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Insights about functions from example-generation tasks: combining e-assessment and written responses

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In this paper we report on student responses to a sequence of tasks that ask for examples of function graphs with various properties. We gathered large samples of student responses to both an e-assessment (N=322) and written (N=332) version of the tasks, and conducted task-based interviews with a further group of students (N=8 for e-assessment, N=11 for written). In the large samples of responses, we identify common families of correct and incorrect responses. Possible explanations for the incorrect responses offer insights into students' (mis)understanding of function concepts, which are supported with observations from the task-based interviews.

Keywords: Example generation, functions, injective, surjective, e-assessment.

Introduction

Prompting students to generate examples of mathematical objects has been suggested as both an effective way to encourage active engagement in mathematics (Watson & Mason, 2005) and as a tool for researchers to gain insight into students' understanding (Zazkis & Leikin, 2007). Central to both claims is the idea of an *example space*: a collection of possible examples of a mathematical object. Watson and Mason (2005, p. 54) emphasise that students' example spaces are personal and situated; in particular, the collection of examples that comes to mind for a given student can depend on the way an example-generation task is posed. Therefore, for teachers, the goal is to design tasks that "challenge learners to explore and extend their example spaces" (p. 62).

Checking the properties of student responses to example-generation tasks can require substantial effort, making e-assessment a practical solution for large undergraduate classes (Kinnear, 2022). However, there are various open questions about e-assessment of example-generation tasks, such as whether students approach them differently to written versions of the same task (Kinnear et al., 2022). The work we report here, making use of both e-assessment and written versions of the same tasks, is part of a larger study (forthcoming), where the focus is on exploring differences in student responses to the e-assessment and the written versions.

The main contribution of this article is to use students' responses to example-generation tasks to shed light on their understanding of injectivity, surjectivity, and the image of a function. While there is relatively little literature on students' difficulties with these concepts (Thoma & Nardi, 2019), we know from experience as lecturers that students often find these concepts hard to grasp. Example-generation tasks provide a means to explore students' understanding of these concepts, since the examples produced by participants "mirror their conceptions of mathematical objects involved" (Zazkis & Leikin, 2007, p. 15). Thus, our overarching research question is: what can we infer about students' understanding of function concepts from their responses to example-generation tasks?

Method

To investigate our research question, we developed a task sequence that could be implemented both through e-assessment and on paper. We gathered responses from three successive cohorts of students taking a first-year pure mathematics module at the same UK university.

Design and implementation of the tasks

We developed a task sequence based on real functions and the notions of injectivity and surjectivity, which students study during week 8 (of 11) in the course. The full task sequence, shown in Figure 1, has four parts; here, we focus on only the first two parts, which highlight the main issues we observed regarding students' understanding of the function image.

In developing the sequence of tasks, we were guided by advice from Watson and Mason (2005, p. 131) to prompt students to "make up an example with some constraints" and to "add constraints sequentially". The first task is intended to be straightforward, assuming the student is familiar with the notation and the concept of a function's image. The second task adds a constraint, requiring students to demonstrate understanding of the meaning of "not injective".

In each case, draw the graph of a function with the given properties, and label important points:

- $f_1 : [0,1] \to [0,1]$ has image $[0, \frac{1}{2}]$.
- $f_2 : [0,1] \to [0,1]$ has image $[0, \frac{1}{2}]$ and is not injective.
- $f_3 : [0,1] \to [0,1]$ is surjective and not injective.
- $f_4 : [0,1] \to [0,1]$ is injective, not surjective, and passes through $(0.2, 0.8)$ and $(0.5, 0.5)$.

Figure 1: The paper-based version of the task sequence

For the e-assessment version, students were provided with an interactive graph for each task, as shown in Figure 2(b). Each graph had four points (labelled A, B, C and D) connected by line segments; by manipulating the points, students could generate examples of piecewise-linear functions.

Participants and procedure

The course lecturer agreed to include our tasks among the weekly assessments issued to the whole class. In 2019/20 we gathered 322 student responses to the e-assessment version, when it was included in the week 8 "reading quiz" that students are asked to complete online after reading the assigned section of the textbook but before any lectures on the material (the best 8 of 10 quiz scores contributed 5% to the course result). In 2020/21, we gathered 332 student responses to the written version, which was included in the week 8 written assignment (the best 8 of 10 written assignments contributed 25% to the course result). Ethical approval for our analysis of the students' work was granted through the School of Mathematics at the University of Edinburgh.

In 2021/22, we conducted task-based interviews: all students taking the course were invited to participate, with a £10 voucher offered for their time. A total of 19 students agreed to take part, with 11 students signing up for an in-person slot (working on the written version with the first author), and 8 students signing up for an online slot (working on the e-assessment version with the second author). Ethical approval for the interviews was granted by the ethics committee at Loughborough University.

Analysis

The written responses were divided evenly between the three authors. For each example, we recorded which type of function was graphed and whether it was correct; for incorrect examples we noted a reason, such as "domain is [0, ½]" or "surjective but should not be", from a shared list of reasons that we built up while coding. We double-coded a sample of 42 scripts: agreement was high (e.g., 97% agreement for whether correct), and disagreements were resolved through discussion.

The e-assessment responses were exported (as a list of the coordinates of points A-D) and analysed systematically using R code to identify the objective properties of each example, such as the domain and image, and whether it is injective. The efficiency of this approach is a particular strength of e-assessment for researchers (e.g., Kinnear, 2022).

The interviews were video recorded and partially transcribed (in particular, noting the examples produced, and any supporting reasoning). From a summary of the examples produced during each interview, we were able to isolate episodes to revisit in greater detail (e.g., where an example was similar to a commonly-observed response).

Results

The number of correct and incorrect responses to each task is shown in Table 1. In the remainder of this section we discuss each task in turn, giving details of the correct then incorrect responses.

Table 1: Number of correct and incorrect responses by format

Task	Format	Correct	Incorrect
Task A	Written	286	45
	E-assessment	177	128
Task B	Written	254	78
	E-assessment	139	153

Task A: a function with image [0, ½]

Out of the 286 correct written responses, 258 (90%) were of $y = ½\,x$, as shown in Figure 2(a), suggesting this is the most accessible example for this group of students. In contrast, out of 177 correct e-assessment responses, only 58 (33%) were of $y = ½\,x$. Instead, the most common e-assessment response was piecewise-linear and strictly increasing (as shown in Figure 2(b)); 84 (47%) of the correct responses were of this type. This is likely because the system constrained the points A-D to be placed on the gridlines shown in Figure 2(b), and B and C were initially located on the x-axis at $x = 0.3$ and $x = 0.7$ respectively: it appears many students only moved B and C vertically, and would therefore be unable to place them to form a graph of $y = ½\,x$.

Out of 45 incorrect written responses, 17 (38%) were examples of surjective functions rather than having the required image of [0, ½]. In many of these cases, students appeared to be confusing the restriction on the image with one on the domain – such as in Figure 2(c), where the domain is [0, ½]. Other students appeared to be attending only to part of the required domain – such as in Figure 2(d), where the student uses a solid line to draw the graph for $0 \leq x \leq ½$ but a dashed line for $½ \leq x \leq 1$.

One student drew a graph similar to Figure 2(d) but with a solid line across the whole domain; they explained their reasoning based on an unconventional definition of image, where an image of [0, ½] "means that for some $A \subseteq [0, 1]$, $\{ f(x) \mid x \in A \} = [0, ½]$". It may be that other students were using similar reasoning without expressing it.

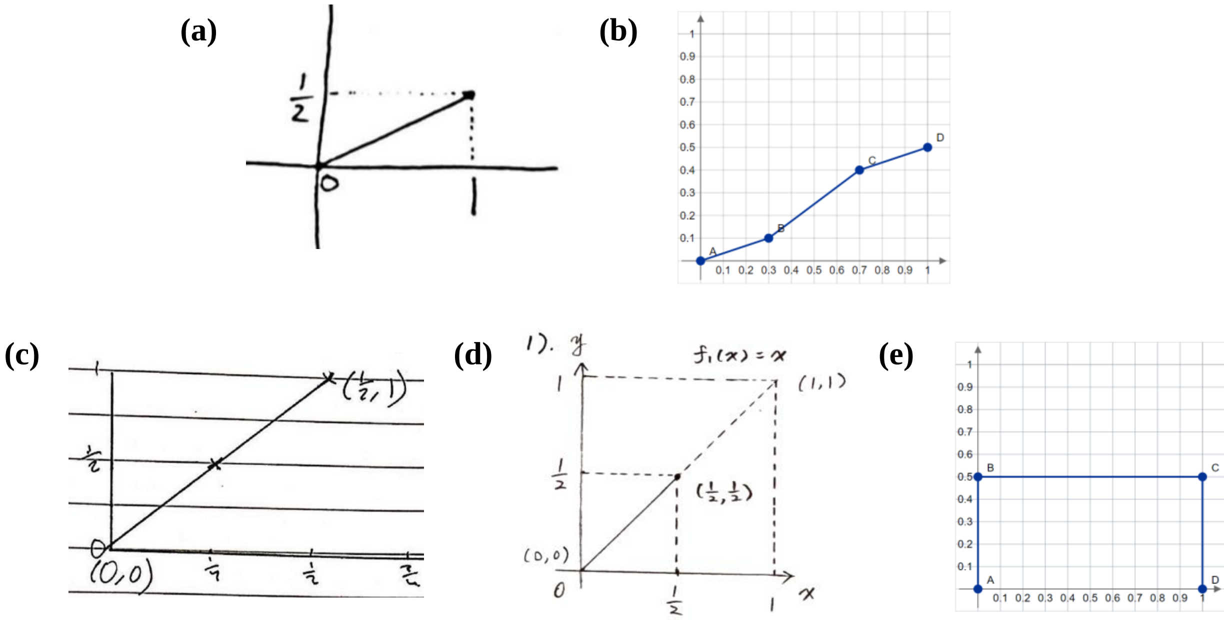


Figure 2: Examples of correct (top row) and incorrect (bottom row) responses to Task A

Similarly, of 128 incorrect e-assessment responses, 41 (32%) were surjective. Among those, 6 had a domain of [0, ½]. A further 15 incorrect examples that were *not* surjective also had a domain of [0, ½], echoing the observation from the written responses that some students appeared to confuse the domain and the image. Another common issue among the incorrect e-assessment responses was the appearance of a vertical segment, meaning that the example was not the graph of a function: 32 (25%) of the incorrect responses had this issue (see Figure 2(e)). This was in contrast with the written responses, where only 7 (15%) of the examples were not graphs of functions; moreover, only one of those featured a vertical segment, with the others mostly consisting of sideways parabolas.

There were two other issues we observed. First, a handful of e-assessment responses were very close to being correct, and may have been a result of poor scale reading or difficulty placing the points precisely. Second, among the e-assessment responses, 6 students placed all the points at (0, ½) and (0, 1), perhaps due to confusing the interval notation used in the task for the coordinates of points.

Task B: a function with image [0, ½] that is not injective

Among the 254 correct written responses, 137 (54%) were classified as quadratics, such as the example shown in Figure 3. Such a response was clearly not possible in the e-assessment version, where the interface was limited to representing a piecewise-linear function. However, it appeared that many students tried to approximate the graph of a parabola: the most common type of correct e-assessment response was a "hump" shape, as shown in Figure 3, with 43 of the 139 correct responses being of this type.

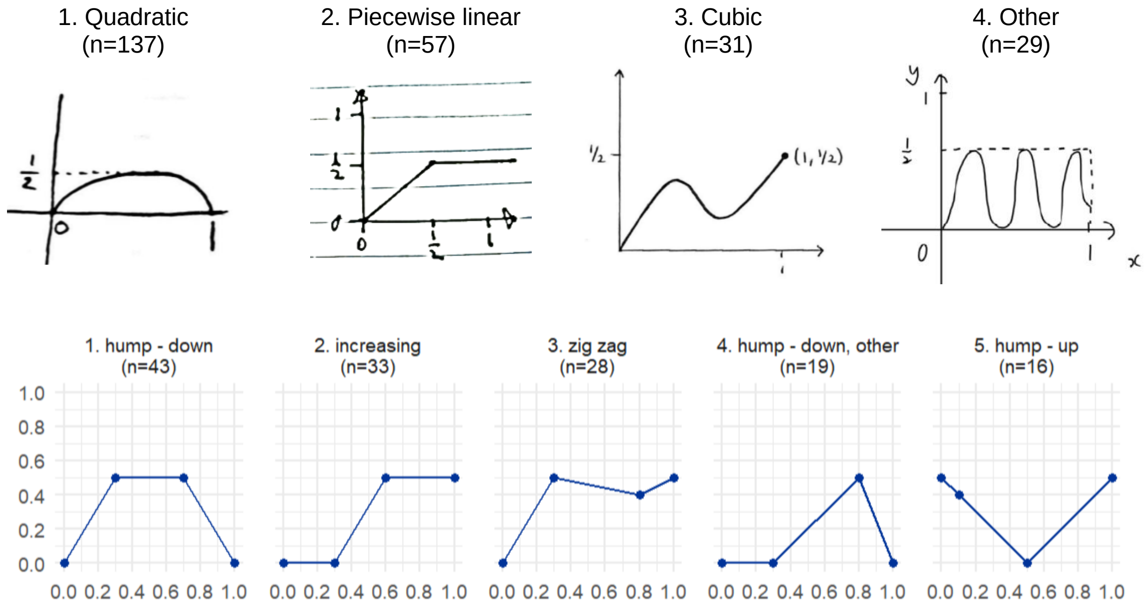


Figure 3: Categories of correct responses to Task B in the written version (top row) and the e-assessment version (bottom row)

One clear difference between the written and e-assessment responses was the presence of flat segments: these were rare in the written work, which tended to use smooth curves, but featured often in the e-assessment examples. In the interviews, students completing the e-assessment task would refer to the flat segments when justifying the non-injectivity of their answers, for instance:

Interviewer:	So, what are you looking for in this task?
Sam:	Em, just like sort of the same thing [as in Task A], but if it's not injective it means that there's either… eh… a value for y that's used twice.
Interviewer:	OK, OK.
Sam:	So, like, here [indicating toward flat segment on their graph]

There were 78 incorrect written responses for Task B, with the most common issue being an incorrect image (50 responses; 64%). Among those, 28 had an image that was too large, 18 had an image that was too small, and 4 had an image of [½ , 1]. Another common issue, that sometimes overlapped with these, was an incorrect domain (18 responses; 23%).

There was a similar pattern with the incorrect e-assessment responses: out of 153 incorrect responses, 115 (75%) had an incorrect image, 61 (40%) had an incorrect domain, and 46 (30%) had both.

For the examples with an image that was too large, we observed similar issues to those already discussed for Task A. Namely, some students appeared to be confusing the domain and the image (such as in Figure 4(a)) while others were attending to only part of the domain (such as in Figure 4(b)). However, for this task, we also found that some students appeared to take no account of the image restriction (such as in Figure 4(c)). This is despite almost all students including ½ as a feature of their example (e.g., an axis annotation, as in Figure 4(c), or as the coordinate of a labelled point), suggesting they were influenced to some extent by the appearance of [0, ½] in the task. Moreover, about half of the students making this error had answered part (a) correctly, suggesting they had some understanding of what the image restriction meant.

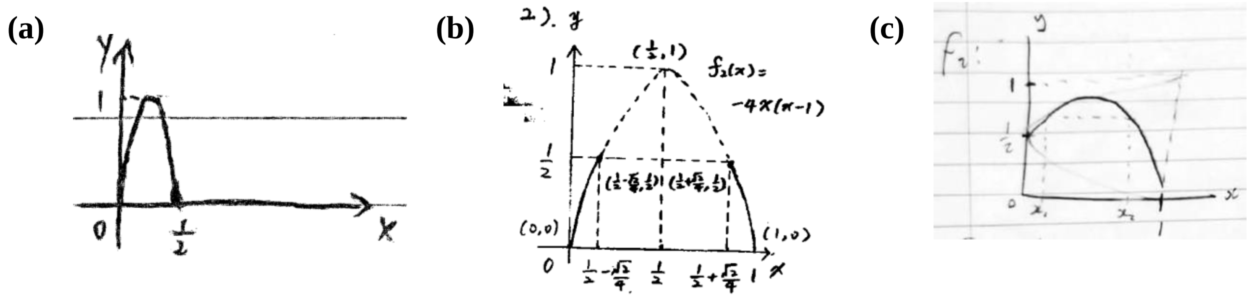
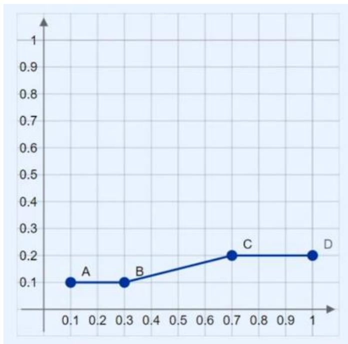


Figure 4: Examples of incorrect responses to Task B

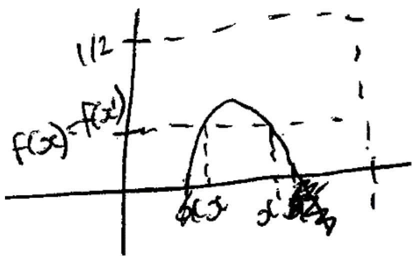
Our observations from the interviews offer three possible explanations for students not satisfying the image requirement. The first is that students (such as the author of Figure 4(c)) were focused on addressing the "not injective" part of the task and simply overlooked the image requirement. For instance, Sally had produced an example for Task A that did have image [0, ½] (although was not defined on the full domain), however for Task B her example did not have the correct image. Instead, she focused her explanation on the non-injectivity:

Sally: it is a non-injective. Injective, is that "every different value of *x* has a different value of *y*"?
Interviewer: Yeah, and this needs to be *not* injective.
Sally: Yeah, so a value of *x*… [moving the points] maybe like this? [resulting in the example shown to the right]
Interviewer: Yeah, so let's… let's see all the… the function needs to go from [0,1] to [0,1] with image [0, ½]
Sally: Yeah.
Interviewer: and it needs to be not injective.
Sally: Yeah, so if values of *x* are 0.1 or 0.3 we have the same value of *y* which is 0.1, and similarly for C and D.



A second possibility is that students were using an unconventional interpretation of "image", as discussed previously for Task A. Such reasoning would be consistent with the 18 written responses we observed where the image was a subset of [0, ½] (such as the graph of $y = (x - ½)^2$), or the 15 e-assessment responses with the same issue. Similar reasoning was offered in one of the interviews by Philip: even with further prompting from the interviewer to consider the other conditions given in the task, Philip was satisfied that their example met all the requirements, "as long as I keep under a half":

Philip: So, *f*(*x*) equals *f*(*x*'). And *x* here, if this is *x*' then, sorry that is *x* and that is *x*' [correcting the labels on the *x*-axis]. And that one's not injective.
Interviewer: OK, so that's the "not injective" part. Are you happy with everything else in the question for this example?
Philip: So we're still in the same interval and we've still got the same image. So as long as I keep under a half [drawing *y* = ½ dashed line] and 1 [drawing *x* = 1 dashed line], I think I'm still complying with the rest.



A third possible reason for students producing examples with an incorrect image is suggested by the explanations given by some interview participants, where they appeared to equate "injective" with "not surjective", for instance:

Steve:	Not injective… so if it *is* injective then there are, there is some part of the domain that's not mapped to.
Interviewer:	So, is the function that you have now on the screen injective do you think? [a representation of $y = ½\ x$]
Steve:	Erm, yes, because this... none of these values [using the mouse to indicate the rectangular area above $y = ½$] here above, well between 1 and ½ are mapped to. So it is, it is injective.

Similarly, in another interview, Serena recalled that "not injective means it's surjective or bijective". This could perhaps have come from a misinterpretation of examples given in the lecture notes (shown in Figure 5), where "Injective" and "Not surjective" are used to label one example, and "Surjective" and "Not injective" used to label another. This could leave the impression that these terms are "opposites", similar to the way that the everyday meaning of "open" and "closed" as antonyms conflicts with the use of these terms in set theory.

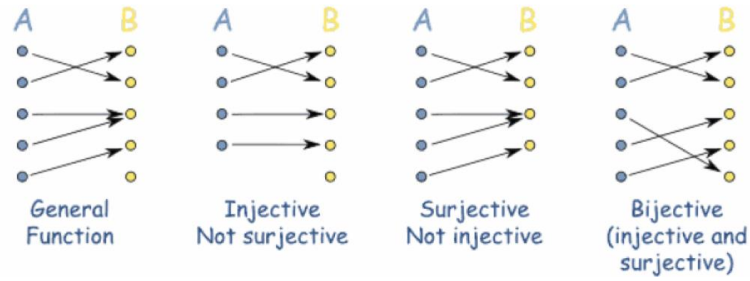


Figure 5: Function examples given in the lecture notes

Discussion

By analysing a large sample of student responses to example-generation tasks, and triangulating with observations from task-based interviews, we have gained insight into students' thinking about function images, injectivity and surjectivity. We found common errors with the image in the large sample of responses, particularly in Task B, whether in writing or through e-assessment. These errors corresponded with answers given during the interviews, where students' explanations suggested three possible reasons for the common errors we observed. In what follows, we review those reasons and elaborate on their implications.

First, some students failed to attend to the requirement on the image, perhaps because it was one of several requirements and students could not hold them all in mind at once. This echoes findings from previous studies of students' example-generation processes, where students often did not check the properties of their examples (Iannone et al., 2011). Such student approaches to example-generation tasks have been suggested as one reason why students may not experience the kinds of learning gains often claimed as an advantage of example-generation tasks (Iannone et al., 2011), which could perhaps be addressed by students becoming more familiar with this type of task (Kinnear, 2022).

Second, some students appeared to view the image (and domain) as constraints that do not need to be satisfied exactly. This demonstrates how example-generation tasks can reveal students' thinking, both for researchers (Zazkis & Leikin, 2007) and for teachers. One open question, particularly for the e-assessment version of the task, is how best to provide students with feedback on incorrect responses (Kinnear et al., 2022); this task may provide a useful test case, where our findings could inform the design of feedback that addresses this way of thinking about image/domain.

Third, some students explicitly equated “injective” with “not surjective”, perhaps influenced by the examples given in the lecture notes (Figure 5). This is similar to the development of figural concepts, that are “inappropriately abstracted due to stressing unintended attributes” (Watson & Mason, 2005, p. 56), and could therefore be addressed by broadening the range of examples used to introduce the concepts to students.

One aspect we have not emphasised here is the comparison between e-assessment and written versions of the task, since this is the focus of our larger study of the same data sources (forthcoming). We note however that students tended to perform less well on the e-assessment version of the tasks (e.g., for Task B only 48% of e-assessment responses were correct, while the success rate on the written version was 77%). The present analysis may help to understand any differences in performance, as the nature of the example spaces probed by the two versions of the task is necessarily different (e.g., as illustrated in Figure 3).

One limitation of our analysis of the large samples is that we only have access to the students’ final response, with no insight into their process or reasoning – and no chance to probe for further examples “by asking for ‘another and another’”, as recommended by Zazkis and Leikin (2007, p. 19). However, our approach of triangulating with the task-based interviews enabled us to identify ways of thinking that may underlie some of the responses we observed. Future research could fruitfully explore the extent to which these ways of thinking are shared by other students.

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