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ESTIMATE ON LOGARITHMIC COEFFICIENTS OF KAMALI-TYPE STARLIKE FUNCTIONS ASSOCIATED WITH FOUR-LEAF SHAPED DOMAIN

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Abstract. In the present paper, the authors introduce a new subclass namely; $R_L(\phi_{4L}, \nu)$ of Kamali-type starlike functions defined in the open unit disk $\mathbb{D}$ connected with four-leaf shaped domain. We investigate the bounds of some initial coefficients, Fekete-Szegö inequality and other results of logarithmic coefficients for the function belonging to above class. Relevant connections of the results derived in this paper with those of earlier works are indicated.

1 Introduction and Motivation

Let $\mathcal{A}$ represent the family of functions h which are holomorphic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ having normalized by $h(0) = h'(0) - 1 = 0$. Thus, each function $h \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of all functions $h \in \mathcal{A}$ that are univalent in $\mathbb{D}$. For any two functions $h,\ g \in \mathcal{A}$, the function h is said to be subordinate to the function g or g is superordinate to the function h, written as $h \prec g$ if there exists an analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{D})$ such that $h(z) = g(w(z))$.

While proving or disproving Bieberbach conjecture, a number of subfamilies of the class $\mathcal{S}$ of normalized univalent functions connected to different image domain arises. The familiar of subclasses were class of starlike, convex, close to-convex and bounded turning functions. By the above notion of subordination, Ma and Minda

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[14] established subclasses of starlike and convex functions respectively as:

$$\mathcal{S}^*(\phi) = \{h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \phi(z) \quad (z \in \mathbb{D})\} \tag{1.2}$$

and

$$C(\phi) = \{h \in \mathcal{A} : 1 + \frac{zh''(z)}{h'(z)} \prec \phi(z) \quad (z \in \mathbb{D})\} \tag{1.3}$$

where the function ϕ is analytic and normalized by $\phi(0) = 1$ and $\phi'(0) > 0$ with $\Re\{\phi(z)\} > 0$ in $\mathbb{D}$ which maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric along the real axis.

For different choices of the function ϕ, we obtain some subfamilies of the class $\mathcal{S}$ which have significant geometrical interpretation.

- If we choose $\phi(z) = \frac{1+Kz}{1+Lz}$ $(-1 \le L < K \le 1)$, then $\mathcal{S}^*[K,L] = \mathcal{S}^*\left(\frac{1+Kz}{1+Lz}\right)$ is the family of Janowski starlike functions [9]. Further, by taking $K = 1 - 2\beta$ $(0 \le \beta < 1)$ and $L = -1$, we get the class $\mathcal{S}^*(\beta) = \mathcal{S}^*(1-2\beta, -1)$ of starlike functions of order β. Clearly, $\mathcal{S}^*(0) = \mathcal{S}^*$ represents well-known class of starlike functions.

- $\mathcal{S}^*_L = \mathcal{S}^*(\phi(z)) = \mathcal{S}^*(\sqrt{1+z})$ was investigated by Sokól and Stankiewicz [31]. The function $\phi(z) = \sqrt{1+z}$ maps the region $\mathbb{D}$ onto the image domain which is bounded by the right half of the Bernoulli lemniscate given by $|w^2 - 1| < 1$.

- If we choose $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, then the class $\mathcal{S}^*(\phi(z))$ reduces the class $\mathcal{S}^*_{car}$ studied by Sharma et al. [26] (also see [37]). It consists of functions $f \in \mathcal{A}$ in such a way that $\frac{zh'(z)}{h(z)}$ lies in the region bounded by cardioid given by $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$.

- By taking $\phi(z) = 1 + sinz$, the class $\mathcal{S}^*(\phi(z)) = \mathcal{S}^*_{sin}$ was introduced by Cho et al. [5].

- Taking $\phi(z) = e^z$, the class $\mathcal{S}^*(e^z) = \mathcal{S}^*_e$ was studied in [15](also see [27]). This function was recently generalized by Srivastava et al. [33].

- Choosing $\phi(z) = 1 + tanhz$, the class $\mathcal{S}^*(\phi(z)) = \mathcal{S}^*_{tanh}$ studied in [41].
  By taking some more particular function $\phi(z)$ in $\mathcal{S}^*(\phi(z))$ we get several subclasses of starlike functions [2, 12, 20].

For each function $h(z) \in \mathcal{S}$, we can define logarithmic function $H_h(z)$ as:

$$H_h(z) = log\frac{h(z)}{z} = 2\sum_{n=1}^{\infty} d_n z^n \quad (z \in \mathbb{D}). \tag{1.4}$$

The numbers d_n are called logarithmic coefficients of h. The logarithmic coefficients are very essential in the problems of univalent functions coefficient. The importance of the logarithmic coefficients of h can transfer to the Taylor-Maclaurin coefficient of univalent function themselves or to their powers via the Lebedev-Milin inequalities. Little exact information is known about the behavior of logarithmic coefficient of h. For instance; the logarithmic coefficient for well-known Koebe function $K(z) = \frac{z}{(1-z)^2}$ are $d_n = \frac{1}{n}$. Because of the extreme properties of the Koebe function, we may expect that $|d_n| \leq \frac{1}{n}$ for each $h \in \mathcal{S}$. However, this conjecture is false even in the case of $n = 2$. For the class $\mathcal{S}$, the sharp estimates of single logarithmic coefficients are known only for $|d_1| \leq 1, \quad |d_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0.6353...$ and for $n \geq 3$ is unknown. Recently, the bounds of logarithmic coefficients of functions in some importance subclasses of $\mathcal{S}$ have been studied by various authors. For recent expository works, see [1, 4, 6, 8, 19, 25, 42].

For a given function $h \in \mathcal{A}$ with the representation of the form (1.1), Pommerenke [21, 22] defined the Hankel determinant $H_{q,n}(h)$ with q as parameter and $n \in \mathbb{N} := \{1, 2, 3, ...\}$ as

$$H_{q,n}(h) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (n, q \in \mathbb{N} = 1, 2, 3...) \,,$$

where $a_1 = 1$. The problem of evaluating the upper bound of $H_{q,n}(h)$ over various subfamilies of $\mathcal{A}$ is an interesting area of research in the field of Geometric Function Theory of Complex Analysis. For fixed values of q and n, the growth rate of $H_{q,n}(h)$ as $n \longrightarrow \infty$ was studied by Noonan and Thomas [17] and Noor [18] for different subfamilies of the univalent function class $\mathcal{S}$. The Hankel determinant $H_{2,1}(h) = a_3 - a_2^2$ and $H_{2,2}(h) = a_2a_4 - a_3^2$ are well-known as Fekete-Szegö functional and second Hankel determinant respectively. The functional $H_{2,1}(h)$ is further generalized as $a_3 - \mu a_2^2$ for some real or complex parameter μ. Finding the upper bounds of $H_{2,1}(h)$ for various subfamilies of class $\mathcal{S}$ were obtained by different researchers (see [32, 34, 36]). Very recently, Srivastava et al. [39] obtained the bounds of second Hankel determinant for q- starlike and q-convex functions. Further, the bounds of some initial coefficients, the Fekete-Szegö inequality and the estimation of Hankel determinant of different orders for various subclasses of univalent and bi-univalent functions are obtained in [16, 28, 30, 35, 38, 40].

From the relation (1.4), the logarithmic coefficients $d_n \quad (n = 1, 2, 3, 4)$ for $h \in \mathcal{S}$ are given by,

$$d_1 = \frac{1}{2}a_2 \tag{1.5}$$

$$d_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2) \tag{1.6}$$

$$d_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3) \tag{1.7}$$

and

$$d_4 = \frac{1}{2}(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4). \tag{1.8}$$

In recent years, Kowalczyk and Lecko [10, 11] studied the Hankel determinant $H_{q,n}(H_h/2)$ whose elements are logarithmic coefficients of h as:

$$H_{q,n}(H_h/2) = \begin{vmatrix} d_n & d_{n+1} & \cdots & d_{n+q-1} \\ d_{n+1} & d_{n+2} & \cdots & d_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n+q-1} & d_{n+q} & \cdots & d_{n+2(q-1)} \end{vmatrix} \quad (n, q \in \mathbb{N} = 1, 2, 3...)\,.$$

It may be noted that;

$$H_{2,1}(H_h/2) = d_1d_3 - d_2^2$$

corresponds to the well-known functional $H_{2,1}(h) = a_3 - a_2^2$ over the class $\mathcal{S}$ or its subclasses.
Using the three leaf function, $Q_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, Shi et al. [29] introduced the class BT_{3L} and investigated the sharp upper bound of $H_{2,1}(H_h/2)$ and $H_{2,2}(H_h/2)$ for the above said classes. Further by virtue of the four leaf function $Q_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$, Alshehry et al. [3] obtained the sharp coefficient type problems of logarithmic function for the families S^*_{4L} and C^*_{4L}.
Motivated by the above researchers and making use of four-leaf shaped domain, we introduce the subclass of the class $\mathcal{A}$ as follows:

Definition 1. *A function $h \in \mathcal{A}$ of the form (1.1) is said to be in the class $R_L(\phi_{4L}, \nu)$ if it satisfies the following subordination condition:*

$$\frac{\nu z^3h'''(z) + (1 + 2\nu)z^2h''(z) + zh'(z)}{\nu z^2h''(z) + zh'(z)} \prec \phi_{4l}(z) \quad (z \in \mathbb{D},\ 0 \le \nu \le 1). \tag{1.9}$$

Putting $\nu = 0$, the relation (1.9) reduces to the class $\mathcal{C}_{4l}$ as:

$$1 + \frac{zh''(z)}{h'(z)} \prec Q_{4L}(z)$$

studied by Alshehry et al. [3].
In the present paper, the authors investigate few coefficient bounds, Fekete-Szegö inequality and other result of logarithmic coefficients for the class $R_L(\phi_{4L}, \nu)$.

2 Preliminaries

Let $\mathcal{P}$ be the class of holomorphic functions $q(z)$ with positive real part and such functions expressed as series expansion as follows:

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \quad (z \in \mathbb{D}). \tag{2.1}$$

Now the following lemmas are useful for further investigations:

Lemma 2. *[7, 13] If $q \in \mathcal{P}$ and be of the form (2.1), then we have*

$$|q_n| \leq 2 \quad (n \geq 1), \tag{2.2}$$

and

$$|q_{n+k} - \mu q_n q_k| \leq 2max\{1, |2\mu - 1|\} = \begin{cases} 2 & 0 \leq \mu \leq 1 \\ 2|2\mu - 1| & otherwise. \end{cases} \tag{2.3}$$

Also, if $B \in [0,1]$ and $B(2B-1) \leq D \leq B$, we obtain

$$|q_3 - 2Bq_1q_2 + Dq_1^3| \leq 2. \tag{2.4}$$

Lemma 3. *(see [23]) Let $q \in \mathcal{P}$ and "has the expan- sion of the form (2.1). Then*

$$|Jq_1^3 - Kq_1q_2 + Lq_3| \leq 2(|J| + |K - 2J| + |J - K + L|). \tag{2.5}$$

Lemma 4. *[24] Let γ, τ, ψ and ς satisfy that τ, $\varsigma \in (0,1)$ and*

$$\begin{aligned} 8(1-\varsigma)\varsigma[(\tau(\varsigma+\tau)-\psi)^2 + (\tau\psi - 2\gamma)^2] + \tau(\psi - 2\varsigma\tau)^2(1-\tau) \\ \leq 4\tau^2\varsigma(1-\varsigma)(1-\tau)^2. \end{aligned} \tag{2.6}$$

If $q \in \mathcal{P}$ has the expansion of the form (2.1) then

$$|\gamma q_1^4 + \varsigma q_2^2 + 2\tau q_1 q_3 - \frac{3}{2}\psi q_1^2 q_2 - q_4| \leq 2. \tag{2.7}$$

3 Main Results

In this section, we start with finding the bounds of the first few initial logarithmic coefficients for the function of the class $R_L(\phi_{4l}, \nu)$.

Theorem 5. *Let the function $h \in \mathcal{A}$ of the form (1.1) be in the class $R_L(\phi_{4l}, \nu)$. Then*

$$|d_1| \leq \frac{5}{24(\nu+1)}, \tag{3.1}$$

$$|d_2| \leq \frac{5}{72(2\nu+1)}, \tag{3.2}$$

$$|d_3| \leq \frac{5}{144(3\nu+1)}, \tag{3.3}$$

and

$$|d_4| \leq \frac{1}{48(4\nu+1)}. \tag{3.4}$$

Proof. Let $h \in \mathcal{A}$ given by (1.1) be in the function class $R_L(\phi_{4L}, \nu)$. Then by Definition 1, there exists an analytic function $u(z)$ satisfying the condition of Schwarz lemma such that

$$\frac{\nu z^3 h'''(z) + (1+2\nu)z^2 h''(z) + zh'(z)}{\nu z^2 h''(z) + zh'(z)} = \phi_{4L}[u(z)] \quad (z \in \mathbb{D}). \tag{3.5}$$

For $q \in \mathcal{P}$ and it may be expressed in terms of Schwarz function $u(z)$ as

$$q(z) = \frac{1+u(z)}{1-u(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + ...$$

Equivalently,

$$\begin{aligned} u(z) =& \frac{q(z)-1}{q(z)+1} \\ =& \frac{1}{2}q_1 z + (\frac{1}{2}q_2 - \frac{1}{4}q_1^2)z^2 + (\frac{1}{8}q_1^3 - \frac{1}{2}q_1 q_2 + \frac{1}{2}q_3)z^3 + \\ & (\frac{1}{2}q_4 - \frac{1}{2}q_1 q_3 - \frac{1}{4}q_2^2 - \frac{1}{16}q_1^4 + \frac{3}{8}q_1^2 q_2)z^4 + ... \end{aligned} \tag{3.6}$$

By simple calculation of $\phi_{4L}(z)$ in terms of $u(z)$, the resulting series expansion is,

$$\begin{aligned} \phi_{4L}(u(z)) =& 1 + \frac{5}{6}u(z) + \frac{1}{6}(u(z))^5 \\ =& 1 + \frac{5}{12}q_1 z + \left(\frac{5}{12}q_2 - \frac{5}{24}q_1^2\right) z^2 + \left(\frac{5}{48}q_1^3 - \frac{5}{12}q_1 q_2 + \frac{5}{12}q_3\right) z^3 + \\ & \left(\frac{5}{12}q_4 - \frac{5}{96}q_1^4 + \frac{5}{16}q_1^2 q_2 - \frac{5}{12}q_1 q_3 - \frac{5}{24}q_2^2\right) z^4 + ... \end{aligned} \tag{3.7}$$

On simplification of left side of relation (3.5) we get

$$\begin{aligned} & \frac{\nu z^3 h'''(z) + (1+2\nu)z^2 h''(z) + zh'(z)}{\nu z^2 h''(z) + zh'(z)} = 1 + 2(1+\nu)a_2 z + \\ & [6(1+2\nu)a_3 - 4(1+\nu)^2 a_2^2]z^2 + \\ & [8(\nu+1)^3 a_2^3 - 18(1+\nu)(1+2\nu)a_2 a_3 + 12(1+3\nu)a_4]z^3 + \\ & [-18(1+2\nu)^2 a_3^2 - 16(1+\nu)^4 a_2^4 + 48(1+\nu)^2(1+2\nu)a_2^2 a_3 - \\ & 32(1+\nu)(1+3\nu)a_2 a_4 + 20(1+4\nu)a_5]z^4 + ... \end{aligned} \tag{3.8}$$

Utilizing (3.7) and (3.8) in the relation (3.5) and equating coefficients of $z,\ z^2, z^3$ and z^4, we get

$$a_2 = \frac{5}{24(\nu+1)} q_1, \tag{3.9}$$

$$a_3 = \frac{5}{72(2\nu+1)} (q_2 - \frac{1}{12} q_1^2), \tag{3.10}$$

$$a_4 = \frac{5}{144(3\nu+1)} [\frac{7}{288} q_1^3 - \frac{3}{8} q_1 q_2 + q_3], \tag{3.11}$$

and

$$a_5 = \frac{1}{4(1+4\nu)} [- \frac{91}{9(24)^3} q_1^4 + \frac{23}{(12)^3} q_1^2 q_2 - \frac{1}{27} q_1 q_3 - \frac{7}{4(72)} q_2^2 + \frac{1}{12} q_4]. \tag{3.12}$$

By making use of (3.9)-(3.12) in (1.5)-(1.8), we get

$$d_1 = \frac{5}{48(1+\nu)} q_1, \tag{3.13}$$

$$d_2 = \frac{5}{144(2\nu+1)} \left[q_2 - \frac{19+38\nu+4\nu^2}{48(1+\nu)^2} q_1^2 \right], \tag{3.14}$$

$$d_3 = \frac{5}{288} \left[\frac{1}{2(12)^2} \left(\frac{14\nu^4 + 79\nu^3 + 283\nu^2 + 210\nu + 42}{(\nu+1)^3(2\nu+1)(3\nu+1)} \right) q_1^3 - \frac{18\nu^2 + 57\nu + 19}{24(\nu+1)(2\nu+1)(3\nu+1)} q_1 q_2 + \frac{1}{3\nu+1} q_3 \right], \tag{3.15}$$

and

$$d_4 = -\frac{1}{96(1+4\nu)} \left[\frac{13109 + 144199\nu + 599600\nu^2 + 1179848\nu^3 + 1133632\nu^4 + 538112\nu^5 + 175776\nu^6 + 26208\nu^7}{248832(1+\nu)^4(1+2\nu)^2(1+3\nu)} q_1^4 + \right.$$
$$\left. \frac{88 + 352\nu + 252\nu^2}{216(1+2\nu)^2} q_2^2 + \frac{57 + 228\nu + 96\nu^2}{72(1+\nu)(1+3\nu)} q_1 q_3 - \frac{2353 + 21177\nu + 66768\nu^2 + 87628\nu^3 + 45120\nu^4 + 9936\nu^5}{5184(1+\nu)^2(1+2\nu)^2(1+3\nu)} q_1^2 q_2 - q_4 \right]. \tag{3.16}$$

Using (2.2) of Lemma 2 in (3.13) we get,

$$|d_1| \leq \frac{5}{24(1+\nu)}. \tag{3.17}$$

Since $0 \leq \frac{19+38\nu+4\nu^2}{48(1+\nu)^2} \leq 1$, by application of (2.3) of Lemma 2 in (3.14) gives

$$|d_2| \leq \frac{5}{72(1+2\nu)}. \tag{3.18}$$

For d_3, relation (3.15) can be written as

$$d_3 = \frac{5}{288(1+3\nu)} \left[q_3 - 2\frac{18\nu^2+57\nu+19}{48(\nu+1)(2\nu+1)} q_1 q_2 + \frac{14\nu^4+79\nu^3+283\nu^2+210\nu+42}{288(\nu+1)^3(2\nu+1)} q_1^3 \right]. \tag{3.19}$$

Comparing the bracket portion of (3.19) with (2.4), we have

$$B = \frac{18\nu^2+57\nu+19}{48(\nu+1)(2\nu+1)}, \quad D = \frac{14\nu^4+79\nu^3+283\nu^2+210\nu+42}{288(\nu+1)^3(2\nu+1)}.$$

Clearly $0 \leq B \leq 1$ and $B \geq D$. Moreover,

$$B(2B-1) = -\frac{5(18\nu^2+57\nu+19)(3\nu^2+3\nu+1)}{1152(1+\nu)^2(1+2\nu)^2} \leq D.$$

All the conditions of Lemma 2 are satisfied. Application of (2.4) in (3.19) we get

$$|d_3| \leq \frac{5}{144(1+3\nu)}.$$

To obtain the bound of d_4, comparing bracket portion of the relation (3.16) with (2.6) we get

$$\gamma = \frac{13109+144199\nu+599600\nu^2+1179848\nu^3+1133632\nu^4+538112\nu^5+175776\nu^6+26208\nu^7}{248832(1+\nu)^4(1+2\nu)^2(1+3\nu)},$$

$$\xi = \frac{88+352\nu+252\nu^2}{216(1+2\nu)^2}, \quad \tau = \frac{57+228\nu+96\nu^2}{144(1+\nu)(1+3\nu)},$$

and

$$\psi = \frac{2353+21177\nu+66768\nu^2+87628\nu^3+45120\nu^4+9936\nu^5}{7776(1+\nu)^2(1+2\nu)^2(1+3\nu)}.$$

Thus, all the conditions of Lemma 4 are satisfied. Application of (2.7) gives

$$|d_4| \leq \frac{1}{48(1+4\nu)}.$$

The proof of Theorem 5 is complete. □

Remark 6. *Putting $\nu = 0$ in Theorem 5 we get the result due to Alshehry et al. ([3], Theorem 12).*

Theorem 7. *Let $h \in R_L(\phi_{4L}, \nu)$ be the series of the form (1.1). Then for any $\lambda \in \mathbb{C}$, we have*

$$|d_2 - \lambda d_1^2| \leq \frac{5}{72(1+2\nu)} max \left\{1, \left|\frac{5[(1+2\nu)(1-3\lambda)+4\nu^2]}{24(1+\nu)^2}\right|\right\}.$$

Proof. Making use of (3.13) and (3.14), one may get

$$\begin{aligned}|d_2-\lambda d_1^2| =& \left|\frac{5}{144(1+2\nu)}q_2 - \frac{5(4\nu^2+38\nu+19)}{(48)(144)(1+\nu)^2(1+2\nu)}q_1^2 - \lambda\frac{25}{(48)^2(1+\nu)^2}q_1^2\right| \\ =& \left|\frac{5}{144(1+2\nu)}q_2 - \frac{5(4\nu^2+38\nu+19)+75\lambda(1+2\nu)}{48(144)(1+\nu)^2(1+2\nu)}q_1^2\right| \\ =& \frac{5}{144(1+2\nu)}\left|q_2 - \frac{4\nu^2+38\nu+19+15\lambda+30\lambda\nu}{48(1+\nu)^2}q_1^2\right| \\ =& \frac{5}{144(1+2\nu)}|q_2-\mu q_1^2|,\end{aligned}$$

where

$$\mu = \frac{4\nu^2+(1+2\nu)(19+15\lambda)}{48(1+\nu)^2}.$$

An application of Lemma 2 gives

$$|d_2-\lambda d_1^2| \le \frac{5}{72(1+2\nu)} max\left\{1, \left|\frac{5[(1+2\nu)(1-3\lambda)+4\nu^2]}{24(1+\nu)^2}\right|\right\}.$$

The proof of Theorem 7 is complete. □

Taking $\nu = 0$ in Theorem 7 we get the following result.

Corollary 8. *Let $h \in R_L(\phi_{4L},\nu)$ be the series of the form (1.1). Then for any $\lambda \in \mathbb{C}$, we have*

$$|d_2-\lambda d_1^2| \le \frac{5}{72} max\left\{1, \frac{5}{24}|3\lambda-1|\right\}$$

Remark 9. *Corollary 8 is a correction of the obtained estimates stated in ([3], Theorem 13).*

Taking $\lambda = 1$, we get the following result in form of corollary:

Corollary 10. *If $h \in R_L(\phi_{4L},\nu)$ then*

$$|d_2-d_1^2| \le \frac{5}{72(1+2\nu)} max\left\{1, \left|\frac{5[4\nu^2-4\nu-2]}{24(1+\nu)^2}\right|\right\}.$$

Remark 11. *Taking $\nu = 0$ in Corollary 10 we get $|d_2-d_1^2| \le \frac{5}{72}$ which coincide the result of Alshehry et al.([3], Corollary 14).*

Theorem 12. *If $h \in R_L(\phi_{4L},\nu)$ and of the form (1.1), then*

$$|d_1d_2-d_3| \le \frac{5}{144}\left[\frac{3112\nu^4+17680\nu^3+24780\nu^2+10205\nu+2930}{1152(1+\nu)^3(1+2\nu)(1+3\nu)}\right].$$

Proof. Using the values of $d_1,\ d_2$ and d_3 from (3.13)-(3.15) in the functional $|d_1 d_2 - d_3|$ and after computation we obtain

$$|d_1 d_2 - d_3| = \frac{5}{288}\left|\frac{56\nu^4 + 376\nu^3 + 1722\nu^2 + 1315\nu + 263}{1152(1+\nu)^3(1+2\nu)(1+3\nu)} q_1^3 - \frac{3\nu^2 + 12\nu + 4}{4(1+\nu)(1+2\nu)(1+3\nu)} q_1 q_2 + \frac{1}{1+3\nu} q_3\right|. \tag{3.20}$$

By application of Lemma 3 gives

$$|d_1 d_2 - d_3| \leq \frac{5}{144}\left[\frac{3112\nu^4 + 17680\nu^3 + 24780\nu^2 + 10205\nu + 2930}{1152(1+\nu)^3(1+2\nu)(1+3\nu)}\right].$$

This proof the result of Theorem 12. □

Letting $\nu = 0$ in Theorem 12 we get the following corollary.

Corollary 13. *Let $h \in \mathcal{A}$ be of the form (1.1). If $h \in \mathcal{C}_{4l}$ then*

$$|d_1 d_2 - d_3| \leq \frac{7325}{82944}.$$

Remark 14. *Corollary 13 is a correction of the obtained estimates stated in ([3], Theorem 15).*

Concluding Remarks:

In the present article, the authors have introduced a subclass of Kamali type starlike function in the open unit disk $\mathbb{D}$ subordinated to four-leaf shaped domain. For function belongs to this class, we have investigated bounds of some of the initial coefficients and the estimate of Fekete-Szegö functional. Researchers can make use of quantum or q-calculus to define the above class and results may derive accordingly.

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