domain $\Omega_b(t)$ as

$$\boldsymbol{\eta}(t,\cdot) = \widehat{\boldsymbol{\eta}}\left(t, \left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1}(t,\cdot)\right), \qquad p(t,\cdot) = \widehat{p}\left(t, \left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1}(t,\cdot)\right), \quad \text{where } \Omega_b(t) = \widehat{\boldsymbol{\Phi}}_b^{\eta}\left(t, \widehat{\Omega}_b\right).$$

We remark that in the last term of the Piola–Kirchhoff stress tensor (5), we have used the Piola transform, which is a transformation that maps tensors in Lagrangian coordinates to corresponding tensors in Eulerian coordinates in such a way that divergence-free tensors in Lagrangian coordinates remain divergence free in Eulerian coordinates [85].

We note that a priori the notion of $\Omega_b(t)$ is not entirely clear, unless $\widehat{\boldsymbol{\eta}}$ is sufficiently regular, and furthermore, the formulation of this problem makes sense only if the map $\widehat{\boldsymbol{\Phi}}_b^{\eta} = \text{Id} + \widehat{\boldsymbol{\eta}}$ is an injective map from $\widehat{\Omega}_b$ to $\Omega_b(t)$. We address these important issues later.

2.2. *The reticular plate subproblem*

The elastodynamics of reticular plates, studied in [2] using homogenization, are governed by a plate-type equation, defined on the equilibrium middle surface of the homogenized plate or shell $\widehat{\Gamma}$. The homogenized equation is given in terms of transverse displacement $\widehat{\boldsymbol{\omega}} = \widehat{\omega}\boldsymbol{e}_y$ from the reference configuration:

$$\widehat{\rho}_p \partial_{tt} \widehat{\omega} + \widehat{\Delta}^2 \widehat{\omega} = \widehat{F}_p, \quad \text{on } \widehat{\Gamma}, \tag{6}$$

where $\widehat{\rho}_p$ is the plate density coefficient and $\widehat{F}_p$ is the external forcing on the plate in y direction, to be specified later in the coupling conditions. The constant $\widehat{\rho}_p$ is the "average" plate density, which depends on the periodic structure. The in-plane bi-Laplacian $\widehat{\Delta}^2$ (Laplace–Beltrami operator for curved $\widehat{\Gamma}$'s) is associated with the elastic energy of the plate. Typically, there is a coefficient $\widetilde{D}$ in front of the bi-Laplacian, which contains information about the periodicity of the structure and its stiffness properties [2]. In the present work, we will assume that it is equal to 1. The source term $\widehat{F}_p$ corresponds to the loading of the poroelastic plate, which will come from the jump in the normal stress (traction) between the free fluid on one side and the thick Biot poroelastic structure on the other, see (7) below.

In our problem, the reticular plate separates the regions of free fluid flow and the Biot poroviscoelastic medium, and is transparent to the flow between the two. The time-dependent configuration of the plate

$$\Gamma(t) = \left\{(x,y) : 0 < x < L,\ y = \widehat{\omega}(t,x)\right\},$$

forms the bottom boundary of the moving Biot domain $\Omega_b(t)$, and the remaining left, top, and right boundaries of the moving Biot domain $\Omega_b(t)$ are fixed in time. Hence, we impose that $\boldsymbol{\eta} = 0$ on the left, top, and right boundaries of $\Omega_b(t)$. See Figure 1. Hence, we can describe the moving domain $\Omega_b(t)$ as

$$\Omega_b(t) = \left\{(x,y) : 0 < x < L,\ \widehat{\omega}(t,x) < y < R\right\}.$$

2.3. *The fluid subproblem*

The free flow of an incompressible, viscous fluid will be modeled by the Navier–Stokes equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi), \quad \text{in } \Omega_f(t), \tag{7}$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \text{in } \Omega_f(t), \tag{8}$$

where $\boldsymbol{u}$ is the fluid velocity and π is the fluid pressure. The Cauchy stress tensor is given by

$$\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi) = 2\nu\boldsymbol{D}(\boldsymbol{u}) - \pi\boldsymbol{I},$$

where π is the fluid pressure and ν is kinematic viscosity coefficient. The moving fluid domain $\Omega_f(t)$ is determined by the plate displacement $\widehat{\omega}$, as follows:

$$\Omega_f(t) = \left\{(x,y) : 0 < x < L, -R < y < \widehat{\omega}(t,x)\right\}.$$

2.4. *The coupling conditions*

The three problems above are fully coupled across the moving interface $\Gamma(t)$ via two sets of coupling conditions: the kinematic and dynamic coupling conditions. To state these conditions, we introduce the following notation:

- The Biot Cauchy stress tensor, defined on the physical domain, is obtained by applying the Piola transform to the Biot Cauchy stress tensor $\widehat{S}_b(\nabla\boldsymbol{\eta},p)$ on the reference domain:

$$\begin{aligned} S_b(\nabla\boldsymbol{\eta},p) &= \left[\det\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)^{-1}\widehat{S}_b\left(\widehat{\nabla}\widehat{\boldsymbol{\eta}},\widehat{p}\right)\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)^t\right]\circ\left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1} \\ &= \left(\frac{1}{\det\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)}\left[2\mu_e\widehat{\boldsymbol{D}}\left(\widehat{\boldsymbol{\eta}}\right) + \lambda_e\left(\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}}\right) + 2\mu_v\widehat{\boldsymbol{D}}\left(\widehat{\boldsymbol{\eta}}_t\right) + \lambda_v\left(\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}}_t\right)\right]\left(\widehat{\nabla}\widehat{\boldsymbol{\Phi}}_b^{\eta}\right)^t\right)\circ\left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1} - \alpha p\boldsymbol{I}. \end{aligned} \tag{9}$$

- The Eulerian structure velocity of the Biot poroviscoelastic matrix is given at each point of the physical domain $\Omega_b(t)$ by

$$\boldsymbol{\xi}(t,\cdot) = \partial_t\widehat{\boldsymbol{\eta}}\left(t,\left(\boldsymbol{\Phi}_b^{\eta}\right)^{-1}(t,\cdot)\right). \tag{10}$$

- The normal unit vector to the moving interface $\Gamma(t)$ will be denoted by $\boldsymbol{n}(t)$, and the normal unit vector to the reference configuration of the interface $\widehat{\Gamma}$ will be denoted by $\widehat{\boldsymbol{n}}$. Note that $\widehat{\boldsymbol{n}} = \boldsymbol{e}_y$. The vectors $\boldsymbol{n}(t)$ and $\widehat{\boldsymbol{n}}$ point outward from $\Omega_f(t)$ and Ω_f, and inward towards $\Omega_b(t)$ and Ω_b.

The two sets of coupling conditions can now be defined as follows.

(I) Kinematic coupling conditions:

- Conservation of mass of the fluid,

$$\boldsymbol{u}\cdot\boldsymbol{n}(t) = (\boldsymbol{q}+\boldsymbol{\xi})\cdot\boldsymbol{n}(t), \qquad \text{on } (0,T)\times\Gamma(t). \tag{11}$$

- Beavers–Joseph–Saffman condition describing tangential fluid slip at $\Gamma(t)$ [86, 87],

$$\beta(\boldsymbol{\xi}-\boldsymbol{u})\cdot\boldsymbol{\tau}(t) = \boldsymbol{\sigma}_f\boldsymbol{n}(t)\cdot\boldsymbol{\tau}(t), \qquad \text{on } (0,T)\times\Gamma(t), \tag{12}$$

  where $\beta \geq 0$ is a constant and $\boldsymbol{\tau}(t)$ is the rightward pointing unit tangent vector to $\Gamma(t)$.
- Continuity of the displacement,

$$\widehat{\boldsymbol{\eta}} = \widehat{\omega}\boldsymbol{e}_y, \qquad \text{on } (0,T)\times\widehat{\Gamma}. \tag{13}$$

(II) Dynamic coupling conditions:

- Balance of forces describing the external forcing on the plate as the difference between the external and internal load,

$$\widehat{F}_p = -\det\left(\nabla\widehat{\boldsymbol{\Phi}}_f^{\omega}\right)\left[\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi)\circ\widehat{\boldsymbol{\Phi}}_f^{\omega}\right]\left(\nabla\widehat{\boldsymbol{\Phi}}_f^{\omega}\right)^{-t}\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{n}} + \widehat{S}_b\left(\widehat{\nabla}\widehat{\boldsymbol{\eta}},\widehat{p}\right)\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{n}}|_{\widehat{\Gamma}}, \qquad \text{on } \widehat{\Gamma}, \tag{14}$$

  where $\widehat{\boldsymbol{\Phi}}_f^{\omega}$ is the Arbitrary Lagrangian–Eulerian (ALE) mapping for the fluid domain:

$$\widehat{\boldsymbol{\Phi}}_f^{\omega}(\widehat{x},\widehat{y}) = \left(\widehat{x}, \widehat{y} + \left(1+\frac{\widehat{y}}{R}\right)\widehat{\omega}\right), \quad (\widehat{x},\widehat{y})\in\widehat{\Omega}_f. \tag{15}$$

- Balance of pressure at the interface,

$$-\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi)\boldsymbol{n}(t)\cdot\boldsymbol{n}(t) + \frac{1}{2}|\boldsymbol{u}|^2 = p, \quad \text{on } (0,T)\times\Gamma(t). \tag{16}$$

2.5. *The initial and boundary conditions*

For the fluid, we will assume rigid walls on $\partial\Omega_f(t)\setminus\Gamma(t)$ and impose a no-slip condition

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega_f(t)\setminus\Gamma(t).$$

Similarly, we will assume that the boundaries of the Biot poroviscoelastic medium, excluding the interface $\Gamma(t)$, are rigid and impose

$$\widehat{\boldsymbol{\eta}} = 0 \quad \text{and} \quad \widehat{p} = 0, \quad \text{on } \partial\widehat{\Omega}_b\setminus\widehat{\Gamma}.$$

Finally, we prescribe the following initial conditions:

$$\begin{aligned}
\boldsymbol{u}(0) = \boldsymbol{u}_0 &\quad \text{in } \Omega_f(0),\\
\widehat{\boldsymbol{\eta}}(0) = \widehat{\boldsymbol{\eta}}_0,\ \partial_t\widehat{\boldsymbol{\eta}}(0) = \widehat{\boldsymbol{\xi}}_0 &\quad \text{in } \widehat{\Omega}_b,\\
\widehat{\omega}(0) = \widehat{\omega}_0,\ \partial_t\widehat{\omega}(0) = \widehat{\zeta}_0 &\quad \text{in } \widehat{\Gamma},\\
\widehat{p}(0) = \widehat{p}_0 &\quad \text{in } \widehat{\Omega}_b.
\end{aligned}$$

3. Definition of weak solutions

To handle the moving domains, it is useful to introduce the mappings that map the reference domains for the fluid, the Biot model, and the interface, onto the moving domains that depend on time and on the solution itself. We introduce these mappings next, and specify how functions and their derivatives transform under those mappings.

3.1. *Mappings between reference and physical domains*

We introduce the *mappings* for the Biot medium, for the plate, and for the fluid by the following:

$$\widehat{\boldsymbol{\Phi}}_b^{\eta}(t,\cdot):\widehat{\Omega}_b\to\Omega_b(t),\qquad \widehat{\boldsymbol{\Phi}}_\Gamma^{\omega}(t,\cdot):\widehat{\Gamma}\to\Gamma(t),\qquad \widehat{\boldsymbol{\Phi}}_f^{\omega}(t,\cdot):\widehat{\Omega}_f\to\Omega_f(t),$$

where

(1) $\widehat{\boldsymbol{\Phi}}_b^{\eta}$ is defined by the Lagrangian mapping (1) introduced in the previous section

$$\widehat{\boldsymbol{\Phi}}_b^{\eta}=\mathrm{Id}+\widehat{\boldsymbol{\eta}}.$$

(2) $\widehat{\boldsymbol{\Phi}}_\Gamma^{\omega}$ is defined by the Lagrangian mapping

$$\widehat{\boldsymbol{\Phi}}_\Gamma^{\omega}(\hat{x},0)=(\hat{x},\widehat{\omega}(\hat{x}))\,. \tag{17}$$

(3) $\widehat{\boldsymbol{\Phi}}_f^{\omega}$ is defined by the Arbitrary Lagrangian–Eulerian (ALE) mapping (15) defined above as

$$\widehat{\boldsymbol{\Phi}}_f^{\omega}(\hat{x},\hat{y})=\left(\hat{x},\hat{y}+\left(1+\frac{\hat{y}}{R}\right)\widehat{\omega}\right),\quad (\hat{x},\hat{y})\in\widehat{\Omega}_f.$$

The inverse is given by

$$\left(\widehat{\boldsymbol{\Phi}}_f^{\omega}\right)^{-1}(x,y)=\left(x,-R+\frac{R}{R+\widehat{\omega}}(R+y)\right),\quad (x,y)\in\Omega_f(t).$$

We will need the following *Jacobians*:

- The Jacobian of $\widehat{\boldsymbol{\Phi}}_f^{\omega}$ on $\widehat{\Omega}_f$ is given by $\widehat{\mathscr{J}}_f^{\omega}=|1+\frac{\hat{\omega}}{R}|$. Because our results will hold up until the time of domain degeneracy when $|\widehat{\omega}|\geq R$, we can get rid of the absolute values and just write
$$\widehat{\mathscr{J}}_f^{\omega}=1+\frac{\widehat{\omega}}{R}. \tag{18}$$
- The Jacobian of $\widehat{\boldsymbol{\Phi}}_b^{\eta}$ on $\widehat{\Omega}_b$ is
$$\widehat{\mathscr{J}}_b^{\eta}=\det\left(\boldsymbol{I}+\widehat{\nabla}\widehat{\boldsymbol{\eta}}\right). \tag{19}$$
- We will also need the arc length measure $\widehat{\mathscr{J}}_\Gamma^{\omega}$ between the reference configuration and the deformed configuration of the plate, defined by
$$\widehat{\mathscr{J}}_\Gamma^{\omega}=\sqrt{1+|\partial_{\hat{x}}\widehat{\omega}|^2}. \tag{20}$$

In the analysis of the full FPSI problem, it is necessary to consider functions on both the reference and the physical domains, and hence, we must examine how functions and derivatives transform under $\widehat{\boldsymbol{\Phi}}_f^{\omega}$ and $\widehat{\boldsymbol{\Phi}}_b^{\eta}$.

Transformations under $\widehat{\boldsymbol{\Phi}}_f^{\omega}$ on $\widehat{\Omega}_f$. Consider, in particular, the fluid velocity $\boldsymbol{u}$. The fluid velocity $\boldsymbol{u}$ defined on $\Omega_f(t)$, transferred to the fixed reference domain $\widehat{\Omega}_f$, is given by

$$\widehat{\boldsymbol{u}}(t,\hat{x},\hat{y})=\boldsymbol{u}\circ\widehat{\boldsymbol{\Phi}}_f^{\omega},\quad \text{for } (\hat{x},\hat{y})\in\widehat{\Omega}_f.$$

- **The divergence free condition.** Recall that on $\Omega_f(t)$, $\nabla\cdot\boldsymbol{u}=0$. However, when we pull the fluid velocity back to the reference domain, $\widehat{\boldsymbol{u}}$ is not necessarily divergence free on $\widehat{\Omega}_f$. Hence, we want to reformulate the divergence free condition on the fixed reference domain. To do this, we look at how derivatives transform under the map $\widehat{\boldsymbol{\Phi}}_f^{\omega}$. In particular, note that for any function g defined on $\Omega_f(t)$,
$$\nabla g=\nabla\left(\widehat{g}\circ\left(\widehat{\boldsymbol{\Phi}}_f^{\omega}\right)^{-1}\right)=\left(\widehat{\nabla}_f^{\omega}\widehat{g}\right)\circ\left(\widehat{\boldsymbol{\Phi}}_f^{\omega}\right)^{-1}$$
for the differential operator
$$\widehat{\nabla}_f^{\omega}=\begin{pmatrix}\partial_{\hat{x}}-(R+y)\partial_{\hat{x}}\widehat{\omega}\frac{R}{(R+\hat{\omega})^2}\partial_{\hat{y}}\\ \frac{R}{R+\hat{\omega}}\partial_{\hat{y}}\end{pmatrix}=\begin{pmatrix}\partial_{\hat{x}}-\frac{(R+\hat{y})\partial_{\hat{x}}\hat{\omega}}{R+\hat{\omega}}\partial_{\hat{y}}\\ \frac{R}{R+\hat{\omega}}\partial_{\hat{y}}\end{pmatrix}, \tag{21}$$

where we used $y = \hat{y} + (1 + \frac{\hat{y}}{R})\hat{\omega}$. Therefore, the divergence free condition on the fixed reference domain $\hat{\Omega}_f$ and the symmetrized gradient on $\hat{\Omega}_f$ transform as follows:

$$\hat{\nabla}_f^{\omega} \cdot \hat{\boldsymbol{u}} = 0 \quad \text{and} \quad \hat{\boldsymbol{D}}_f^{\omega}(\hat{\boldsymbol{u}}) = \frac{1}{2}\left(\hat{\nabla}_f^{\omega}\hat{\boldsymbol{u}} + \left(\hat{\nabla}_f^{\omega}\hat{\boldsymbol{u}}\right)^t\right).$$

- **The time derivatives.** Next, we consider how time derivatives are transformed under the map $\hat{\boldsymbol{\Phi}}_f^{\omega}$. We have that

$$\partial_t \boldsymbol{u}(t,x,y) = \partial_t \left(\hat{\boldsymbol{u}}(t, \left(\boldsymbol{\Phi}_f^{\omega}\right)^{-1}(\hat{x},\hat{y})\right) = \partial_t \hat{\boldsymbol{u}} - \partial_{\hat{y}} \hat{\boldsymbol{u}} \frac{(R+\hat{y})\partial_t \hat{\omega}}{R+\hat{\omega}}.$$

By introducing the domain velocity

$$\hat{\boldsymbol{w}} = \frac{R+\hat{y}}{R}\partial_t \hat{\omega} \boldsymbol{e}_y, \tag{22}$$

we have that

$$\partial_t \boldsymbol{u} = \partial_t \hat{\boldsymbol{u}} - \left(\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^{\omega}\right)\hat{\boldsymbol{u}}. \tag{23}$$

Transformations under $\hat{\boldsymbol{\Phi}}_b^{\eta}$ on $\hat{\Omega}_b$. Recall that $\hat{\boldsymbol{\Phi}}_b^{\eta}(\hat{x},\hat{y}) = (\hat{x},\hat{y}) + \hat{\boldsymbol{\eta}}(\hat{x},\hat{y})$ for $(\hat{x},\hat{y}) \in \hat{\Omega}_b$. So given a scalar function g defined on $\Omega_b(t)$, we want to see how ∇g transforms when pulled back to the reference domain. We define the pull back of g to the reference domain $\hat{\Omega}_b$ by

$$\hat{g} = g \circ \hat{\boldsymbol{\Phi}}_b^{\eta}.$$

Similarly to what we had above, we claim that for some differential operator $\hat{\nabla}_b^{\eta}$,

$$\nabla g = \nabla\left(\hat{g} \circ \left(\hat{\boldsymbol{\Phi}}_b^{\eta}\right)^{-1}\right) = \left(\hat{\nabla}_b^{\eta}\hat{g}\right) \circ \left(\hat{\boldsymbol{\Phi}}_b^{\eta}\right)^{-1}.$$

We emphasize that ∇ is a gradient on the physical domain, $\hat{\nabla}$ is a gradient on the reference domain, and $\hat{\nabla}_b^{\eta}$ is a differential operator (different from $\hat{\nabla}$) on the reference domain. For any function g defined on the physical domain, we have that

$$\hat{\nabla}\left(g \circ \hat{\boldsymbol{\Phi}}_b^{\eta}\right) = \left[(\nabla g) \circ \hat{\boldsymbol{\Phi}}_b^{\eta}\right] \cdot \left(\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}\right).$$

Hence, for

$$\hat{\nabla}_b^{\eta}\hat{g} = (\nabla g) \circ \hat{\boldsymbol{\Phi}}_b^{\eta},$$

we have the following explicit formula for the differential operator $\hat{\nabla}_b^{\eta}$ on the reference domain:

$$\hat{\nabla}_b^{\eta}\hat{g} = \left(\frac{\partial \hat{g}}{\partial \hat{x}}, \frac{\partial \hat{g}}{\partial \hat{y}}\right) \cdot \left(\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}\right)^{-1}. \tag{24}$$

We remark that the invertibility of the matrix $\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}$ will be related to whether the map $(\hat{x},\hat{y}) \to (\hat{x},\hat{y}) + \hat{\boldsymbol{\eta}}(\hat{x},\hat{y})$ is a bijection between $\hat{\Omega}_b$ and $\Omega_b(t)$.

3.2. *Weak formulation*

To derive a weak formulation of the coupled FPSI problem (2), (3), (5), (6), (7), and (8) with the coupling conditions (11)-(16), we introduce the test function $\boldsymbol{v}$ for the fluid velocity $\boldsymbol{u}$, the test function $\hat{\boldsymbol{\psi}}$ for the structure displacement $\hat{\boldsymbol{\eta}}$, the test function $\hat{\varphi}$ for the plate displacement $\hat{\omega}$, and the test function $\hat{r}$ for the Biot pressure $\hat{p}$, multiply the corresponding equations by the test functions, integrate by parts, and use the Reynold's transport theorem and the coupling conditions to obtain the following definition of a weak solution.

Definition 1. *The ordered four-tuple* $(\boldsymbol{u}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p})$ *is a weak solution if for every test function* $(\boldsymbol{v}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r})$ *that is* C_c^1 *in time on* $[0,T]$ *taking values in the test space, satisfying* $\widehat{\boldsymbol{\psi}} = \widehat{\varphi}\boldsymbol{e}_y$ *on* $\widehat{\Gamma}$, *we have that*

$$
\begin{aligned}
&-\int_0^T\int_{\Omega_f(t)} \boldsymbol{u}\cdot\partial_t\boldsymbol{v} + \frac{1}{2}\int_0^T\int_{\Omega_f(t)}\left[((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v} - ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{u}\right] + \frac{1}{2}\int_0^T\int_{\Gamma(t)}(\boldsymbol{u}\cdot\boldsymbol{n} - 2\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{v}\\
&+2\nu\int_0^T\int_{\Omega_f(t)}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v}) + \int_0^T\int_{\Gamma(t)}\left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \beta\int_0^T\int_{\Gamma(t)}(\xi_\tau - u_\tau)(\psi_\tau - v_\tau)\\
&-\rho_p\int_0^T\int_{\hat{\Gamma}}\partial_t\widehat{\omega}\cdot\partial_t\widehat{\varphi} + \int_0^T\int_{\hat{\Gamma}}\widehat{\Delta}\widehat{\omega}\cdot\widehat{\Delta}\widehat{\varphi} - \rho_b\int_0^T\int_{\hat{\Omega}_b}\partial_t\widehat{\boldsymbol{\eta}}\cdot\partial_t\widehat{\boldsymbol{\psi}} + 2\mu_e\int_0^T\int_{\hat{\Omega}_b}\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}})\\
&+\lambda_e\int_0^T\int_{\hat{\Omega}_b}(\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}}) + 2\mu_v\int_0^T\int_{\hat{\Omega}_b}\widehat{\boldsymbol{D}}(\partial_t\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}}) + \lambda_v\int_0^T\int_{\hat{\Omega}_b}(\widehat{\nabla}\cdot\partial_t\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}})\\
&-\alpha\int_0^T\int_{\Omega_b(t)}p\nabla\cdot\boldsymbol{\psi} - c_0\int_0^T\int_{\hat{\Omega}_b}\widehat{p}\cdot\partial_t\widehat{r} - \alpha\int_0^T\int_{\Omega_b(t)}\frac{D}{Dt}\boldsymbol{\eta}\cdot\nabla r - \alpha\int_0^T\int_{\Gamma(t)}(\boldsymbol{\xi}\cdot\boldsymbol{n})r\\
&+\kappa\int_0^T\int_{\Omega_b(t)}\nabla p\cdot\nabla r - \int_0^T\int_{\Gamma(t)}((\boldsymbol{u}-\boldsymbol{\xi})\cdot\boldsymbol{n})r\\
&\qquad = \int_{\Omega_f(0)}\boldsymbol{u}_0\cdot\boldsymbol{v}(0) + \rho_p\int_{\hat{\Gamma}}\partial_t\widehat{\omega}_0\cdot\widehat{\varphi}(0) + \rho_b\int_{\hat{\Omega}_b}\partial_t\widehat{\boldsymbol{\eta}}_0\cdot\widehat{\boldsymbol{\psi}}(0) + c_0\int_{\hat{\Omega}_b}\widehat{p}_0\cdot\widehat{r}(0). \qquad (25)
\end{aligned}
$$

Details of the derivation of this weak formulation will be presented in [1].

Remark 2. We will omit explicit mention of function compositions with the mappings $\widehat{\boldsymbol{\Phi}}_f^\omega$, $\widehat{\boldsymbol{\Phi}}_\Gamma^\omega$, and $\widehat{\boldsymbol{\Phi}}_b^\eta$ (defined in (15), (17), and (1)), and their inverses throughout the paper, as the function compositions needed will be clear from the context. In particular, we follow this convention in the weak formulation above. For example, since the pore pressure $\widehat{p}$ and the test function $\widehat{\boldsymbol{\psi}}$ are defined on the reference domain $\widehat{\Omega}_b$, the integral

$$-\alpha\int_0^T\int_{\Omega_b(t)}p\nabla\cdot\boldsymbol{\psi} \quad \text{means} \quad -\alpha\int_0^T\int_{\Omega_b(t)}\left(\widehat{p}\circ\left(\boldsymbol{\Phi}_b^\eta\right)^{-1}\right)\nabla\cdot\left(\widehat{\boldsymbol{\psi}}\circ\left(\boldsymbol{\Phi}_b^\eta\right)^{-1}\right).$$

As another example, the integral

$$-\int_0^T\int_{\Gamma(t)}\left((\boldsymbol{u}-\zeta\boldsymbol{e}_y)\cdot\boldsymbol{n}\right)r \quad \text{means} \quad -\int_0^T\int_{\Gamma(t)}\left(\left(\boldsymbol{u}-\left(\zeta\circ\left(\boldsymbol{\Phi}_\Gamma^\omega\right)^{-1}\right)\boldsymbol{e}_y\right)\cdot\boldsymbol{n}\right)\left(\widehat{r}\circ\left(\boldsymbol{\Phi}_b^\eta\right)^{-1}\right).$$

Remark 3. The above weak formulation *is inadequate* for the finite energy weak solutions. By the energy estimates, see Section 4.1, the regularity of the structure displacement $\widehat{\boldsymbol{\eta}}$ on $\widehat{\Omega}_b$ in the finite energy space is $L^2(0,T,H^1(\widehat{\Omega}_b))$, which is not enough regularity to interpret the term

$$\alpha\int_{\Omega_b(t)}p\nabla\cdot\boldsymbol{\psi},$$

since the test function has regularity $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega}_b)$ on the *fixed reference domain*, due to the corresponding finite energy regularity of $\widehat{\boldsymbol{\eta}}$. Hence, after changing variables, which adds an extra factor of $\det(\boldsymbol{I}+\widehat{\nabla}\widehat{\boldsymbol{\eta}})$ arising from the Jacobian, which is only in $L^1(0,T;L^1(\widehat{\Omega}_b))$ in two dimensions, there is not enough regularity to guarantee that this integral is finite.

Therefore, we cannot interpret the above notion of weak solution properly in the space of finite energy solutions, as the finite energy space does not have enough regularity to make sense of certain integrals in the weak formulation involving the deformed domain $\Omega_b(t)$. This is why we introduce a regularized problem, which is consistent with the original problem in the sense that weak solutions to the regularized problem converge, as the regularization parameter tends to zero, to a smooth solution of the original, nonregularized problem, when a smooth solution exists. This weak-classical consistency will be shown in [1]. In this manuscript, we show that the regularized, nonlinearly coupled FPSI problem has a weak solution which satisfies an energy estimate “consistent” with the original, nonregularized problem.

4. Weak solutions to the regularized moving boundary FPSI problem

We recall that the main difficulty in defining a weak formulation is the lack of regularity of $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$ from the energy estimate. Therefore, we modify our weak formulation appropriately to give $\hat{\boldsymbol{\eta}}$ more regularity, by convolving with a smooth compactly supported function with support of width δ. Because we are working on a bounded domain $\hat{\Omega}_b$, the convolution must be defined in a way that preserves the Dirichlet condition on the left, top, and right boundaries of $\hat{\Omega}_b$. To do this, we extend the domain $\hat{\Omega}_b$ by L in each horizontal direction, and by R in the vertical direction to define an extended domain $\tilde{\Omega}_b$ as follows:

$$\tilde{\Omega}_b = [-L, 2L] \times [-R, 2R].$$

Assuming that the convolution parameter $\delta < \min(L, R)$, the resulting convolved function will be defined on $\hat{\Omega}_b$. Furthermore, to satisfy the Dirichlet boundary condition we must use an odd extension of $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$ to the larger domain $\tilde{\Omega}_b$ along the lines $\hat{x} = 0$, $\hat{x} = L$, $\hat{y} = 0$ and $\hat{y} = R$:

Definition 4. *Given $\hat{\boldsymbol{\eta}}$ defined on $\hat{\Omega}_b$ satisfying $\hat{\boldsymbol{\eta}} = 0$ on $\hat{x} = 0$, $\hat{x} = L$, and $\hat{y} = R$ and $\hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y$ on Γ, we define the* odd extension of $\hat{\boldsymbol{\eta}}$ to $\tilde{\Omega}_b$ *as follows.*

(1) *On $[0, L] \times [-R, 0]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = \hat{\omega}(\hat{x})\boldsymbol{e}_y + (\hat{\omega}(\hat{x})\boldsymbol{e}_y - \hat{\boldsymbol{\eta}}(\hat{x}, -\hat{y}))$.*
(2) *On $[0, L] \times [R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(\hat{x}, 2R - \hat{y})$.*
(3) *On $[-L, 0] \times [-R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(-\hat{x}, \hat{y})$.*
(4) *On $[L, 2L] \times [-R, 2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = -\hat{\boldsymbol{\eta}}(2L - \hat{x}, \hat{y})$.*

To define the convolution of $\hat{\boldsymbol{\eta}}$, we use a radially symmetric smooth function σ on $\mathbb{R}^2$ with compact support in the closed ball of radius one, with the property that $\int_{\mathbb{R}^2} \sigma = 1$, and define

$$\sigma_\delta = \delta^{-2}\sigma(\delta^{-1}\boldsymbol{x}), \quad \text{on } \mathbb{R}^2.$$

Regularized (in space) functions, which are spatially smooth on $\hat{\Omega}_b$:

- The regularized Biot displacement $\hat{\boldsymbol{\eta}}$ is then defined by extending $\hat{\boldsymbol{\eta}}$ to $\tilde{\Omega}_b$ by odd extension and defining:
$$\hat{\boldsymbol{\eta}}^\delta = \hat{\boldsymbol{\eta}} * \sigma_\delta, \quad \text{on } \hat{\Omega}_b. \tag{26}$$
- The regularized Lagrangian mapping is defined by:
$$\hat{\boldsymbol{\Phi}}_b^{\eta^\delta} = \mathrm{Id} + \hat{\boldsymbol{\eta}}^\delta, \tag{27}$$
- The regularized moving Biot domain is defined by
$$\Omega_b^\delta(t) = \hat{\boldsymbol{\Phi}}_b^{\eta^\delta}\left(t, \hat{\Omega}_b\right). \tag{28}$$
- The regularized moving interface is defined by
$$\Gamma^\delta(t) = \hat{\boldsymbol{\Phi}}_b^{\eta^\delta}(t, \hat{\Gamma}).$$
The motivation for this comes from the need to satisfy the continuity of displacement for the regularized quantities at the fluid-structure interface, which otherwise would not have been satisfied.

Note that the following property remains to be true for the regularized displacement:

$$\hat{\boldsymbol{\eta}}^\delta = 0 \ \text{ on } \ \partial\hat{\Omega}_b \setminus \hat{\Gamma}.$$

We can now define a *weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter δ*. We start by defining the solution and test space, which is motivated by the energy estimates in Section 4.1, and then we state the regularized weak formulation. We will formulate the solution space, test space, and regularized weak formulation for both the moving fluid/Biot domain (Eulerian) and the fixed reference fluid/Biot domain (Lagrangian) formulations.

Definition 5 (Solution and test spaces for the regularized problem).

- Fluid function space (moving domain-Eulerian formulation).

$$V_f(t) = \left\{\boldsymbol{u} = (u_x, u_y) \in H^1\left(\Omega_f(t)\right) : \nabla\cdot\boldsymbol{u} = 0, \ \textit{and}\ \boldsymbol{u} = 0 \ \textit{when}\ x = 0, x = L, y = -R\right\}, \tag{29}$$

$$\mathcal{V}_f = L^\infty\left(0, T; L^2\left(\Omega_f(t)\right)\right) \cap L^2\left(0, T; V_f(t)\right). \tag{30}$$

- Fluid function space (fixed domain- Lagrangian formulation).

$$V_f^\omega = \left\{\widehat{\boldsymbol{u}} = \left(\widehat{u}_x, \widehat{u}_y\right) \in H^1\left(\widehat{\Omega}_f\right) : \widehat{\nabla}_f^\omega\cdot\widehat{\boldsymbol{u}} = 0, \ \textit{and}\ \widehat{\boldsymbol{u}} = 0 \ \textit{when}\ \widehat{x} = 0, \widehat{x} = L, \widehat{y} = -R\right\}, \tag{31}$$

$$\mathcal{V}_f^\omega = L^\infty\left(0, T; L^2\left(\widehat{\Omega}_f\right)\right) \cap L^2\left(0, T; V_f^\omega\right). \tag{32}$$

- Plate function space.

$$\mathcal{V}_\omega = W^{1,\infty}\left(0, T; L^2(\widehat{\Gamma})\right) \cap L^\infty\left(0, T; H_0^2(\widehat{\Gamma})\right). \tag{33}$$

- Biot displacement function space.

$$V_d = \left\{\widehat{\boldsymbol{\eta}} = \left(\widehat{\eta}_x, \widehat{\eta}_y\right) \in H^1\left(\widehat{\Omega}_b\right) : \widehat{\boldsymbol{\eta}} = 0 \ \textit{for}\ \widehat{x} = 0, \widehat{x} = L, \widehat{y} = R, \ \textit{and}\ \widehat{\eta}_x = 0 \ \textit{on}\ \widehat{\Gamma}\right\}, \tag{34}$$

$$\mathcal{V}_b = W^{1,\infty}\left(0, T; L^2\left(\widehat{\Omega}_b\right)\right) \cap L^\infty\left(0, T; V_d\right) \cap H^1\left(0, T; V_d\right). \tag{35}$$

- Biot pore pressure function space.

$$V_p = \left\{\widehat{p} \in H^1\left(\widehat{\Omega}_b\right) : \widehat{p} = 0 \ \textit{for}\ \widehat{x} = 0, \widehat{x} = L, \widehat{y} = R\right\}, \tag{36}$$

$$\mathcal{Q}_b = L^\infty\left(0, T; L^2\left(\widehat{\Omega}_b\right)\right) \cap L^2\left(0, T; V_p\right). \tag{37}$$

- Weak solution space (moving domain).

$$\mathcal{V}_{sol} = \left\{\left(\boldsymbol{u}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p}\right) \in \mathcal{V}_f \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \widehat{\boldsymbol{\eta}} = \widehat{\omega}\boldsymbol{e}_y \ \textit{on}\ \widehat{\Gamma}\right\}. \tag{38}$$

- Weak solution space (fixed domain).

$$\mathcal{V}_{sol}^\omega = \left\{\left(\widehat{\boldsymbol{u}}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p}\right) \in \mathcal{V}_f^\omega \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \widehat{\boldsymbol{\eta}} = \widehat{\omega}\boldsymbol{e}_y \ \textit{on}\ \widehat{\Gamma}\right\}. \tag{39}$$

- Test space (moving domain).

$$\mathcal{V}_{test} = \left\{\left(\boldsymbol{v}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r}\right) \in C_c^1\left([0, T); V_f(t) \times H_0^2(\widehat{\Gamma}) \times V_d \times V_p\right) : \widehat{\boldsymbol{\psi}} = \widehat{\varphi}\boldsymbol{e}_y \ \textit{on}\ \widehat{\Gamma}\right\}. \tag{40}$$

- Test space (fixed domain).

$$\mathcal{V}_{test}^\omega = \left\{\left(\widehat{\boldsymbol{v}}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r}\right) \in C_c^1\left([0, T); V_f^\omega \times H_0^2(\widehat{\Gamma}) \times V_d \times V_p\right) : \widehat{\boldsymbol{\psi}} = \widehat{\varphi}\boldsymbol{e}_y \ \textit{on}\ \widehat{\Gamma}\right\}. \tag{41}$$

Remark 6. Because $\widehat{\Gamma}$ is one dimensional, for plate displacements $\widehat{\omega} \in \mathcal{V}_\omega$, we have that $\widehat{\omega} \in C(0, T; C^1(\widehat{\Gamma}))$ and hence, there is a one-to-one correspondence between functions in $\mathcal{V}_{\text{sol}}$ and $\mathcal{V}_{\text{sol}}^\omega$ and functions in $\mathcal{V}_{\text{test}}$ and $\mathcal{V}_{\text{test}}^\omega$, given by composition with the ALE mapping (15) for the fluid domain, $\widehat{\boldsymbol{\Phi}}_f^\omega : \widehat{\Omega}_f \to \Omega_f(t)$, for the component involving fluid velocities.

To write a weak formulation for the regularized problem, we introduce the reticular plate velocity $\widehat{\zeta} = \partial_t\widehat{\omega}$, which satisfies the kinematic coupling condition $\widehat{\zeta}\boldsymbol{e}_y = \widehat{\boldsymbol{\xi}}$ on $\widehat{\Gamma} \times (0, T)$, which follows from (13). With this notation, we have the following definition of a weak solution for the regularized problem.

Definition 7 (Weak solution to regularized problem, moving fluid domain formulation). *An ordered four-tuple* $(\boldsymbol{u}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p}) \in \mathcal{V}_{sol}$ *is a* weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter δ *if for every test function* $(\boldsymbol{v}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r}) \in \mathcal{V}_{test}$,

$$\begin{aligned}
&-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u}\cdot\partial_t \boldsymbol{v} + \frac{1}{2}\int_0^T \int_{\Omega_f(t)} \left[((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v} - ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{u}\right] + \frac{1}{2}\int_0^T\int_{\Gamma(t)} (\boldsymbol{u}\cdot\boldsymbol{n} - 2\zeta\boldsymbol{e}_y\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{v} \\
&+ 2\nu\int_0^T\int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v}) + \int_0^T\int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{n} + \beta\int_0^T\int_{\Gamma(t)} (\zeta\boldsymbol{e}_y\cdot\boldsymbol{\tau} - \boldsymbol{u}\cdot\boldsymbol{\tau})(\boldsymbol{\psi}\cdot\boldsymbol{\tau} - \boldsymbol{v}\cdot\boldsymbol{\tau}) \\
&- \rho_p\int_0^T\int_{\hat\Gamma} \partial_t\widehat{\omega}\cdot\partial_t\widehat{\varphi} + \int_0^T\int_{\hat\Gamma}\widehat{\Delta}\widehat{\omega}\cdot\widehat{\Delta}\widehat{\varphi} - \rho_b\int_0^T\int_{\hat\Omega_b}\partial_t\widehat{\boldsymbol{\eta}}\cdot\partial_t\widehat{\boldsymbol{\psi}} + 2\mu_e\int_0^T\int_{\hat\Omega_b}\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}}) \\
&+ \lambda_e\int_0^T\int_{\hat\Omega_b}(\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}}) + 2\mu_v\int_0^T\int_{\hat\Omega_b}\widehat{\boldsymbol{D}}(\partial_t\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}}) + \lambda_v\int_0^T\int_{\hat\Omega_b}(\widehat{\nabla}\cdot\partial_t\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}}) \\
&- \alpha\int_0^T\int_{\Omega_b^\delta(t)} p\nabla\cdot\boldsymbol{\psi} - c_0\int_0^T\int_{\hat\Omega_b}\widehat{p}\,\partial_t\widehat{r} - \alpha\int_0^T\int_{\Omega_b^\delta(t)}\frac{D^\delta}{Dt}\boldsymbol{\eta}\cdot\nabla r - \alpha\int_0^T\int_{\Gamma^\delta(t)}\left(\zeta\boldsymbol{e}_y\cdot\boldsymbol{n}^\delta\right)r \\
&+ \kappa\int_0^T\int_{\Omega_b^\delta(t)}\nabla p\cdot\nabla r - \int_0^T\int_{\Gamma(t)}\left((\boldsymbol{u}-\zeta\boldsymbol{e}_y)\cdot\boldsymbol{n}\right)r \\
&\qquad = \int_{\Omega_f(0)}\boldsymbol{u}_0\cdot\boldsymbol{v}(0) + \rho_p\int_{\hat\Gamma}\partial_t\widehat{\omega}_0\cdot\widehat{\varphi}(0) + \rho_b\int_{\hat\Omega_b}\partial_t\widehat{\boldsymbol{\eta}}_0\cdot\widehat{\boldsymbol{\psi}}(0) + c_0\int_{\hat\Omega_b}\widehat{p}_0\cdot\widehat{r}(0), \qquad (42)
\end{aligned}$$

where $\frac{D^\delta}{Dt} = \frac{d}{dt} + (\boldsymbol{\xi}^\delta\cdot\nabla)$ *is the material derivative with respect to the regularized displacement,* $\boldsymbol{n}$ *denotes the upward pointing normal vector to* $\Gamma(t)$*, and* $\boldsymbol{n}^\delta$ *denotes the upward pointing normal vector to* $\Gamma^\delta(t)$*.*

Notice that only four terms contain regularization via convolution with parameter δ. While there are many different ways to write the regularized weak formulation, the regularization presented above is a regularization that deviates from the original, nonregularized problem, in the smallest possible number of terms, and is still consistent (in the weak-classical consistency sense [1]) with the original, nonregularized problem.

Remark 8. We remark that, while the solution to the regularized problem above depends on the regularization parameter δ implicitly, we will not use δ to denote this implicit dependence for simplicity of notation.

Remark 9. As in Remark 2, in this weak formulation for the regularized problem, we simplify notation by omitting the explicit compositions with the maps $\widehat{\boldsymbol{\Phi}}_f^\omega$, $\widehat{\boldsymbol{\Phi}}_\Gamma^\omega$, $\widehat{\boldsymbol{\Phi}}_b^\eta$, and $\widehat{\boldsymbol{\Phi}}_b^{\eta^\delta}$, (defined in (15), (17), (1), and (27) respectively), and their inverses. The necessary compositions with such mappings, which are omitted for notational simplicity, will be clear from the context. To clarify this, as an example, we show the explicit compositions with these mappings for the two terms in the weak formulation for the regularized problem, which correspond to the two terms mentioned in Remark 2:

$$-\alpha\int_0^T\int_{\Omega_b^\delta(t)} p\nabla\cdot\boldsymbol{\psi} \quad \text{means} \quad -\alpha\int_0^T\int_{\Omega_b^\delta(t)}\left(\widehat{p}\circ\left(\boldsymbol{\Phi}_b^{\eta^\delta}\right)^{-1}\right)\nabla\cdot\left(\widehat{\boldsymbol{\psi}}\circ\left(\boldsymbol{\Phi}_b^{\eta^\delta}\right)^{-1}\right),$$

and the second integral from Remark 2 (which also appears in the weak formulation for the regularized problem) has the same meaning here:

$$-\int_0^T\int_{\Gamma(t)}\left((\boldsymbol{u}-\zeta\boldsymbol{e}_y)\cdot\boldsymbol{n}\right)r \quad \text{means} \quad -\int_0^T\int_{\Gamma(t)}\left(\left(\boldsymbol{u}-\left(\zeta\circ(\boldsymbol{\Phi}_\Gamma^\omega)^{-1}\right)\boldsymbol{e}_y\right)\cdot\boldsymbol{n}\right)\left(\widehat{r}\circ(\boldsymbol{\Phi}_b^\eta)^{-1}\right).$$

Next we reformulate the definition of a weak solution to the regularized moving boundary FPSI problem by stating the weak formulation on the fixed reference domain. In particular, we will need to handle any integrals dealing with time-dependent domains by using a change of variables, which will introduce the Jacobians of the transformations, $\widehat{\mathcal{J}}_f^\omega$, $\widehat{\mathcal{J}}_b^\eta$, and $\widehat{\mathcal{J}}_\Gamma^\omega$ defined in (18), (19), and (20), respectively. Furthermore, we have to use the transformation of time derivatives, given by (22) and (23), and the transformed gradient (21), where we assume that $|\omega| < R$ so that there is no domain degeneracy. Finally, by introducing the following notation for the renormalized normal and tangent vectors:

$$\widehat{\boldsymbol{n}}^\omega = (-\partial_{\hat{x}}\widehat{\omega}, 1), \qquad \widehat{\boldsymbol{\tau}}^\omega = (1, \partial_{\hat{x}}\widehat{\omega}), \qquad \widehat{\boldsymbol{n}}^{\omega^\delta} = \left(-\partial_{\hat{x}}\left(\left(\widehat{\boldsymbol{\eta}}^\delta\right)_y|_{\hat{\Gamma}}\right), 1\right), \tag{43}$$

after the calculation which can be found in [1], we can now define a weak solution for the regularized problem, defined on the fixed reference domain as follows.

Definition 10 (Weak solution to regularized problem, fixed fluid domain formulation). *An ordered four-tuple* $(\widehat{\boldsymbol{u}}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p}) \in \mathcal{V}_{sol}^\omega$ *is a* weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter δ *if for all test functions* $(\widehat{\boldsymbol{v}}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r}) \in \mathcal{V}_{test}^\omega$*, the following equality holds:*

$$\begin{aligned}
&-\int_0^T\int_{\hat{\Omega}_f}\left(1+\frac{\widehat{\omega}}{R}\right)\widehat{\boldsymbol{u}}\cdot\partial_t\widehat{\boldsymbol{v}} + \frac{1}{2}\int_0^T\int_{\hat{\Omega}_f}\left(1+\frac{\widehat{\omega}}{R}\right)\left[\left((\widehat{\boldsymbol{u}}-\widehat{\boldsymbol{w}})\cdot\widehat{\nabla}_f^\omega\widehat{\boldsymbol{u}}\right)\cdot\widehat{\boldsymbol{v}} - \left((\widehat{\boldsymbol{u}}-\widehat{\boldsymbol{w}})\cdot\widehat{\nabla}_f^\omega\widehat{\boldsymbol{v}}\right)\cdot\widehat{\boldsymbol{u}}\right]\\
&-\frac{1}{2R}\int_0^T\int_{\hat{\Omega}_f}(\partial_t\widehat{\omega})\,\widehat{\boldsymbol{u}}\cdot\widehat{\boldsymbol{v}} + \frac{1}{2}\int_0^T\int_{\hat{\Gamma}}\left(\widehat{\boldsymbol{u}}\cdot\widehat{\boldsymbol{n}}^\omega - \widehat{\zeta}\boldsymbol{e}_y\cdot\widehat{\boldsymbol{n}}^\omega\right)\widehat{\boldsymbol{u}}\cdot\widehat{\boldsymbol{v}} + 2\nu\int_0^T\int_{\hat{\Omega}_f}\left(1+\frac{\widehat{\omega}}{R}\right)\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{u}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{v}})\\
&+\int_0^T\int_{\hat{\Gamma}}\left(\frac{1}{2}|\widehat{\boldsymbol{u}}|^2-\widehat{p}\right)\left(\widehat{\boldsymbol{\psi}}-\widehat{\boldsymbol{v}}\right)\cdot\widehat{\boldsymbol{n}}^\omega + \frac{\beta}{\widehat{\mathcal{J}}_\Gamma^\omega}\int_0^T\int_{\hat{\Gamma}}\left(\widehat{\zeta}\boldsymbol{e}_y-\widehat{\boldsymbol{u}}\right)\cdot\widehat{\boldsymbol{\tau}}^\omega\left(\widehat{\boldsymbol{\psi}}-\widehat{\boldsymbol{v}}\right)\cdot\widehat{\boldsymbol{\tau}}^\omega\\
&-\rho_p\int_0^T\int_{\hat{\Gamma}}\partial_t\widehat{\omega}\cdot\partial_t\widehat{\varphi} + \int_0^T\int_{\hat{\Gamma}}\widehat{\Delta}\widehat{\omega}\cdot\widehat{\Delta}\widehat{\varphi} - \rho_b\int_0^T\int_{\hat{\Omega}_b}\partial_t\widehat{\boldsymbol{\eta}}\cdot\partial_t\widehat{\boldsymbol{\psi}} + 2\mu_e\int_0^T\int_{\hat{\Omega}_b}\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}})\\
&+\lambda_e\int_0^T\int_{\hat{\Omega}_b}(\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}}) + 2\mu_v\int_0^T\int_{\hat{\Omega}_b}\widehat{\boldsymbol{D}}(\partial_t\widehat{\boldsymbol{\eta}}):\widehat{\boldsymbol{D}}(\widehat{\boldsymbol{\psi}}) + \lambda_v\int_0^T\int_{\hat{\Omega}_b}(\widehat{\nabla}\cdot\partial_t\widehat{\boldsymbol{\eta}})(\widehat{\nabla}\cdot\widehat{\boldsymbol{\psi}})\\
&-\alpha\int_0^T\int_{\hat{\Omega}_b}\widehat{\mathcal{J}}_b^{\eta^\delta}\,\widehat{p}\,\widehat{\nabla}_b^{\eta^\delta}\cdot\widehat{\boldsymbol{\psi}} - c_0\int_0^T\int_{\hat{\Omega}_b}\widehat{p}\,\partial_t\widehat{r} - \alpha\int_0^T\int_{\hat{\Omega}_b}\widehat{\mathcal{J}}_b^{\eta^\delta}\,\partial_t\widehat{\boldsymbol{\eta}}\cdot\widehat{\nabla}_b^{\eta^\delta}\widehat{r}\\
&-\alpha\int_0^T\int_{\hat{\Gamma}}\left(\widehat{\zeta}\boldsymbol{e}_y\cdot\widehat{\boldsymbol{n}}^{\omega^\delta}\right)\widehat{r} + \kappa\int_0^T\int_{\hat{\Omega}_b}\widehat{\mathcal{J}}_b^{\eta^\delta}\,\widehat{\nabla}_b^{\eta^\delta}\widehat{p}\cdot\widehat{\nabla}_b^{\eta^\delta}\widehat{r} - \int_0^T\int_{\hat{\Gamma}}\left((\widehat{\boldsymbol{u}}-\widehat{\zeta}\boldsymbol{e}_y)\cdot\widehat{\boldsymbol{n}}^\omega\right)\widehat{r}\\
&\qquad= \int_{\Omega_f(0)}\boldsymbol{u}(0)\cdot\boldsymbol{v}(0) + \rho_p\int_{\hat{\Gamma}}\partial_t\widehat{\omega}(0)\cdot\widehat{\varphi}(0) + \rho_b\int_{\hat{\Omega}_b}\partial_t\widehat{\boldsymbol{\eta}}(0)\cdot\widehat{\boldsymbol{\psi}}(0) + c_0\int_{\hat{\Omega}_b}\widehat{p}(0)\cdot\widehat{r}(0).
\end{aligned}\tag{44}$$

For completeness, we list the definitions of all of the relevant expressions below:

$$\widehat{\boldsymbol{w}} = \frac{R+\widehat{y}}{R}\partial_t\widehat{\omega}\boldsymbol{e}_y, \qquad \widehat{\nabla}_f^\omega = \left(\partial_{\hat{x}} - \frac{(R+\widehat{y})\partial_{\hat{x}}\widehat{\omega}}{R+\widehat{\omega}}\partial_{\hat{y}}, \frac{R}{R+\widehat{\omega}}\partial_{\hat{y}}\right), \qquad \widehat{\nabla}_b^{\eta^\delta}\widehat{g} = \left(\frac{\partial\widehat{g}}{\partial\widehat{x}}, \frac{\partial\widehat{g}}{\partial\widehat{y}}\right)\cdot\left(\boldsymbol{I}+\widehat{\nabla}\widehat{\boldsymbol{\eta}}^\delta\right)^{-1}$$

$$\widehat{\mathcal{J}}_b^{\eta^\delta} = \det\left(\boldsymbol{I}+\widehat{\nabla}\widehat{\boldsymbol{\eta}}^\delta\right), \quad \widehat{\mathcal{J}}_\Gamma^\omega = \sqrt{1+|\partial_{\hat{x}}\widehat{\omega}|^2}, \quad \widehat{\boldsymbol{n}}^\omega$$
$$= (-\partial_{\hat{x}}\widehat{\omega}, 1), \quad \widehat{\boldsymbol{\tau}}^\omega \qquad = (1, \partial_{\hat{x}}\widehat{\omega}), \quad \widehat{\boldsymbol{n}}^{\omega^\delta} = \left(-\partial_{\hat{x}}\left((\widehat{\boldsymbol{\eta}}^\delta)_y|_{\hat{\Gamma}}\right), 1\right).$$

4.1. *Formal energy equality for the regularized problem*

In this subsection we show that our regularization is defined in a way to preserve the variational structure of the problem. More precisely, we formally prove that a weak solution to the regularized problem satisfies an energy equality in the same way as for the original problem. To do this, we recall the weak formulation for the regularized problem (44) defined on the fixed reference domain and formally substitute $(\widehat{\boldsymbol{v}}, \widehat{\varphi}, \widehat{\boldsymbol{\psi}}, \widehat{r}) = (\widehat{\boldsymbol{u}}, \partial_t\widehat{\omega}, \partial_t\widehat{\boldsymbol{\eta}}, \widehat{p})$ for the test function. After integration by parts and after using the Reynold's transport theorem, we obtain the following energy equality:

$$E^K(T) + E^E(T) + \int_0^T \left(D_f^V(t) + D_b^V(t) + D_{f_b}^V(t) + D_\beta^V(t)\right) dt = E^K(0) + E^E(0) \tag{45}$$

where

$$E^K(t) = \frac{1}{2}\int_{\Omega_f(t)} |\boldsymbol{u}(t)|^2 + \frac{1}{2}\rho_b \int_{\widehat{\Omega}_b} \left|\partial_t\widehat{\boldsymbol{\eta}}(t)\right|^2 + \frac{1}{2}c_0 \int_{\widehat{\Omega}_b} \left|\widehat{p}(t)\right|^2 + \frac{1}{2}\rho_p \int_{\widehat{\Gamma}} |\partial_t\widehat{\omega}(t)|^2$$

represents the kinetic energy of the fluid, the kinetic energy of the Biot poroviscoelastic matrix motion, the kinetic energy of the filtrating fluid flow in the Biot medium, and the kinetic energy of the plate motion, respectively.

$$E^E(t) = 2\mu_e \int_{\widehat{\Omega}_b} \left|\widehat{\boldsymbol{D}}\left(\widehat{\boldsymbol{\eta}}\right)(t)\right|^2 + 2\lambda_e \int_{\widehat{\Omega}_b} \left|\widehat{\nabla}\cdot\widehat{\boldsymbol{\eta}}(t)\right|^2 + \int_{\widehat{\Gamma}} \left|\widehat{\Delta}\widehat{\omega}(t)\right|^2$$

represents the elastic energy of the Biot poroviscoelastic matrix and the elastic energy of the plate, and

$$D_f^V(t) = 2\nu \int_{\Omega_f(t)} |\boldsymbol{D}(\boldsymbol{u})|^2, \quad D_b^V(t) = 2\mu_v \int_{\widehat{\Omega}_b} \left|\widehat{\boldsymbol{D}}\left(\partial_t\widehat{\boldsymbol{\eta}}\right)\right|^2 + \lambda_v \int_{\widehat{\Omega}_b} \left|\widehat{\nabla}\cdot\partial_t\widehat{\boldsymbol{\eta}}\right|^2,$$

$$D_{f_b}^V(t) = \kappa \int_{\Omega_b^\delta(t)} |\nabla p|^2, \qquad D_\beta^V(t) = \beta \int_{\Gamma(t)} |(\boldsymbol{\xi} - \boldsymbol{u})\cdot\boldsymbol{\tau}|^2$$

represents dissipation due to fluid viscosity, viscosity of the Biot poroviscoelastic matrix, dissipation due to permeability effects, and dissipation due to friction in the Beavers–Joseph–Saffman slip condition. Details of the calculation can be found in [1].

5. Statement of the main result and construction of approximate solutions

To show the existence of a weak solution to the regularized problem, we use a constructive existence proof, which involves a splitting scheme [82]. This is an approach that has been used for constructive existence of weak solutions for a variety of FSI problems, see for example [3–5,27,28]. Here, we design a splitting scheme which consists of two subproblems: one for the fluid-Biot medium, and one for the plate.

We now state the main result.

Theorem 11 (Main Result). *Consider initial data for the plate displacement* $\widehat{\omega}_0 \in H_0^2(\widehat{\Gamma})$, *plate velocity* $\widehat{\zeta}_0 \in L^2(\widehat{\Gamma})$, *Biot displacement* $\widehat{\boldsymbol{\eta}}_0 \in H^1(\widehat{\Omega}_b)$, *Biot velocity* $\widehat{\boldsymbol{\xi}}_0 \in H^1(\widehat{\Omega}_b)$, *Biot pore pressure* $\widehat{p}_0 \in L^2(\widehat{\Omega}_b)$, *and fluid velocity* $\boldsymbol{u}_0 \in H^1(\Omega_f(0))$ *which is divergence-free. Suppose further that* $|\widehat{\omega}_0| \le R_0 < R$ *for some* R_0, $\widehat{\boldsymbol{\eta}}_0|_\Gamma = \widehat{\omega}_0\boldsymbol{e}_y$, *and* $\widehat{\boldsymbol{\xi}}_0|_\Gamma = \widehat{\zeta}_0\boldsymbol{e}_y$, *and for some arbitrary but fixed regularization parameter* $\delta > 0$, *suppose that* $Id + \widehat{\boldsymbol{\eta}}_0^\delta$ *is an invertible map with* $\det(\boldsymbol{I} + \nabla\widehat{\boldsymbol{\eta}}_0^\delta) > 0$. *Then, there exists a weak solution* $(\boldsymbol{u}, \widehat{\omega}, \widehat{\boldsymbol{\eta}}, \widehat{p})$ *to the regularized FPSI problem with regularization parameter* δ *on some time interval* $[0, T]$, *for some* $T > 0$.

While T in general depends on δ, we will show in [1] that if there exists a smooth solution to the nonregularized FPSI problem, then this time T for the regularized problem is independent of δ. This will allow us to pass to the limit as $\delta \to 0$ and show that weak solutions to the regularized FPSI problems constructed in this manuscript, converge to a smooth solution of the original, nonregularized problem, when a smooth solution to the nonregularized problem exists.

Remark 12. The result above is a local result, since it holds up to some time $T > 0$, which needs to be sufficiently small. However, it is easy to show that this $T > 0$ can be made maximal, in the sense that it holds until the time for which $\mathrm{Id} + \widehat{\boldsymbol{\eta}}^\delta$ fails to be invertible or $|\widehat{\omega}(\cdot, x)| = R$ for some

$x \in \widehat{\Gamma}$ when the reticular plate collides with the boundary. This can be shown using a standard method, see e.g. [88, pg. 397-398], or the proof of [3, Theorem 7.1].

In the rest of this manuscript we present the *main steps of the proof* of Theorem 11. The proof is based on constructing a sequence of approximate solutions and showing that, up to a subsequence, they converge to a weak solution of the regularized FPSI problem. The approximate solutions are constructed by semi-discretizing the problem in time and using a Lie operator splitting scheme [82].

An important notational convention. In the rest of the manuscript we will drop the "hat" notation for the Lagrangian variables and the reference domain. For example, we will use Ω_b instead of $\widehat{\Omega}_b$, and keep $\Omega_b(t)$ for the Eulerian, physical domain. The distinction between the functions defined on these two domains will be clear from the context. Moreover, we will use the following notation for the regularization $(f)^\delta := f^\delta$, where f is a function defined on $\widehat{\Omega}_b$.

5.1. *The splitting scheme*

Let $\Delta t = T/N$. The splitting scheme is given in terms of the fluid velocity, plate displacement and velocity, and Biot poroviscoelastic material displacement and pressure:

$$\left(\boldsymbol{u}_N^{n+\frac{i}{2}}, \omega_N^{n+\frac{i}{2}}, \zeta_N^{n+\frac{i}{2}}, \boldsymbol{\eta}_N^{n+\frac{i}{2}}, p_N^{n+\frac{i}{2}}\right), \qquad \text{for } n = 0, 1, \dots, N \text{ and } i = 0, 1.$$

We semidiscretize the regularized weak formulation (44) on the fixed reference domain using Euler discretization to approximate time derivatives, with the following shorthand notation

$$\dot{f}_N^{n+\frac{i}{2}} = \frac{f_N^{n+\frac{i}{2}} - f_N^{n+\frac{i}{2}-1}}{\Delta t}.$$

The semidiscretized problem is split into two subproblems: a plate subproblem and a fluid-Biot subproblem.

The plate subproblem. Only the plate displacement and velocity $\omega_N^{n+\frac{1}{2}}$ and $\zeta_N^{n+\frac{1}{2}}$ are updated in this step, leaving the remaining variables unchanged:

$$\boldsymbol{u}_N^{n+\frac{1}{2}} = \boldsymbol{u}_N^n, \qquad \boldsymbol{\eta}_N^{n+\frac{1}{2}} = \boldsymbol{\eta}_N^n, \qquad p_N^{n+\frac{1}{2}} = p_N^n.$$

The weak formulation reads: find $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ and $\zeta_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$, such that

$$\begin{cases} \displaystyle\int_\Gamma \left(\frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t}\right) \cdot \phi = \int_\Gamma \zeta_N^{n+\frac{1}{2}} \cdot \phi, & \text{for all } \phi \in L^2(\Gamma), \\ \displaystyle\rho_p \int_\Gamma \left(\frac{\zeta_N^{n+\frac{1}{2}} - \zeta_N^n}{\Delta t}\right) \cdot \varphi + \int_\Gamma \Delta\omega_N^{n+\frac{1}{2}} \cdot \Delta\varphi = 0, & \text{for all } \varphi \in H_0^2(\Gamma). \end{cases} \tag{46}$$

When $n = 0$, we set $\omega_N^{-\frac{1}{2}} = \omega_0$ and $\zeta_N^0 = \zeta_0$. Recall that $\omega_0 \boldsymbol{e}_y = \boldsymbol{\eta}_0|_\Gamma$ and $\zeta_0 \boldsymbol{e}_y = \boldsymbol{\xi}_0|_\Gamma$.

Using the Lax–Milgram Lemma one can show that this problem *has a unique solution* (see [1] for details), and it satisfies the following *energy equality*:

$$\frac{1}{2}\rho_p \int_\Gamma \left|\zeta_N^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\rho_p \int_\Gamma \left|\zeta_N^{n+\frac{1}{2}} - \zeta_N^n\right|^2 + \frac{1}{2}\int_\Gamma \left|\Delta\omega_N^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\int_\Gamma \left|\Delta\left(\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}\right)\right|^2$$
$$= \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^n|^2 + \frac{1}{2}\int_\Gamma \left|\Delta\omega_N^{n-\frac{1}{2}}\right|^2. \tag{47}$$

The fluid and Biot subproblem. For the fluid and Biot subproblem, we update the quantities related to the fluid and the Biot medium. Due to the kinematic coupling between the Biot medium displacement and the plate displacement, we must also update the plate velocity, as the dynamics of the Biot medium affect the kinematics of the plate. In this step, only the plate displacement remains unchanged:

$$\omega_N^{n+1} = \omega_N^{n+\frac{1}{2}}.$$

To state the weak formulation of the fluid and Biot subproblem, we define the solution and test spaces, respectively:

$$\mathcal{V}_N^{n+1} = \left\{ \left(\boldsymbol{u}, \zeta, \boldsymbol{\eta}, p \right) \in \mathcal{V}_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p \right\}, \tag{48}$$

$$\mathcal{Q}_N^{n+1} = \left\{ \left(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r \right) \in V_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma \right\}, \tag{49}$$

where V_f^{ω}, V_d, and V_p are defined in (31), (34), and (36).

The weak formulation now reads: find $(\boldsymbol{u}_N^{n+1}, \zeta_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_N^{n+1}$ defined on the reference domain, such that for all test functions $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{Q}_N^{n+1}$ defined on the reference domain, the following holds:

$$\begin{aligned}
&\int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \dot{\boldsymbol{u}}_N^{n+1} \cdot \boldsymbol{v} + 2\nu \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{D}_f^{\omega_N^n}\left(\boldsymbol{u}_N^{n+1}\right) : \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + \int_\Gamma \left(\frac{1}{2} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{u}_N^n - p_N^{n+1}\right) (\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{n}^{\omega_N^n} \\
&+ \frac{1}{2} \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left[\left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{v} \right) \cdot \boldsymbol{u}_N^{n+1} \right] \\
&+ \frac{1}{2R} \int_{\Omega_f} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{v} + \frac{1}{2} \int_\Gamma \left(\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1} \right) \cdot \boldsymbol{n}^{\omega_N^n} \left(\boldsymbol{u}_N^n \cdot \boldsymbol{v} \right) \\
&+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} \int_\Gamma \left(\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1} \right) \cdot \boldsymbol{\tau}^{\omega_N^n} \left(\boldsymbol{\psi} - \boldsymbol{v} \right) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} \left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n}{\Delta t} \right) \cdot \boldsymbol{\psi} \\
&+ \rho_p \int_\Gamma \left(\frac{\zeta_N^{n+1} - \zeta_N^{n+\frac{1}{2}}}{\Delta t} \right) \varphi + 2\mu_e \int_{\Omega_b} \boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+1}\right) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_e \int_{\Omega_b} \left(\nabla \cdot \boldsymbol{\eta}_N^{n+1} \right) (\nabla \cdot \boldsymbol{\psi}) \\
&+ 2\mu_v \int_{\Omega_b} \boldsymbol{D}\left(\dot{\boldsymbol{\eta}}_N^{n+1}\right) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_{\Omega_b} \left(\nabla \cdot \dot{\boldsymbol{\eta}}_N^{n+1} \right) (\nabla \cdot \boldsymbol{\psi}) - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} \\
&+ c_0 \int_{\Omega_b} \frac{p_N^{n+1} - p_N^n}{\Delta t} r - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \dot{\boldsymbol{\eta}}_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha \int_\Gamma \left(\dot{\boldsymbol{\eta}}_N^{n+1} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta} \right) r \\
&+ \kappa \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \int_\Gamma \left[\left(\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1} \right) \cdot \boldsymbol{n}^{\omega_N^n} \right] r = 0,
\end{aligned} \tag{50}$$

and

$$\int_\Gamma \left(\frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t} \right) \cdot \boldsymbol{\phi} = \int_\Gamma \zeta_N^{n+1} \boldsymbol{e}_y \cdot \boldsymbol{\phi}, \quad \text{for all } \boldsymbol{\phi} \in L^2(\Gamma). \tag{51}$$

The proof of the *existence of a unique solution* for this subproblem can be obtained under the following two assumptions:

(1) ASSUMPTION 1A: *Boundedness of the plate displacement away from R.* There exists a positive constant R_{max} such that

$$\left| \omega_N^{k+\frac{i}{2}} \right| \le R_{max} < R, \quad \text{for all } k = 0, 1, \ldots, n \text{ and } i = 0, 1. \tag{52}$$

(2) ASSUMPTION 2A: *Invertibility of the map from fixed to moving Biot domain.* The map

$$\text{Id} + \left(\boldsymbol{\eta}_N^n\right)^\delta : \Omega_b \to (\Omega_b)_N^{n,\delta} \quad \text{is invertible,} \tag{53}$$

where we define $(\Omega_b)_N^{n,\delta}$ to be the image of Ω_b under the map $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta$.

The proof is based on using the Lax–Milgram Lemma. However, in this case the proof is more involved for two reasons. First, the bilinear form associated with problem (50) and (51) is not coercive on the Hilbert space $\mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$, because of a mismatch between the hyperbolic and parabolic scaling in the problem. The second reason is that it is not a priori clear that Korn's inequality, which is needed in the proof of the existence, holds for the Biot domain. To deal with the first difficulty and recover the coercive structure of the problem, the test functions can be rescaled so that

$$\boldsymbol{v} \to (\Delta t)\boldsymbol{v}, \qquad r \to (\Delta t) r.$$

This scaling of the test functions is valid because if $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, v) \in \mathscr{Q}_N^{n+1}$, then the rescaled test function satisfies $((\Delta t)^{-1}\boldsymbol{v}, \varphi, \boldsymbol{\psi}, (\Delta t)^{-1} r) \in \mathscr{Q}_N^{n+1}$ also. To deal with the second difficulty, one can show by explicit calculation that the following Korn's inequality holds for this problem:

Proposition 13 (Korn's inequality for the Biot poroviscoelastic domain). *For all* $\boldsymbol{\eta} \in V_d$,

$$\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 \ge \frac{1}{2}\int_{\Omega_b} |\nabla \boldsymbol{\eta}|^2.$$

Details of the proof can be found in [1].

The Lax Milgram Lemma implies the existence of a unique solution $(\boldsymbol{u}_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$ to the weak formulation (50), (51), which satisfies the following *energy equality*:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^{n+1}}{R}\right)|\boldsymbol{u}_N^{n+1}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+1}|^2 + \mu_e\int_{\Omega_b}\left|\boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+1}\right)\right|^2 \\
&+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot\boldsymbol{\eta}_N^{n+1}\right|^2 + \frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+1}\right|^2 + 2\mu_v(\Delta t)\int_{\Omega_b}\left|\boldsymbol{D}\left(\dot{\boldsymbol{\eta}}_N^{n+1}\right)\right|^2 + \lambda_v(\Delta t)\int_{\Omega_b}\left|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}\right|^2 \\
&+\kappa(\Delta t)\int_{\Omega_b}\mathscr{J}_b^{(\eta_N^n)^\delta}\left|\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}\right|^2 + \frac{\beta(\Delta t)}{\mathscr{J}_\Gamma^{\omega_N^n}}\int_\Gamma\left|\left(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{\tau}^{\omega_N^n}\right|^2 \\
&+\frac{1}{2}\rho_b\int_{\Omega_b}\left|\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n\right|^2 + \frac{1}{2}c_0\int_{\Omega_b}\left|p_N^{n+1}-p_N^n\right|^2 \\
&+\mu_e\int_{\Omega_b}\left|\boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n\right)\right|^2 + \frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot\left(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n\right)\right|^2 \\
&= \frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)|\boldsymbol{u}_N^n|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^n|^2 + \mu_e\int_{\Omega_b}\left|\boldsymbol{D}\left(\boldsymbol{\eta}_N^n\right)\right|^2 \\
&+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot\boldsymbol{\eta}_N^n\right|^2 + \frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+\frac{1}{2}}\right|^2.
\end{aligned}$$

The monolithic semidiscrete problem and uniform energy estimates. To obtain uniform energy estimates for approximate solutions of our semidiscretized scheme it is useful to present the scheme in monolithic form:

$$\begin{aligned}
&\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\dot{\boldsymbol{u}}_N^{n+1}\cdot\boldsymbol{v} + 2\nu\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\boldsymbol{D}_f^{\omega_N^n}\left(\boldsymbol{u}_N^{n+1}\right):\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + \int_\Gamma\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{u}_N^n - p_N^{n+1}\right)(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{n}^{\omega_N^n} \\
&+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left[\left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{v}\right)\cdot\boldsymbol{u}_N^{n+1}\right] \\
&+\frac{1}{2R}\int_{\Omega_f}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{v} + \frac{1}{2}\int_\Gamma\left(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1}\right)\cdot\boldsymbol{n}^{\omega_N^n}\left(\boldsymbol{u}_N^n\cdot\boldsymbol{v}\right) + \frac{\beta}{\mathscr{J}_\Gamma^{\omega_N^n}}\int_\Gamma\left(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{\tau}^{\omega_N^n} \\
&+\rho_b\int_{\Omega_b}\left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n}{\Delta t}\right)\cdot\boldsymbol{\psi} + \rho_p\int_\Gamma\left(\frac{\zeta_N^{n+1}-\zeta_N^n}{\Delta t}\right)\varphi + 2\mu_e\int_{\Omega_b}\boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+1}\right):\boldsymbol{D}(\boldsymbol{\psi}) + \lambda_e\int_{\Omega_b}\left(\nabla\cdot\boldsymbol{\eta}_N^{n+1}\right)(\nabla\cdot\boldsymbol{\psi}) \\
&+2\mu_v\int_{\Omega_b}\boldsymbol{D}\left(\dot{\boldsymbol{\eta}}_N^{n+1}\right):\boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v\int_{\Omega_b}\left(\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}\right)(\nabla\cdot\boldsymbol{\psi})
\end{aligned}$$

$$
-\alpha\int_{\Omega_b}\mathscr{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}+c_0\int_{\Omega_b}\frac{p_N^{n+1}-p_N^n}{\Delta t}r
$$

$$
-\alpha\int_{\Omega_b}\mathscr{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r-\alpha\int_\Gamma\left(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta}\right)r+\kappa\int_{\Omega_b}\mathscr{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r
$$

$$
-\int_\Gamma\left[\left(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1}\right)\cdot\boldsymbol{n}^{\omega_N^n}\right]r+\int_\Gamma\Delta\omega_N^{n+\frac{1}{2}}\cdot\Delta\varphi=0,\ \forall\,(\boldsymbol{v},\varphi,\boldsymbol{\psi},r)\in\mathscr{Q}_N^{n+1}, \tag{54}
$$

$$
\int_\Gamma\left(\frac{\omega_N^{n+\frac{1}{2}}-\omega_N^{n-\frac{1}{2}}}{\Delta t}\right)\phi=\int_\Gamma\zeta_N^{n+\frac{1}{2}}\phi,\qquad\int_\Gamma\left(\frac{\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n}{\Delta t}\right)\cdot\boldsymbol{\phi}=\int_\Gamma\zeta_N^{n+1}\boldsymbol{e}_y\cdot\boldsymbol{\phi},\ \forall\,\phi,\boldsymbol{\phi}\in L^2(\Gamma). \tag{55}
$$

This formulation implies uniform energy estimates for the following discrete energy and discrete dissipation:

$$
\begin{aligned}
E_N^{n+\frac{i}{2}}&=\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{u}_N^{n+\frac{i}{2}}\right|^2+\frac{1}{2}\rho_b\int_{\Omega_b}\left|\dot{\boldsymbol{\eta}}_N^{n+\frac{i}{2}}\right|^2+\frac{1}{2}c_0\int_{\Omega_b}\left|p_N^{n+\frac{i}{2}}\right|^2+\mu_e\int_{\Omega_b}\left|\boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+\frac{i}{2}}\right)\right|^2,\\
&\quad+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot\boldsymbol{\eta}_N^{n+\frac{i}{2}}\right|^2+\frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+\frac{i}{2}}\right|^2+\frac{1}{2}\int_\Gamma\left|\Delta\omega_N^{n+\frac{i}{2}}\right|^2,\ i=0,1.\\
D_N^{n+1}&=2\nu(\Delta t)\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{D}_f^{\omega_N^n}\left(\boldsymbol{u}_N^{n+1}\right)\right|^2+2\mu_v(\Delta t)\int_{\Omega_b}\left|\boldsymbol{D}\left(\dot{\boldsymbol{\eta}}_N^{n+1}\right)\right|^2+\lambda_v(\Delta t)\int_{\Omega_b}\left|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}\right|^2\\
&\quad+\kappa(\Delta t)\int_{\Omega_b}\mathscr{J}_b^{(\eta_N^n)^\delta}\left|\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\right|^2+\frac{\beta(\Delta t)}{\mathscr{J}_\Gamma^{\omega_N^n}}\int_\Gamma\left|\left(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{\tau}^{\omega_N^n}\right|^2.
\end{aligned} \tag{56}
$$

The following *discrete energy equalities* hold:

$$
E_N^{n+\frac{1}{2}}+\frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+\frac{1}{2}}-\zeta_N^n\right|^2+\frac{1}{2}\int_\Gamma\left|\Delta\left(\omega_N^{n+\frac{1}{2}}-\omega_N^{n-\frac{1}{2}}\right)\right|^2=E_N^n \tag{57}
$$

$$
\begin{aligned}
&E_N^{n+1}+D_N^{n+1}+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n\right|^2+\frac{1}{2}\rho_b\int_{\Omega_b}\left|\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n\right|^2+\frac{1}{2}c_0\int_{\Omega_b}\left|p_N^{n+1}-p_N^n\right|^2\\
&\quad+\mu_e\int_{\Omega_b}\left|\boldsymbol{D}\left(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n\right)\right|^2+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot\left(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n\right)\right|^2+\frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+1}-\zeta_N^{n+\frac{1}{2}}\right|^2=E_N^{n+\frac{1}{2}}.
\end{aligned} \tag{58}
$$

We remark that the terms not included in the definition of $E_N^{n+\frac{i}{2}}$ and D_N^{n+1}, appearing in (57) and (58), are numerical dissipation terms.

These energy identities immediately imply that $E_N^{n+\frac{i}{2}}$ and $\sum_{n=1}^N D_N^n$ are *uniformly bounded* by a constant C independent of n and N.

The semidiscretized splitting scheme defines semidiscretized approximations of the solution to the regularized problem at discrete time points. To work with approximate functions and show that they converge to the solution of the continuous problem, we need to extend the semidiscrete approximations to the entire time interval and investigate uniform boundedness of those approximate solution functions. This is done next.

6. Uniform bounds on approximate solutions to the regularized problem

We introduce two extensions to the whole time interval $[0,T]$ of semidiscrete approximate solutions constructed in the previous sections. One is the following piecewise constant extension:

$$
\boldsymbol{u}_N(t)=\boldsymbol{u}_N^n,\quad\boldsymbol{\eta}_N(t)=\boldsymbol{\eta}_N^n,\quad p_N(t)=p_N^n,\quad\omega_N(t)=\omega_N^{n-\frac{1}{2}},\quad\zeta_N(t)=\zeta_N^{n-\frac{1}{2}},\quad\zeta_N^*(t)=\zeta_N^n,
$$
$$
\text{for }(n-1)\Delta t<t\le n\Delta t,
$$

and the other one is a linear interpolation that will be used to estimate time derivatives for the compactness arguments later in the paper:

$$\overline{\boldsymbol{\eta}}_N(n\Delta t) = \boldsymbol{\eta}_N^n, \qquad \overline{p}_N(n\Delta t) = p_N^n, \qquad \overline{\omega}_N(n\Delta t) = \omega_N^{n-\frac{1}{2}}, \quad \text{for } n = 0, 1, \dots, N,$$

where we formally set $\omega_N^{-\frac{1}{2}} = \omega_0$. Note that $\partial_t \overline{\omega}_N = \zeta_N$ and $\partial_t \overline{\boldsymbol{\eta}}_N|_\Gamma = \zeta_N^* \boldsymbol{e}_y$.

From the preceding energy estimates, the following lemma stating uniform boundedness holds.

Lemma 14 (Uniform boundedness of approximate solutions). *Assume:*

(1) ASSUMPTION 1B: UNIFORM BOUNDEDNESS OF PLATE DISPLACEMENTS. *There exists a positive constant R_{max} such that for all N,*

$$\left|\omega_N^{n-\frac{1}{2}}\right| \le R_{max} < R, \quad \textit{for all } n = 0, 1, \dots, N, \tag{59}$$

$$\left|(\boldsymbol{\eta}_N^n)^\delta|_\Gamma\right| \le R_{max} < R, \quad \textit{for all } n = 0, 1, \dots, N. \tag{60}$$

(2) ASSUMPTION 2B: UNIFORM INVERTIBILITY OF THE ALE MAPPING (JACOBIAN). *There exists a positive constant c_0 such that for all N,*

$$\det\left(\boldsymbol{I} + \nabla\left(\boldsymbol{\eta}_N^n\right)^\delta\right) \ge c_0 > 0, \quad \textit{for all } n = 0, 1, \dots, N. \tag{61}$$

(3) ASSUMPTION 2C: UNIFORM BOUNDEDNESS OF THE ALE MAPPING (MATRIX NORM). *There exists positive constants c_1 and c_2 such that for all N,*

$$\left|\left(\boldsymbol{I} + \nabla\left(\boldsymbol{\eta}_N^n\right)^\delta\right)^{-1}\right| \le c_1, \qquad \left|\boldsymbol{I} + \nabla\left(\boldsymbol{\eta}_N^n\right)^\delta\right| \le c_2, \quad \textit{for all } n = 0, 1, \dots, N. \tag{62}$$

Then for all N:

- $\boldsymbol{u}_N$ *is uniformly bounded in* $L^\infty(0, T; L^2(\Omega_f))$ *and* $L^2(0, T; H^1(\Omega_f))$.
- $\boldsymbol{\eta}_N$ *is uniformly bounded in* $L^\infty(0, T; H^1(\Omega_b))$.
- p_N *is uniformly bounded in* $L^\infty(0, T; L^2(\Omega_b))$ *and* $L^2(0, T; H^1(\Omega_b))$.
- ω_N *is uniformly bounded in* $L^\infty(0, T; H_0^2(\Gamma))$.

In addition, we have the following estimates on the linear interpolations.

- $\overline{\boldsymbol{\eta}}_N$ *is uniformly bounded in* $W^{1,\infty}(0, T; L^2(\Omega_b))$.
- $\overline{\boldsymbol{\omega}}_N$ *is uniformly bounded in* $W^{1,\infty}(0, T; L^2(\Gamma))$.

Remark 15 (A crucial remark about invertibility). At first, it would appear that to show the uniform boundedness results above, we also need to have a fourth assumption, which is Assumption 2A (53) from before, that the map $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta : \Omega_b \to \mathbb{R}^2$ is injective (and is hence a bijection onto its image), for each $n = 0, 1, \dots, N$ and for all N. However, this is implied by an injectivity theorem, see Ciarlet [85, Theorem 5-5-2]. Note also that Assumption 1A (52) from before is automatically satisfied once we verify Assumption 1B (59), (60). In particular, this injectivity theorem is as follows. Since $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) > 0$ by Assumption 2B (61), it suffices to show that $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta = \boldsymbol{\varphi}_0$ on $\partial\Omega_b$, for some injective mapping $\boldsymbol{\varphi}_0 : \overline{\Omega_b} \to \mathbb{R}^2$, for example a standard ALE mapping $\boldsymbol{\varphi}_0(x, y) = \left(x, y + \left(1 - \frac{y}{R}\right)\omega\right)$ can be used. This implies the very useful fact that $(\text{Id} + (\boldsymbol{\eta}_N^n)^\delta)(\overline{\Omega_b}) = \boldsymbol{\varphi}_0(\overline{\Omega_b})$, which means that the *deformed configuration is fully determined by the behavior on the boundary.*

The results of Lemma 14 follows from the uniform energy estimates. More details can be found in [1].

An immediate consequence of the uniform boundedness result is the following proposition.

Proposition 16. *Assume that the three assumptions listed in Lemma 14 hold. Then, there exists a subsequence such that the following weak convergence results hold:*

- $\boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ *weakly* in* $L^\infty(0,T;L^2(\Omega_f))$, $\quad \boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ *weakly in* $L^2(0,T;H^1(\Omega_f))$,
- $\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta}$ *weakly* in* $L^\infty(0,T;H^1(\Omega_b))$, $\quad \overline{\boldsymbol{\eta}}_N \rightharpoonup \overline{\boldsymbol{\eta}}$ *weakly* in* $W^{1,\infty}(0,T;L^2(\Omega_b))$,
- $p_N \rightharpoonup p$ *weakly* in* $L^\infty(0,T;L^2(\Omega_b))$, $\quad p_N \rightharpoonup p$ *weakly in* $L^2(0,T;H^1(\Omega_b))$,
- $\omega_N \rightharpoonup \omega$ *weakly* in* $L^\infty(0,T;H_0^2(\Gamma))$, $\quad \overline{\omega}_N \rightharpoonup \overline{\omega}$ *weakly* in* $W^{1,\infty}(0,T;L^2(\Gamma))$.

Furthermore, $\boldsymbol{\eta} = \overline{\boldsymbol{\eta}}$ *and* $\omega = \overline{\omega}$.

To use these results and to be able to construct approximate solutions, we must verify that the assumptions from Lemma 14 hold. This is given by the following lemma.

Lemma 17. *Suppose that the initial data satisfies* $|\omega_0| \le R_0 < R$ *for some* R_0*, and suppose that* $\boldsymbol{\eta}_0$ *has the property that* $Id + (\boldsymbol{\eta}_0)^\delta$ *is invertible with* $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) \ge c_0 > 0$ *on* Ω_b *for some positive constant* c_0*. Then,* there exists a sufficiently small time $T > 0$ *such that for all N, all three assumptions in Lemma 14 hold and the splitting scheme is well defined until time T.*

Proof. First, notice that the assumptions on the initial data immediately imply that the three assumptions from Lemma 14 hold for the initial data, i.e., for $n = 0$.

Next, we want to choose an appropriate time T such that the three assumptions hold uniformly for all N and $n\Delta t$ up to time T. To do this, note that the energy estimates imply

$$E_N^{k+\frac{1}{2}} \le E_0, \qquad E_N^{k+1} \le E_0,$$

and, after completing both subproblems of the scheme on the time step $[k\Delta t, (k+1)\Delta t]$, it follows that

$$\|\dot{\boldsymbol{\eta}}_N^n\|_{L^2(\Omega_b)} \le C, \qquad \text{for } n = 0,1,\dots,k+1, \tag{63}$$

$$\left\|\omega_N^{n+\frac{1}{2}}\right\|_{H_0^2(\Gamma)} \le C, \qquad \text{for } n = 0,1,\dots,k, \tag{64}$$

$$\left\|\zeta_N^{n+\frac{i}{2}}\right\|_{L^2(\Gamma)} \le C, \qquad \text{for } 0 \le n + \frac{i}{2} \le k+1 \quad \text{and } i = 0,1, \tag{65}$$

for a constant C that is independent of k and N, since C depends only on the initial energy E_0.

The proof of (59)*:* Suppose that the linear interpolation $\overline{\omega}_N$ is defined up to time $(k+1)\Delta t$. Then, by (64) and (65), it satisfies $\|\overline{\omega}_N\|_{W^{1,\infty}(0,(k+1)\Delta t;L^2(\Gamma))} \le C$ and $\|\overline{\omega}_N\|_{L^\infty(0,(k+1)\Delta t;H_0^2(\Gamma))} \le C$. Thus, following the method in [3], we obtain by an interpolation inequality that for all $t, t+\tau \in [0,(k+1)\Delta t]$ with $\tau > 0$,

$$\|\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)\|_{H^1(\Gamma)} \le C\|\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)\|_{L^2(\Gamma)}^{1/2}\|\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)\|_{H^2(\Gamma)}^{1/2}. \tag{66}$$

Here, we used a Sobolev interpolation inequality, see for example [89, Theorem 4.17 (pg. 79)]. By the Lipschitz continuity of $\overline{\omega}_N$ taking values in $L^2(\Gamma)$ and by the boundedness of $\overline{\omega}_N$ in $H_0^2(\Gamma)$, we get $\|\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)\|_{H^1(\Gamma)} \le C \cdot \tau^{1/2}$, for a constant C depending only on E_0 (and in particular, not depending on k and N). Therefore, setting $t = 0$ and $\tau = (k+1)\Delta t$ and using the continuous embedding of $H^1(\Gamma)$ into $C(\Gamma)$,

$$\left\|\omega_N^{k+1} - \omega_0\right\|_{C(\Gamma)} \le C \cdot [(k+1)t]^{1/2} \le C \cdot T^{1/2}, \tag{67}$$

for a constant C that is independent of k and N. Because $|\omega_0| < R$, we can choose $T > 0$ sufficiently small so that

$$C \cdot T^{1/2} < R - \|\omega_0\|_{C(\Gamma)}. \tag{68}$$

This gives (59).

The proof of (60), (61), *and* (62). Here, we want to control the behavior of $\boldsymbol{\eta}$. To do this, first note that $\|\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0\|_{L^2(\Omega_b)} \le (\Delta t)\sum_{n=1}^{k+1} \|\dot{\boldsymbol{\eta}}_N^n\|_{L^2(\Omega_b)} \le (k+1)(\Delta t)C \le CT$, for a constant C depending only on E_0. By regularization, we then have that for a constant depending only on δ and E_0,

$$\left\| \left(\boldsymbol{\eta}_N^{k+1}\right)^{\delta} - (\boldsymbol{\eta}_0)^{\delta} \right\|_{H^3(\Omega_b)} \le CT.$$

By using the trace theorem and the continuous embedding of $H^2(\Gamma)$ into $C(\Gamma)$, and by using the continuous embedding of $H^2(\Omega_b)$ into $C(\Omega_b)$, we obtain

$$\left\| \left(\boldsymbol{\eta}_N^{k+1}\right)^{\delta}|_{\Gamma} - (\boldsymbol{\eta}_0)^{\delta}|_{\Gamma} \right\|_{C(\Gamma)} \le CT \quad \text{and} \quad \left\| \nabla\left(\boldsymbol{\eta}_N^{k+1}\right)^{\delta} - \nabla(\boldsymbol{\eta}_0)^{\delta} \right\|_{C(\Omega_b)} \le CT, \tag{69}$$

where C depends only on δ and E_0. Since $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^{\delta})$, $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^{\delta}|$, and $|(\boldsymbol{I} + (\nabla(\boldsymbol{\eta}_N^n)^{\delta})^{-1}|$ are continuous functions of $\nabla(\boldsymbol{\eta}_N^n)^{\delta}$, from (69), we get that the assumptions (60), (61), and (62) are satisfied, where we can choose the constants c_0, c_1, c_2, and R_{max} (defined in the statement of those assumptions) independently of N and $n = 0, 1, \dots, N$, because of the fact that the constant C in our estimates does not depend on k (satisfying $(k+1)\Delta t \le T$) or N. □

7. Compactness for the regularized problem

Since the problem is nonlinear, to pass to the limit in the weak formulation and show that the approximate solutions converge to a weak solution of the regularized problem, we need stronger convergence than just weak and weak* convergence in Proposition 16. For this reason, we will use compactness arguments. In particular, for functions defined on fixed domains, such as the Biot poroelastic matrix displacement and plate displacement, we use the classical Aubin-Lions compactness theorem. For functions defined on moving domains, such as the fluid velocity, we use the generalized Aubin–Lions compactness theorem presented in [90], see also [3]. The reason we must use a *generalized Aubin–Lions compactness theorem* is that the approximate fluid velocities are defined on different time-dependent fluid domains. To be able to compare functions defined on different domains we introduce a maximal domain Ω_f^M containing all of the physical approximate fluid domains $\Omega_{f,N}^n$:

$$\Omega_f^M = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le L, -R \le y \le M(x)\right\},$$

where $M(x)$ is an upper bound for the location of the interface displacement, defined by

$$\omega_N^n(x) \le M(x), \qquad \text{for all } x \in [0,L], n = 0, 1, \dots, N, \text{ and for all } N. \tag{70}$$

The existence of such a maximal domain Ω_f^M, namely the existence of $M(x)$, follows by arguments similar to [91, Lemma 2.5] and [90, Lemma 4.5], established in the context of incompressible FSI. Once the maximal fluid domain is defined, we can extend the fluid velocities $\boldsymbol{u}_N^n$ from $\Omega_{f,N}^n$ to this common maximal domain Ω_f^M, using extensions by zero in $\Omega_f^M \cap (\Omega_{f,N}^n)^c$. Notice that since $\omega_N^n(x)$ are all uniformly Lipschitz, the *extensions by zero* of the H^1 functions $\boldsymbol{u}_N^n$ defined on Lipschitz domains to Ω_f^M are uniformly bounded in $H^s(\Omega_f^M)$ for all s such that $0 < s < 1/2$. The strong convergence result below is stated in terms of the velocity functions defined on the maximal domain Ω_f^M.

Theorem 18. *The following strong convergence results hold:*

(1) The Biot poroviscoelastic matrix displacement: *There exists a subsequence such that* $\overline{\boldsymbol{\eta}}_N \to \boldsymbol{\eta}$ *strongly in* $C(0,T;L^2(\Omega_b))$.
(2) The plate displacement: *Given arbitrary* $0 < s < 2$, *there exists a subsequence such that* $\overline{\omega}_N \to \omega$ *in* $C(0,T;H^s(\Gamma))$, *and* $\omega_N \to \omega$ *in* $L^\infty(0,T;H^s(\Gamma))$.
(3) The Biot poroviscoelastic matrix velocity and plate velocity: *For* $-1/2 < s < 0$, *there exists a subsequence such that* $(\boldsymbol{\xi}_N, \zeta_N) \to (\boldsymbol{\xi}, \zeta)$ *strongly in* $L^2(0,T;H^{-s}(\Omega_b)\times H^{-s}(\Gamma))$.
(4) The pore pressure: *There exists a subsequence such that* $p_N \to p$, *strongly in* $L^2(0,T;L^2(\Omega_b))$.
(5) The free fluid velocity: *Let* Ω_f^M *be a maximal domain containing all of the physical approximate fluid domains* $\Omega_{f,N}^n$, *and let* $\boldsymbol{u}_N$ *be defined on* Ω_f^M *via a zero extension from* $\Omega_{f,N}^n$ *to* Ω_f^M. *Then, there exists a subsequence such that* $\boldsymbol{u}_N \to \boldsymbol{u}$ *in* $L^2(0,T;L^2(\Omega_f^M))$.

Precompactness for the approximate Biot poroviscoelastic medium displacements follows from the compactness of the embedding $W^{1,\infty}(0,T;L^2(\Omega_b)) \cap L^\infty(0,T;H^1(\Omega_b)) \subset\subset C(0,T;L^2(\Omega_b))$, which is proved by a standard Aubin–Lions argument.

Precompactness for the approximate plate displacements follows from the Arzela–Ascoli theorem and the fact that $H^{2\alpha}$ embeds compactly into any $H^{2\alpha-\epsilon}$ for $\epsilon > 0$, once we choose $\alpha \in (0,1)$ and $\epsilon > 0$ appropriately so that $2\alpha - \epsilon = s$ for a given arbitrary $0 < s < 2$.

Precompactness for the approximate Biot velocity and plate velocity follow by using a compactness criterion for piecewise constant functions due to Dreher and Jüngel [92].

Precompactness for the approximate pore pressure follows similarly by using arguments based on the Dreher–Jüngel compactness criterion for piecewise constant functions [92].

Precompactness for the approximate fluid velocity follows from the generalized Aubin–Lions compactness theorem [90] for functions defined on moving domains, except for one additional wrinkle. Namely, since in this work we are considering a moving fluid domain with a moving plate as a boundary, we must modify the arguments presented in [90] by considering extra arguments to handle the additional perturbative terms that appear in this particular FPSI model. Details can be found in [1].

8. Passing to the limit

With the convergence results just established, we pass to the limit in the semidiscrete formulation (54) and (55). There is, however, another difficulty that needs to be overcome, which is related to dealing with the test functions for the fluid velocity. In particular, on the fixed reference fluid domain Ω_f on which (54) and (55) are defined, the test functions for the fluid velocity in $\mathcal{V}_{\text{test}}^{\omega}$, defined by (41), satisfy $\nabla_f^{\omega} \cdot \boldsymbol{v} = 0$ on Ω_f, where ω is the solution for the plate displacement. However, the test functions for the fluid velocity in the semidiscrete formulation belong to the semidiscrete test space $\mathcal{Q}_N^{n+1}$, defined by (49), and satisfy $\nabla^{\omega_N^n} \cdot \boldsymbol{v} = 0$ on Ω_f. Hence, we need a way of comparing the test functions from $\mathcal{Q}_N^{n+1}$ and $\mathcal{V}_{\text{test}}^{\omega}$. To do this, we go back to the physical space and consider the velocity test functions in the *physical space* where the classical divergence free condition holds, and extend the test functions to the maximal domain Ω_f^M in the way specified below. We look at the restrictions of those test function onto the physical fluid domains that are bounded by ω and by ω_N, and define the test functions on the fixed, reference fluid domain to be those restrictions in the physical domain, pulled back to the reference domain via the corresponding ALE mappings. These test functions defined on the reference fluid domain now depend on N. However, for the test functions constructed this way, we can show that they are dense in the original test spaces, and we can get the desired convergence properties as $N \to \infty$ to be able to pass to the limit in the corresponding weak formulation.

More precisely, consider the following *test space* $\mathscr{X}$, which consists of functions $\boldsymbol{v}$ in $C_c^1([0,T) \times \Omega_f^M)$ defined on the physical space, satisfying the following properties for each $t \in [0,T)$:

(1) For each $t \in [0,T)$, $\boldsymbol{v}(t)$ is a smooth vector-valued function on Ω_f^M.
(2) $\nabla \cdot \boldsymbol{v}(t) = 0$ on Ω_f^M for all $t \in [0,T)$.
(3) $\boldsymbol{v}(t) = 0$ on $x = 0$, $x = L$, and $y = -R$, for all $t \in [0,T)$.

Given $\boldsymbol{v} \in \mathscr{X}$, define

$$\widetilde{\boldsymbol{v}} = \boldsymbol{v}|_{\Omega_f^\omega} \circ \boldsymbol{\Phi}_f^\omega \quad \text{and} \quad \widetilde{\boldsymbol{v}}_N = \boldsymbol{v}|_{\Omega_f^{\omega_N}} \circ \boldsymbol{\Phi}_f^{\omega_N}.$$

The test functions $\widetilde{\boldsymbol{v}}$ are dense in the fluid velocity test space $\mathscr{V}_f^\omega$ associated with the fixed domain formulation, and the test functions $\widetilde{\boldsymbol{v}}_N$ restricted to $[n\Delta t, (n+1)\Delta t)$ are dense in $V_f^{\omega_N^n}$, where $V_f^{\omega_N^n}$ is the velocity test space for the semidiscretized problem(s) given in (49). Therefore, for each fixed N, we can consider the semidiscrete formulation with the test function $\widetilde{\boldsymbol{v}}_N$, which we emphasize is discontinuous in time, due to the jumps in ω_N at each $n\Delta t$. To pass to the limit as $N \to \infty$ we can use the same approach as in [3, Lemma 7.1] and [91, Lemma 2.8], to obtain the following strong convergence results of the velocity test functions $\widetilde{\boldsymbol{v}}_N$ and their gradients, which will allow us to pass to the limit in the semidiscrete weak formulations:

Proposition 19. *Consider $\boldsymbol{v} \in \mathscr{X}$. As $N \to \infty$,*

$$\widetilde{\boldsymbol{v}}_N \to \boldsymbol{v}, \qquad \nabla \widetilde{\boldsymbol{v}}_N \to \nabla \boldsymbol{v},$$

pointwise, uniformly on $[0,T] \times \Omega_f$, as $N \to \infty$.

With this construction of velocity test functions defined on approximate domains, we can pass to the limit in the semidiscrete weak formulation (54) and (55) where $\boldsymbol{v}$ is replaced with $\widetilde{\boldsymbol{v}}_N$. By using the strong convergence results from Theorem 18, along with the strong convergence of test functions stated in Proposition 19, and after using discrete integration by parts as in

$$\int_0^T \int_{\Omega_f} \left(1 + \frac{\tau_{\Delta t}\omega_N}{R}\right) \partial_t \overline{\boldsymbol{u}}_N \cdot \widetilde{\boldsymbol{v}}_N \to -\int_0^T \int_{\Omega_f} \left(1 + \frac{\omega}{R}\right) \boldsymbol{u} \cdot \partial_t \widetilde{\boldsymbol{v}} - \frac{1}{R} \int_0^T \int_{\Omega_f} (\partial_t \omega) \boldsymbol{u} \cdot \widetilde{\boldsymbol{v}} - \int_{\Omega_f} \left(1 + \frac{\omega_0}{R}\right) \boldsymbol{u}(0) \cdot \widetilde{\boldsymbol{v}}(0),$$

for $\widetilde{\boldsymbol{v}} = \boldsymbol{v}|_{\Omega_f^\omega} \circ \boldsymbol{\Phi}_f^\omega$, where $\boldsymbol{v} \in \mathscr{X}$ (see, e.g., [91, pg. 79-81]), we can pass to the limit in the semidiscrete weak formulation of the regularized problem. The limiting weak formulation holds for all velocity test functions in the smooth test space, which can be extended to the general test space $\mathscr{V}_{\text{test}}^\omega$ defined in (41) by using a density argument. Therefore, we have shown that the approximate weak solutions converge, up to a subsequence, to a weak solution to the regularized problem, as stated in Theorem 11.

This completes the main result of this manuscript, stated in Theorem 11 providing existence of a weak solution to the nonlinearly coupled, regularized fluid-poroviscoelastic structure interaction problem, given in Definition 10.

We conclude this manuscript by making the important observation that the weak solution constructed here satisfies the following energy estimate.

Proposition 20. *The weak solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ to the regularized problem, constructed above, satisfies the following energy estimate:*

$$\frac{1}{2}\int_{\Omega_f(t)} |\boldsymbol{u}|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}|^2 + \frac{1}{2} c_0 \int_{\Omega_b} |p|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2$$
$$+ \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}|^2 + \frac{1}{2}\rho_p \int_\Gamma |\zeta|^2 + \frac{1}{2}\int_\Gamma |\Delta \omega|^2 + 2\nu \int_0^t \int_{\Omega_f(s)} |\boldsymbol{D}(\boldsymbol{u})|^2$$

$$+2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi})|^2 + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}|^2 + \kappa \int_0^t \int_{\Omega_b^\delta(s)} \left|\nabla p\right|^2 + \beta \int_0^t \int_{\Gamma(s)} \left|\left(\zeta \boldsymbol{e}_y - \boldsymbol{u}\right)\cdot \boldsymbol{\tau}\right)\right|^2 \le E_0, \quad (71)$$

for almost every $t \in [0, T]$.

Proof. The approximate solutions $(\boldsymbol{u}_N, \boldsymbol{\eta}_N, p_N, \omega_N)$ satisfy the following energy inequality:

$$\frac{1}{2}\int_{\Omega_f^{\omega_N}(t)} |\boldsymbol{u}_N|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}_N|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N|^2 + \mu_e \int_{\Omega_b} \left|\boldsymbol{D}(\boldsymbol{\eta}_N)\right|^2$$
$$+ \frac{1}{2}\lambda_e \int_{\Omega_b} \left|\nabla \cdot \boldsymbol{\eta}_N\right|^2 + \frac{1}{2}\rho_p \int_\Gamma |\zeta_N|^2 + \frac{1}{2}\int_\Gamma |\Delta \omega_N|^2 + 2\nu \int_0^t \int_{\Omega_{f,N}(s)} |\boldsymbol{D}(\boldsymbol{u}_N)|^2$$
$$+2\mu_v \int_0^t \int_{\Omega_b} \left|\boldsymbol{D}(\boldsymbol{\xi}_N)\right|^2 + \lambda_v \int_0^t \int_{\Omega_b} \left|\nabla \cdot \boldsymbol{\xi}_N\right|^2 + \kappa \int_0^t \int_{\Omega_{b,N}^\delta(s)} |\nabla p_N|^2 + \beta \int_0^t \int_{\Gamma(s)} \left|\left(\widehat{\zeta}_N \boldsymbol{e}_y - \boldsymbol{u}_N\right)\cdot \boldsymbol{\tau}\right|^2 \le E_0,$$

where E_0 is the initial energy of the problem. We can then use the weak convergences of the approximate solutions, stated in Proposition 16, and weak lower semicontinuity in order to pass to the limit in the energy inequality. □

9. Conclusions and Related Upcoming Work

We provided the main steps in the existence proof of a weak solution to a regularized FPSI problem consisting of multi-layered poroelastic media interacting with the free flow of an incompressible, viscous fluid. The benchmark *nonlinearly coupled* FPSI model that we considered consists of an incompressible, viscous fluid modeled by the 2D Navier–Stokes equations interacting with a multilayered structure consisting of a poroviscoelastic medium modeled by the 2D Biot equations and a reticular plate. The reticular plate that is at the interface between the fluid and the Biot poroviscoelastic medium associates mass (inertia) and elastic energy to the interface, and it is "transparent" to fluid flow. The free fluid flow and the poroviscoelastic medium are coupled via the *moving (plate) interface* through the kinematic and dynamic coupling conditions. Though we have presented the main results for the case of a Biot poroviscoelastic medium, the existence result holds for both the elastic and viscoelastic Biot model.

We emphasize that this existence result is the first such result for a *nonlinearly coupled* FPSI model. The nonlinear coupling over the moving interface presents a major difficulty since both the fluid domain *and* the poroviscoelastic medium domain are functions of time. Because the Biot medium is a 2D thick medium having the same dimension as the fluid domain, there are substantial difficulties in defining the time-dependent moving Biot domain $\Omega_b(t)$. Namely, we would want to define the moving Biot domain $\Omega_b(t)$ as the image of the reference Biot domain $\widehat{\Omega}_b$ under the map $\text{Id} + \widehat{\boldsymbol{\eta}}$, where $\widehat{\boldsymbol{\eta}}$ is the Biot medium displacement, and we would furthermore want this map $\text{Id} + \widehat{\boldsymbol{\eta}}$ to be a bijection between $\widehat{\Omega}_b$ and $\Omega_b(t)$. However, in the finite energy space for the problem, the Biot medium displacement $\widehat{\boldsymbol{\eta}}$ is not even a continuous function on $\widehat{\Omega}_b$, which makes it difficult to properly interpret the moving Biot domain $\Omega_b(t)$. Furthermore, the weak formulation for the original FPSI problem (without regularization) contains integrals over the time-dependent Biot domain $\Omega_b(t)$ which do not necessarily converge for solutions in the finite energy space. This is due to the low regularity of the factor $\widehat{\mathscr{J}}_b^\eta = \det(\boldsymbol{I} + \nabla\widehat{\boldsymbol{\eta}})$ for $\widehat{\boldsymbol{\eta}}$ in the finite energy space appearing in the weak formulation, where the Jacobian $\det(\boldsymbol{I} + \nabla\widehat{\boldsymbol{\eta}})$ arises from a change of variables from $\Omega_b(t)$ to $\widehat{\Omega}_b$.

To address this difficulty, in this manuscript, we considered a regularized problem, which consists of a regularized Biot matrix displacement $\boldsymbol{\eta}^\delta$ via convolution with a smooth kernel, which defines a regularized moving Biot domain $\Omega_b^\delta(t)$. We formulated a weak formulation to the regularized FPSI problem, which differs from the original formulation in the smallest number of terms. In this manuscript, we proved the existence of a weak solution to the regularized problem, which is difficult to study in its own right. The proof was constructive, and was based on the time discretization via operator splitting. Upon fixing an arbitrary regularization parameter $\delta > 0$, we showed that there exists a uniform time T independent of the time step Δt, for which the splitting scheme remains valid up until time T, so that we could construct approximate solutions for the regularized FPSI problem on a uniform time interval $[0, T]$. We used compactness arguments and an analysis of the test space, which depends on the solution itself, in order to pass to the limit in these approximate solutions to obtain a resulting weak solution to the regularized moving boundary FPSI problem.

While the constructive existence proof in this manuscript was carried out in the case of a viscoelastic Biot medium, the existence result continues to hold in the case of a purely elastic Biot medium. The main difference is that rather than having the Biot velocity $\widehat{\boldsymbol{\xi}} \in L^\infty(0,T;L^2(\widehat{\Omega}_b)) \cap L^2(0,T;H^1(\widehat{\Omega}_b))$ as in the case of a poroviscoelastic Biot medium, we instead have $\widehat{\boldsymbol{\xi}} \in L^\infty(0,T;L^2(\widehat{\Omega}_b))$ for a poroelastic Biot medium. Thus, in the case of a poroelastic Biot medium, the Biot velocity $\widehat{\boldsymbol{\xi}}$ does not have a well-defined trace along $\widehat{\Gamma}$ and thus, it no longer makes sense to additionally impose that $\widehat{\boldsymbol{\xi}}|_{\widehat{\Gamma}} = \widehat{\zeta}\boldsymbol{e}_y$ where $\widehat{\zeta}$ is the plate velocity, and we instead just impose continuity of displacements $\widehat{\boldsymbol{\eta}}|_{\widehat{\Gamma}} = \widehat{\omega}\boldsymbol{e}_y$, where $\widehat{\boldsymbol{\eta}}$ is the Biot medium displacement and $\widehat{\omega}$ is the plate displacement. Due to the reticular plate separating the Biot medium and the fluid flow, considering the case of a purely elastic Biot medium does not present additional difficulties in the constructive existence proof described in this manuscript. This is because even though the Biot velocity $\widehat{\boldsymbol{\xi}}$ does not have a well-defined trace along the interface $\widehat{\Gamma}$ in the case of a purely elastic Biot material, the presence of the reticular plate, which has a well-defined velocity $\widehat{\zeta} \in L^\infty(0,T;L^2(\widehat{\Gamma}))$ in the finite energy space, allows us to still be able to interpret any terms in the weak formulation involving the velocity of the interface between the Biot medium and the free fluid flow.

In an associated manuscript [1], we prove that the regularized weak formulation that we have constructed in this manuscript is consistent with the original, nonregularized problem in the case of the viscoelastic Biot model, in the sense that weak solutions of the regularized problem converge (as the regularization parameter $\delta \to 0$) to a classical (smooth) solution of the original nonregularized problem when such a classical solution exists, where this convergence takes place on a uniform time interval that is independent of the regularization parameter δ. We call this property weak-classical consistency, and remark that viscoelasticity of the Biot medium is needed in this weak-classical consistency proof. However, the existence proof presented in this manuscript holds for both the case of a purely elastic and viscoelastic Biot medium.

We conclude this manuscript with a remark that for the regularized moving boundary FPSI problem that we have considered, we are studying the singular limit in which the reticular plate thickness tends to zero [1]. We have preliminary results which indicate that under the assumption of viscoelasticity in the Biot model, the singular limit can be taken, establishing the existence of a weak solution to the corresponding regularized moving boundary fluid-structure interaction problem between the flow of an incompressible, viscous fluid modeled by the Navier–Stokes equations, and a poroviscoelastic structure modeled by the Biot equations, *without* a reticular plate between the Biot medium and the fluid flow [1]. The existence of a weak solution to the regularized, moving-boundary problem studied in *this manuscript*, is a crucial stepping stone for the singular limit proof to be carried out.

Conflicts of interest

The authors declare no competing financial interest.

Dedication

The manuscript was written through contributions of all authors. All authors have given approval to the final version of the manuscript.

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