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Research article

Dynamic bipolar fuzzy aggregation operators: A novel approach for emerging technology selection in enterprise integration

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Abstract: Emerging technology selection is crucial for enterprise integration, driving innovation, competitiveness, and streamlining operations across diverse sectors like finance and healthcare. However, the decision-making process for technology adoption is often complex and fraught with uncertainties. Bipolar fuzzy sets offer a nuanced representation of uncertainty, allowing for simultaneous positive and negative membership degrees, making them valuable in decision-making and expert systems. In this paper, we introduce dynamic averaging and dynamic geometric operators under bipolar fuzzy environment. We also establish some of the fundamental crucial features of these operators. Moreover, we present a step by step mechanism to solve MADM problem under bipolar fuzzy dynamic aggregation operators. In addition, these new techniques are successfully applied for the selection of the most promising emerging technology for enterprise integration. Finally, a comparative study is conducted to show the validity and practicability of the proposed techniques in comparison to existing methods.

Keywords: bipolar fuzzy sets; bipolar fuzzy dynamic weighted averaging (BFDWA) operator; bipolar fuzzy dynamic weighted geometric (BFDWG) operator; decision making; optimization; algorithms
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1. Introduction

Multi-attribute decision-making (MADM) challenges emerge across a wide spectrum of situations, requiring the assessment and choice of various alternatives, actions, or candidates based on predetermined criteria. The adoption of MADM facilitated by aggregation operators is gaining prominence due to its adaptability to address practical issues across various domain such as science, engineering, environmental sciences, social sciences etc [1–3]. Aggregation operators play a pivotal role by amalgamating scattered values into a unique entity in a specified framework, thereby enabling the resultant aggregation outcome to represent all individual values. This versatility makes MADM through aggregation operators an attractive approach for handling real-world problems. Before the introduction of aggregation operators, decisions were usually based on crisp sets, which operated on the binary premise that an element either belonged to a set or did not belong to the set. This approach has limitations when dealing with the complexities of the real world, where there is confusion regarding belongingness in a set. In response to this challenge, Zadeh [4] introduced the notion of fuzzy sets (FS). An FS A of the universal set X is represented by the function $\mu_A: X \to [0, 1]$, called the membership function. Later, many mathematicians conducted various studies on this interesting concept. Kahne [5] formulated a decision-making framework to address scenarios that demand evaluations of alternatives predicated upon multiple attributes, each characterized by distinct degrees of importance. In [6], Jain developed an efficient approach to decision-making. Dubois and Prade [7] discussed fuzzy set operations such as union, intersection, and complement. Yager [8] presented fuzzy set aggregation operations in 1978. These operations were essential for combining fuzzy sets to make decisions or draw inferences using different fuzzy data points.

In the ongoing evolution of decision-making processes, the need to contend with imprecise, ambiguous, and uncertain information remains a constant challenge. In response to this, Atanassov proposed the concept of intuitionistic fuzzy sets (IFS) in [9]. An IFS contains both the membership function $\mu_A: X \to [0, 1]$ and the non-membership function $\nu_A: X \to [0, 1]$, which is a robust extension of the fuzzy set. References [10] and [11] introduce weighted and order-weighted aggregation operators, while reference [12] explores geometric aggregation operators in the intuitionistic fuzzy (IF) environment. Xu [13] developed novel averaging operators in the IF framework. To address MADM problems in an IF context, generalized aggregation operators were proposed in [14]. Subsequently, Xu and Wang [15] introduced induced generalized aggregation operators for IF knowledge. Huang's work [16] explored Hamacher aggregation operators for decision-making within IF settings. Additionally, in [17], Bonferroni mean operators designed for intuitionistic fuzzy sets (IFS) were introduced. The reference [18] presented a hybrid integrated decision-making paradigm for the computing framework that utilized soft and complex intuitionistic fuzzy information. Zhang [19] introduced another extension of fuzzy sets known as bipolar fuzzy sets (BFS). These BF sets are distinguished by a dual-component structure, with one component representing positive membership degrees falling within the range of [0, 1], and the other component denoting negative membership degrees within the range of [-1, 0]. BFS is an advanced tool for handling uncertainty in decision science. In 2004, Zhang and Zhang [20] progressed the field by applying BFS to bipolar logical reasoning and set theory. BFS has been crucial in computational psychiatry, as shown by Zhang et al. in 2011 [21]. It has also proved valuable in medical science with contributions from Zhang et al. in 2009 [22]. Moreover, the concept of BFS was successfully applied by the authors in the fields of quantum cellular combinatorics in 2013 [23] and organizational modeling in 2015 [24]. Gul [25] introduced the concept of bipolar fuzzy aggregation operators. Within this framework, he defined two pivotal operators: the bipolar fuzzy weighted averaging (BFWA) operator and the bipolar fuzzy weighted geometric (BFWG) operator. In 2017, Wei et al. [26] introduced the hesitant bipolar fuzzy weighted averaging and geometric operators. Xu and Wei [27] presented Hamacher aggregation operators, and they also investigated the characteristics and special cases of these operators. In 2019, Jana [28] introduced Dombi aggregation operators designed for BFS. These operators were applied to the development of solutions for multi-attribute group decision-making challenges. Jan et al. [29] developed a viable hybrid decision-making framework for human-computer interaction in the context of bipolar complex picture fuzzy soft sets based on their findings. Mani et al. [30] introduced the concept of intuitionistic fuzzy bipolar metric spaces and solved integral equations through this novel notion.

The technology of emerging enterprises evolves in collaboration with business. Enterprise growth promotes innovation and progress within an industry. As they expand, these businesses stimulate innovation and progress across industries. Nevertheless, in this era of rapid technological advancement, unforeseen challenges cannot be disregarded. Pioneering technological enterprises encounter novel challenges and uncertainties in their pursuit of innovation. The dynamic nature of the digital realm presents both opportunities and risks. Privacy issues, data intrusions, and cybersecurity concerns necessitate vigilant safeguarding and reduction. Flexibility and proactivity are critical qualities in the dynamic realm of technological advancement. These organizations remain competitive by embracing new technologies and methods and demonstrating agility. By utilizing AI, IoT, and blockchain, they may strengthen their position in the market, expedite operations, and enhance the experiences of visitors. Similar to how purified water is indispensable for life, companies developing new technologies must be innovative and progressive. By implementing strategic refinement, process development, and emerging technology, these businesses have the potential to flourish and influence entire industries and society. Pursuing excellence in the development of technological enterprises is an admirable endeavor. This study presents an innovative approach to managing emerging technological challenges in dynamic environments. The strategy incorporates evolving data and enhances the precision and assurance of optimal solutions in practical situations.

1.1. Motivation

The combination of the BF set and the dynamic operators is the basic purpose of this study. The primary justification for conducting this research is given below.

1) In managing ambiguous data, dynamic operators provide exceptional adaptability.

2) These operators demonstrate outstanding ability to convert inconsistent information into a unified value, thereby efficiently tackling the complexities of decision-making in a constantly evolving scenario.

3) Bipolar fuzzy sets are significant due to their capacity to capture both positive and negative opinions in the data simultaneously. Bipolar fuzzy sets offer an intricate depiction in dynamic environments characterized by constantly changing circumstances.

1.2. Research gap, objectives and contributions of the study

Previous studies have mostly focused on decision-making scenarios when all initial choice data is collected at the same time. However, in many decision-making scenarios, it is standard practice to collect the relevant data at different periods. The term "time interval" is employed in dynamic aggregation operators within the MADM context to assess the decision-maker's information preference over time intervals. This is achieved by utilizing a time-dependent function. By incorporating variables into the dynamic framework, it becomes possible to monitor the changes in membership degrees over time and analyze the fluctuations within specific time intervals. This feature enhances the precision of decision-making, offers an understanding of alterations, and assesses the dynamics of fuzzy sets. A variety of approaches are required to tackle these issues. The use of fuzzy dynamic weighted averaging and geometric operators in decision-making models is crucial, as they facilitate dynamic changes and effectively capture intricate interactions within systems that are impacted by imprecision and uncertainty. Although these operators have been defined for intuitionistic and classical fuzzy environments. These environments are unable to manage uncertainty from dual perspectives, encompassing both positive and negative aspects. Therefore, it is crucial to describe these operators for bipolar fuzzy sets in order to handle such scenarios and close this gap.

The following are the primary objectives of the theoretical framework that our research revolves around:

1) To define BF dynamic variable and to develop fundamental laws for these numbers.

2) To introduce BFDWA and BFDWG operators for these numbers and to formulate their structural properties.

3) To design step by step algorithm for solving MADM problems using these proposed techniques and to present an illustration of solving specific MADM problems under framework of BF dynamic environment.

The following are the major key goals of the theoretical framework that our research focuses on:

1) Two innovative aggregation operators, the BFDWA and BFDWG operators, have been introduced for decision-making context to handle bipolar fuzzy information.

2) A comprehensive exposition is provided on the fundamental attributes of the operators under consideration, encompassing their idempotency, monotonicity, and boundedness.

3) A systematic approach to manage MADM issues using BF dynamic aggregation operators is developed.

4) The proposed approach is implemented to select the most suitable emerging technology enterprise. It demonstrates the significance of the developed strategy and proposed operators in the process of decision-making.

5) The proposed methodology is compared to various other approaches described in the literature. As demonstrated by the comparison outcomes, the developed methodology is consistent and reliable.

The following is the manuscript's structure: Basic definitions are presented in Section 2. In Section 3, we introduce BF dynamic aggregation operators and examine their basic characteristics. In Section 4, a technique for using BF dynamic aggregation operators to solve MADM-related issues is presented. In Section 5, the newly introduced operators are utilized to facilitate the process of selecting rising technological company. Moreover, we include a comparative analysis that demonstrates the usefulness and applicability of these novel strategies in comparison with existing methodologies. Section 6 summarizes the key findings of this research.

2. Preliminaries

In this section, we present key definitions that are important for understanding the major content of this study.

Definition 1. ([4]) A fuzzy set of the universe X is defined as

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}. \tag{2.1}$$

Here, $\mu_A: X \to [0, 1]$ is called membership function and $\mu_A(x)$ signifies the degree of membership of $x \in X$.

Definition 2. ([27]). The bipolar fuzzy set (BFS) of the universe X is represented as follows:

$$B = \{\langle x, \mu_B^+(x), \nu_B^-(x)\rangle | x \in X\}, \tag{2.2}$$

wherein $\mu_B^+: X \to [0, 1]$ and $\nu_B^-: X \to [-1,0]$ are called the positive and negative membership functions, respectively. The notation $\langle(\mu_B^+, \nu_B^-)\rangle$ is used to represent bipolar fuzzy number (BFN).

Definition 3. ([28]) Consider a set of bipolar fuzzy numbers (BFNs), denoted as $\alpha_k = (\mu_k^+, \nu_k^-)$, where $k = 1,2, \dots, ɳ$. A BFWA operator is a function BFWA: $\psi^{ɳ} \to \psi$ defined in the following manner:

$$BFWA(\alpha_1, \alpha_2, \dots, \alpha_{ɳ}) = \oplus_{k=1}^{ɳ} \mathrm{w}_k . \alpha_k$$

$$= \left(1 - \prod_{k=1}^{ɳ}(1 - \mu_k^+)^{\mathrm{w}_k}, - \prod_{k=1}^{ɳ}(|\nu_k^-|)^{\mathrm{w}_k}\right). \tag{2.3}$$

Here, $\mathrm{w}_k = [\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_{ɳ}]^T$ represents the associated weight vector of these BFNs for $k = 1,2, \dots, ɳ$, with the condition that $\mathrm{w}_k \in [0,1]$, and the constraint $\sum_{k=1}^{ɳ} \mathrm{w}_k = 1$ holds.

Definition 4. ([28]) Consider a set of bipolar fuzzy numbers (BFNs), denoted as $\alpha_k = (\mu_k^+, \nu_k^-)$, for $k = 1,2, \dots, ɳ$. A BFWG operator is a function BFWG: $\psi^{ɳ} \to \psi$ defined in the following manner:

$$BFWG(\alpha_1, \alpha_2, \dots, \alpha_{ɳ}) = \otimes_{k=1}^{ɳ} {\alpha_k}^{\mathrm{w}_k}$$

$$= \left(\prod_{k=1}^{ɳ}(\mu_k^+)^{\mathrm{w}_k}, -1 + \prod_{k=1}^{ɳ}(1 + \nu_k^-)^{\mathrm{w}_k}\right). \tag{2.4}$$

Here, $\mathrm{w}_k = [\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_{ɳ}]^T$ represents the associated weight vector of these BFNs for $k = 1,2, \dots, ɳ$, with the condition that $\mathrm{w}_k \in [0, 1]$, and the constraint $\sum_{k=1}^{ɳ} \mathrm{w}_k = 1$ holds.

Definition 5. ([27]). Let $\alpha = (\mu_\alpha^+, \nu_\alpha^-)$ denote a BFN. The score function of α is characterized as follows: $\check{S}(\alpha) = \frac{1+\mu_\alpha^+ + \nu_\alpha^-}{2}$, where $\check{S}(\alpha) \in [0, 1]$. The accuracy function is determined as: $\breve{A}(\alpha) = \frac{\mu^+ + \nu^-}{2}$, where $\breve{A}(\alpha) \in [0,1]$.

In accordance with the preceding definitions of Š and Ă, the ordering relations between two BFNs α_1 and α_2 are delineated as follows:

i) If $\check{S}(\alpha_1) < \check{S}(\alpha_2)$ then $\alpha_1 \prec \alpha_2$.

ii) If $\check{S}(\alpha_1) > \check{S}(\alpha_2)$ then $\alpha_1 \succ \alpha_2$.

iii) When $\check{S}(\alpha_1) = \check{S}(\alpha_2)$, further comparison is accomplished through the accuracy function as follows:

a) If $\breve{A}(\alpha_1) < \breve{A}(\alpha_2)$, then $\alpha_1 \prec \alpha_2$.

b) If $\breve{A}(\alpha_1) > \breve{A}(\alpha_2)$, then $\alpha_1 \succ \alpha_2$.

c) If $\breve{A}(\alpha_1) = \breve{A}(\alpha_2)$, then $\alpha_1 \sim \alpha_2$.

3. Dynamic operations on bipolar fuzzy numbers

This section introduces BFDWA and BFDWG operators developed for the BFNs. Fundamental operational laws are formulated with respect to BF dynamic numbers. Additionally, we examine the structural characteristics of the BF dynamic numbers, namely idempotency, boundedness, and monotonicity, within the framework of the BFDWG and BFDWA operators. The structural flowchart of BFDWA and BFDWG operators is displayed in Figure 1.

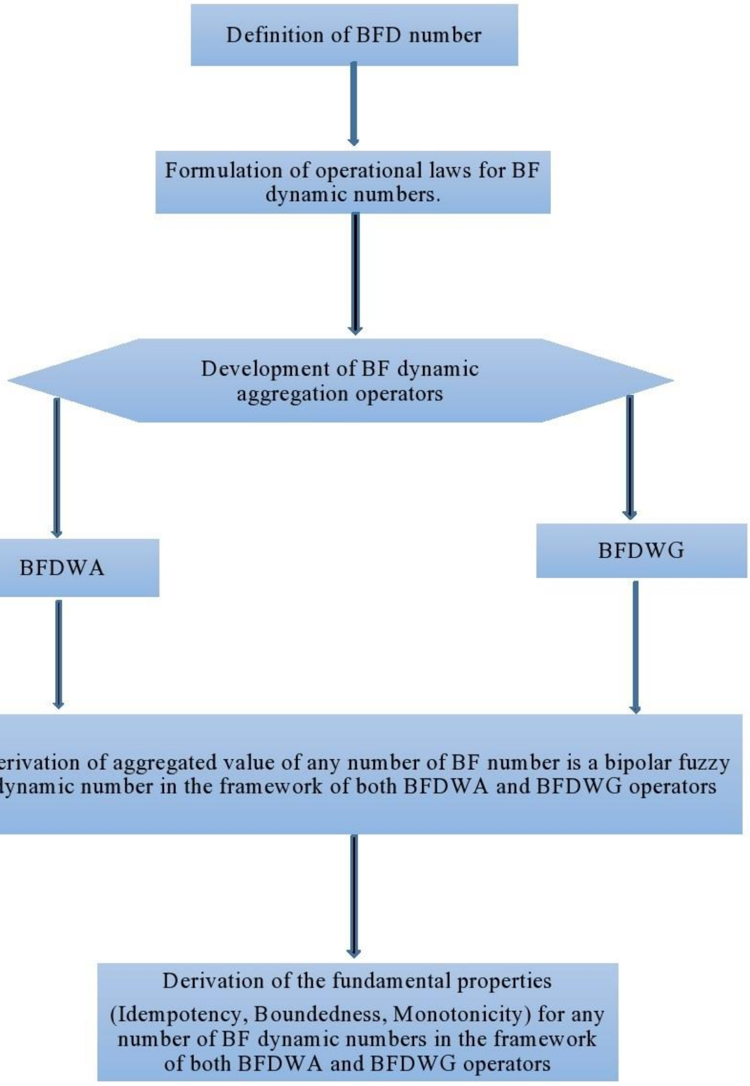

Figure 1. Structural flowchart of BFDWA and BFDWG operators.

Definition 6. For the time variable t, the BFN is formally represented as follows:

$$\alpha_t = (\mu_t^+, \nu_t^-), \text{ where, } \mu_t^+ \in [0,1] \text{ and } \nu_t^- \in [-1,0].$$

Furthermore, if $t = t_1, t_2, \dots, t_\eta$, then $\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}$ signifies η distinct BFNs, each linked with a different time period.

Definition 7. Consider two BFNs, $\alpha_{t_1} = \langle(\mu_{t_1}^+, \nu_{t_1}^-)\rangle$ and $\alpha_{t_2} = \langle(\mu_{t_2}^+, \nu_{t_2}^-)\rangle$. The following are the operational laws that govern their relationship:
i) $\alpha_{t_1} \leq \alpha_{t_2}$ if $\mu_{t_1}^+ \leq \mu_{t_2}^+$ and $\nu_{t_1}^- \geq \nu_{t_2}^-$;
ii) $\alpha_{t_1} = \alpha_{t_2}$ if and only if $\alpha_{t_1} \subseteq \alpha_{t_2}$ and $\alpha_{t_2} \subseteq \alpha_{t_1}$;
iii) $\alpha_{t_1}^c = \{\langle x, 1 - \mu_{t_1}^+, |\nu_{t_1}^-| - 1 | x \in X\rangle\}$.
Definition 8. Let us consider three BFNs, $\alpha_t = \langle(\mu_t^+, \nu_t^-)\rangle$, $\alpha_{t_1} = \langle(\mu_{t_1}^+, \nu_{t_1}^-)\rangle$ and $\alpha_{t_2} = \langle(\mu_{t_2}^+, \nu_{t_2}^-)\rangle$ over X and $\lambda_{t_k} > 0$. We define the following dynamic operations on these BFNs:
i) $\alpha_{t_1} \oplus \alpha_{t_2} = (\langle \mu_{t_1}^+ + \mu_{t_2}^+ - \mu_{t_1}^+\mu_{t_2}^+, -|\nu_{t_1}^-||\nu_{t_2}^-|\rangle)$;
ii) $\alpha_{t_1} \otimes \alpha_{t_2} = (\langle \mu_{t_1}^+\mu_{t_2}^+, \nu_{t_1}^- + \nu_{t_2}^- - \nu_{t_1}^-\nu_{t_2}^-\rangle)$;
iii) $\lambda_t\alpha_t = \left(1 - (1 - \mu_t^+)^{\lambda_t}, -|\nu_t^-|^{\lambda_t}\right)$;
iv) ${\alpha_t}^{\lambda_t} = \left(\mu_t^{\lambda_t}, -1 + |1 + \nu_t^-|^{\lambda_t}\right)$.

We propose dynamic weighted averaging operator for BFNs, referred to as (BFDWA) in the following definition.

Definition 9. Let $\alpha_{t_k} = (\mu_{t_k}^+, \nu_{t_k}^-)$, for $k = 1,2, \dots, \eta$, be a collection of BFNs. A BFDWA operator is a function $BFDWA: \psi^{\eta} \to \psi$ defined as follows:

$$BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \oplus_{k=1}^{\eta} \lambda_{t_k}.\alpha_{t_k}$$
$$= \left(1 - \prod_{k=1}^{\eta}\left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}}, -\prod_{k=1}^{\eta}\left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}}\right).$$

Here $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_\eta}\right]^T$ is a weight vector linked with t_k for $k = 1, 2, \dots, \eta$, and it satisfies the conditions $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$.

Theorem 1. Given a collection of BFNs, denoted as $\alpha_{t_k} = (\mu_{t_k}^+, \nu_{t_k}^-)$, for $k = 1, 2, \dots, \eta$, existing at η distinct time periods. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_\eta}\right]^T$ be the associated weight vector of BFNs α_{t_k}, for $k = 1,2, \dots, \eta$, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$, then the aggregated value obtained through the BFDWA operation also constitutes a BFN. Mathematically, this can be expressed as:

$$BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \left(1 - \prod_{k=1}^{\eta}\left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}}, -\prod_{k=1}^{\eta}\left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}}\right).$$

Proof: The proof of this theorem employs the method of mathematical induction. Specifically, when $\eta = 2$, the BFDWA operation is demonstrated as follows:

$$BFDWA(\alpha_{t_1}, \alpha_{t_2}) = \lambda_{t_1}\alpha_{t_1} \oplus \lambda_{t_2}\alpha_{t_2}$$
$$= \left(1 - \left(1 - \mu_{t_1}^+\right)^{\lambda_{t_1}}, -\left(|\nu_{t_1}^-|\right)^{\lambda_{t_1}}\right) \oplus \left(1 - \left(1 - \mu_{t_2}^+\right)^{\lambda_{t_2}}, -\left(|\nu_{t_2}^-|\right)^{\lambda_{t_2}}\right)$$
$$= \langle 1 - \left\{\left(1 - \mu_{t_1}^+\right)^{\lambda_{t_1}} + \left(1 - \mu_{t_2}^+\right)^{\lambda_{t_2}}\right\}, -\left\{|\nu_{t_1}^-|^{\lambda_{t_1}} + |\nu_{t_2}^-|^{\lambda_{t_2}}\right\}\rangle.$$

Consequently,

$$BFDWA(\alpha_{t_1}, \alpha_{t_2}) = \left(1 - \prod_{k=1}^{2}\left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}}, -\prod_{k=1}^{2}\left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}}\right).$$

This show that the result is true for $\eta = 2$. Let the result holds for $\eta = p$, where p is a natural number.

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_p}\right)=\oplus_{k=1}^{p}\lambda_{t_k}.\alpha_{t_k}=\left(1-\prod_{k=1}^{p}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}},-\prod_{k=1}^{p}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}}\right).$$

Next, we investigate the case where $\eta=p+1$:

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_p},\alpha_{t_{p+1}}\right)=\oplus_{k=1}^{p}\lambda_{t_k}.\alpha_{t_k}\oplus\lambda_{t_{p+1}}.\alpha_{t_{p+1}}=\left(1-\prod_{k=1}^{p}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}},-\prod_{k=1}^{p}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}}\right)\oplus\left(1-\left(1-\mu_{t_{p+1}}^{+}\right)^{\lambda_{t_{p+1}}},-\left(\left|\nu_{t_{p+1}}^{-}\right|\right)^{\lambda_{t_{p+1}}}\right).$$

This implies that,

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_p},\alpha_{t_{p+1}}\right)=\left(1-\prod_{k=1}^{p+1}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}},-\prod_{k=1}^{p+1}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}}\right).$$

Henceforth, the assertion holds for $\eta=p+1$. Therefore, it can be established that the statement is true for all values of natural numbers η.

Example 1. Consider a scenario involving four BFNs $\alpha_{t_1}=(0.5,-0.4)$, $\alpha_{t_2}=(0.6,-0.4)$, $\alpha_{t_3}=(0.7,-0.3)$ and $\alpha_{t_4}=(0.2,-0.3)$ each assigned a weight vector $\lambda_{t_k}=(0.2,0.1,0.3,0.4)^T$, Then

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\oplus_{k=1}^{\eta}\lambda_{t_k}.\alpha_{t_k}$$

$$=\left(1-\prod_{k=1}^{4}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}},-\prod_{k=1}^{4}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}}\right)=$$

$$\begin{pmatrix}1-\{(1-0.5)^{0.2}\times(1-0.6)^{0.1}\times(1-0.7)^{0.3}\times(1-0.2)^{0.4}\},\\ -\{(|-0.4|)^{0.2}\times(|-0.4|)^{0.1}\times(|-0.3|)^{0.3}\times(|0.3|)^{0.4}\}\end{pmatrix}=(0.4937,-0.3270).$$

Theorem 2. (Idempotency Property) Consider a collection of BFNs denoted as $\alpha_{t_k}=\left(\mu_{t_k}^{+},\nu_{t_k}^{-}\right)$ for $k=1,2,\dots,\eta$, where all members in this collection are identical, i.e., for all values of k, $\alpha_{t_k}=\alpha_{t_j}$ for some $j\in\{1,2,\dots,\eta\}$, where, $\alpha_{t_j}=\left(\mu_{t_j}^{+},\nu_{t_j}^{-}\right)$. Let $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2},\dots,\lambda_{t_\eta}\right]^T$ represents the weight vector linked with t_k, for $k=1,2,\dots,\eta$, such that $\lambda_{t_k}\in[0,1]$ and $\sum_{k=1}^{\eta}\lambda_{t_k}=1$. Then, it can be established that:

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\alpha_{t_j}.$$

Proof: Given that $\alpha_{t_k}=\left(\mu_{t_k}^{+},\nu_{t_k}^{-}\right)$, for $k=1,2,\dots,\eta$, and $\alpha_{t_k}=\alpha_{t_j}$ for some j. Then, we proceed to demonstrate the following equality:

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\oplus_{k=1}^{\eta}\lambda_{t_k}\alpha_{t_k}$$

$$=\left(1-\prod_{k=1}^{\eta}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}},-\prod_{k=1}^{\eta}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}}\right)$$

$$=\left(1-\left(1-\mu_{t_j}^{+}\right)^{\sum_{k=1}^{\eta}\lambda_{t_k}},-\left|\nu_{t_j}^{-}\right|^{\sum_{k=1}^{\eta}\lambda_{t_k}}\right)$$

$$=\left(1-\left(1-\mu_{t_j}^{+}\right),-\left|\nu_{t_j}^{-}\right|\right)$$

$$=\left(\mu_{t_j}^{+},-\left|\nu_{t_j}^{-}\right|\right)=\left(\mu_{t_j}^{+},\nu_{t_j}^{-}\right).$$

Consequently,

$$BFDWA\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\alpha_{t_j}.$$

Theorem 3. (Boundedness property) Consider a collection of BFNs denoted as $\alpha_{t_k}=\left(\mu_{t_k}^{+},\nu_{t_k}^{-}\right)$ for $k=1,2,\dots,\eta$. Let $\alpha_t^{-}=\min\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\left(\mu_t^{-},\nu_t^{-}\right)$, $\alpha_t^{+}=\max\left(\alpha_{t_1},\alpha_{t_2},\dots,\alpha_{t_\eta}\right)=\left(\mu_t^{+},\nu_t^{+}\right)$

and $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_ꞃ}\right]^T$ represents the weight vector linked with t_k, for $k = 1,2,\dots,ꞃ$, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{ꞃ} \lambda_{t_k} = 1$, then $\alpha_t^- \le BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_ꞃ}\right) \le \alpha_t^+$.

Proof: Let $\alpha_{t_k} = \left(\mu_{t_k}^+, \nu_{t_k}^-\right)$ represent the collection of BFNs. We define $\acute{\mu}_t^- = \min\left\{\mu_{t_k}^+\right\}, \acute{\nu}_t^- = \max\left\{\nu_{t_k}^-\right\}$, $\acute{\mu}_t^+ = \max\left\{\mu_{t_k}^+\right\}$ and $\acute{\nu}_t^+ = \min\left\{\nu_{t_k}^-\right\}$.

Consequently, we can derive the following inequalities:

$$\acute{\mu}_t^- \le \mu_{t_k}^+ \le \acute{\mu}_t^+$$

$$\Rightarrow 1 - \acute{\mu}_t^- \ge 1 - \mu_{t_k}^+ \ge 1 - \acute{\mu}_t^+$$

$$\Rightarrow \prod_{k=1}^{ꞃ} (1 - \acute{\mu}_t^-)^{\lambda_{t_k}} \ge \prod_{k=1}^{ꞃ} \left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \ge \prod_{k=1}^{ꞃ} (1 - \acute{\mu}_t^+)^{\lambda_{t_k}}$$

$$\Rightarrow 1 - \prod_{k=1}^{ꞃ} (1 - \acute{\mu}_t^-)^{\lambda_{t_k}} \le 1 - \prod_{k=1}^{ꞃ} \left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \le 1 - \prod_{k=1}^{ꞃ} (1 - \acute{\mu}_t^+)^{\lambda_{t_k}}$$

$$\Rightarrow 1 - (1 - \acute{\mu}_t^-)^{\sum_{k=1}^{ꞃ} \lambda_{t_k}} \le 1 - \prod_{k=1}^{ꞃ} \left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \le 1 - (1 - \acute{\mu}_t^+)^{\sum_{k=1}^{ꞃ} \lambda_{t_k}}$$

$$\Rightarrow 1 - (1 - \acute{\mu}_t^-) \le 1 - \prod_{k=1}^{ꞃ} \left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \le 1 - (1 - \acute{\mu}_t^+)$$

$$\Rightarrow \acute{\mu}_t^- \le 1 - \prod_{k=1}^{ꞃ} \left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \le \acute{\mu}_t^+. \tag{3.1}$$

Moreover,

$$\acute{\nu}_t^+ \le \nu_{t_k}^- \le \acute{\nu}_t^-$$

$$\Rightarrow \prod_{k=1}^{ꞃ} (|\acute{\nu}_t^+|)^{\lambda_{t_k}} \le \prod_{k=1}^{ꞃ} \left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}} \le \prod_{k=1}^{ꞃ} (|\acute{\nu}_t^-|)^{\lambda_{t_k}}$$

$$\Rightarrow \prod_{k=1}^{ꞃ} (|\acute{\nu}_t^+|)^{\lambda_{t_k}} \le \prod_{k=1}^{ꞃ} \left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}} \le \prod_{k=1}^{ꞃ} (|\acute{\nu}_t^-|)^{\lambda_{t_k}}$$

$$\Rightarrow (|\acute{\nu}_t^+|)^{\sum_{k=1}^{ꞃ} \lambda_{t_k}} \le \prod_{k=1}^{ꞃ} \left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}} \le (|\acute{\nu}_t^-|)^{\sum_{k=1}^{ꞃ} \lambda_{t_k}}$$

$$\Rightarrow (|\acute{\nu}_t^+|) \le \prod_{k=1}^{ꞃ} \left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}} \le (|\acute{\nu}_t^-|)$$

$$\Rightarrow (|\acute{\nu}_t^-|) \le -\prod_{k=1}^{ꞃ} \left(|\nu_{t_k}^-|\right)^{\lambda_{t_k}} \le (|\acute{\nu}_t^+|). \tag{3.2}$$

By comparing inequality (3.1) and (3.2)

$$\alpha_t^- \le BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_ꞃ}\right) \le \alpha_t^+.$$

Theorem 4. (Monotonicity property) Consider α_{t_k}, for $k = 1,2,\dots,ꞃ$, and $\acute{\alpha}_{t_k}$, for $k = 1,2,\dots,ꞃ$, be two sets of BFNs. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_ꞃ}\right]^T$ represents the weight vector linked with t_k, for $k = 1,2,\dots,ꞃ$, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{ꞃ} \lambda_{t_k} = 1$. If $\mu_{t_k}^+ \le \acute{\mu}_{t_k}^+$ and $\nu_{t_k}^- \ge \acute{\nu}_{t_k}^-$ for all k, then $BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_ꞃ}\right) \le BFDWA\left(\acute{\alpha}_{t_1}, \acute{\alpha}_{t_2}, \dots, \acute{\alpha}_{t_ꞃ}\right)$.

Proof: Assuming that $\mu_{t_k}^+ \le \acute{\mu}_{t_k}^+$ for all k, then

$$\left(1 - \mu_{t_k}^+\right)^{\lambda_{t_k}} \ge (1 - \acute{\mu}_t^+)^{\lambda_{t_k}}$$

$$\Rightarrow \prod_{k=1}^{\eta} \left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}} \geq \prod_{k=1}^{\eta} \left(1-\acute{\mu}_{t_k}^{+}\right)^{\lambda_{t_k}}$$

$$\Rightarrow 1-\prod_{k=1}^{\eta}\left(1-\mu_{t_k}^{+}\right)^{\lambda_{t_k}} \leq 1-\prod_{k=1}^{\eta}\left(1-\acute{\mu}_{t_k}^{+}\right)^{\lambda_{t_k}}.$$

Similarly, by considering $\nu_{t_k}^{-} \geq \acute{\nu}_{t_k}^{-}$, we can derive:

$$-\prod_{k=1}^{\eta}\left(\left|\nu_{t_k}^{-}\right|\right)^{\lambda_{t_k}} \geq -\prod_{k=1}^{\eta}\left(\left|\acute{\nu}_{t_k}^{-}\right|\right)^{\lambda_{t_k}}.$$

Hence,

$$BFDWA\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) \leq BFDWA\left(\acute{\alpha}_{t_1}, \acute{\alpha}_{t_2}, \dots, \acute{\alpha}_{t_\eta}\right).$$

Definition 10. Let $\alpha_{t_k} = \left(\mu_{t_k}^{+}, \nu_{t_k}^{-}\right)$, for $k = 1,2,\dots,\eta$, be a collection of BFNs. A bipolar fuzzy dynamic weighted geometric (BFDWG) operator is a function $BFDWG: \psi^{\eta} \to \psi$ defined as follows:

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \otimes_{k=1}^{\eta} {\alpha_{t_k}}^{\lambda_{t_k}}$$

$$= \left(\prod_{k=1}^{\eta} (\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{\eta} \left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right).$$

Where $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_\eta}\right]^T$ is a weight vector linked with t_k for $k = 1, 2, \dots, \eta$, and it satisfies the conditions $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$.

Theorem 5. Given a collection of BFNs, denoted as $\alpha_{t_k} = \left(\mu_{t_k}^{+}, \nu_{t_k}^{-}\right)$, for $k = 1, 2, \dots, \eta$, existing at η distinct time periods. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_\eta}\right]^T$ be the associated weight vector of BFNs α_{t_k}, for $k = 1,2,\dots,\eta$, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$, then the aggregated value obtained through the BFDWG operation also constitutes a BFN. Mathematically, this can be expressed as:

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \otimes_{k=1}^{\eta} {\alpha_{t_k}}^{\lambda_{t_k}}$$

$$= \left(\prod_{k=1}^{\eta} (\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{\eta} \left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right).$$

Proof: We will use the mathematical induction approach to prove this result. Specifically, when $\eta = 2$, the BFDWG operation is demonstrated as follows:

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}\right) = {\alpha_{t_1}}^{\lambda_{t_1}} \otimes {\alpha_{t_2}}^{\lambda_{t_2}}$$

$$= \left((\mu_{t_1}^{+})^{\lambda_{t_1}}, -1 + \left(1+\nu_{t_1}^{-}\right)^{\lambda_{t_1}}\right) \otimes \left((\mu_{t_2}^{+})^{\lambda_{t_2}}, -1 + \left(1+\nu_{t_2}^{-}\right)^{\lambda_{t_2}}\right)$$

$$= \langle\left\{(\mu_{t_1}^{+})^{\lambda_{t_1}} + (\mu_{t_2}^{+})^{\lambda_{t_2}}\right\}, -1\left\{\left(1+\nu_{t_1}^{-}\right)^{\lambda_{t_1}} + \left(1+\nu_{t_2}^{-}\right)^{\lambda_{t_2}}\right\}\rangle.$$

Consequently,

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}\right) = \left(\prod_{k=1}^{2} (\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{2} \left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right).$$

Hence, confirming the validity of the theorem for the case when $\eta = 2$. Let the result be true for $\eta = p$, where $p \in \mathbb{N}$.

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_p\right) = \otimes_{k=1}^{p} {\alpha_{t_k}}^{\lambda_{t_k}}$$

$$= \left(\prod_{k=1}^{p}(\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{p}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right).$$

Next, we investigate the case where $\eta = p + 1$:

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_{p+1}}\right) = \otimes_{k=1}^{p} {\alpha_{t_k}}^{\lambda_{t_k}} \otimes {\alpha_{t_{p+1}}}^{\lambda_{t_{p+1}}}$$

$$= \left(\prod_{k=1}^{p}(\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{p}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right)$$

$$\otimes \left((\mu_{t_{p+1}}^{+})^{\lambda_{t_{p+1}}}, -1 + \left(1+\nu_{t_{p+1}}^{-}\right)^{\lambda_{t_{p+1}}}\right).$$

This implies that,

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_{p+1}}\right) = \left(\prod_{k=1}^{p+1}(\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{p+1}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right).$$

Henceforth, the assertion holds for $\eta = p + 1$. Thus, the result is true for all $\eta \in \mathbb{N}$.

Example 2. There are four BFNs $\alpha_{t_1} = (0.7, -0.3)$, $\alpha_{t_2} = (0.6, -0.4)$, $\alpha_{t_3} = (0.6, -0.5)$ and $\alpha_{t_4} = (0.1, -0.2)$ and $\lambda_{t_k} = (0.2, 0.1, 0.3, 0.4)^T$ are weight vectors for these BFNs α_{t_k}, then,

$$= \left(\prod_{k=1}^{4}(\mu_{t_k}^{+})^{\lambda_{t_k}}, -1 + \prod_{k=1}^{4}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right)$$

$$= \begin{pmatrix} \{(0.7)^{02} \times (0.6)^{0.1} \times (0.6)^{0.3} \times (0.1)^{0.4}\}, \\ -1 + \{(1-0.3)^{0.2} \times (1-0.4)^{0.1} \times (1-0.5)^{0.3} \times (1-0.2)^{0.4}\} \end{pmatrix}$$

$$= (0.3021, -0.3427).$$

Theorem 6. (Idempotency Property) Consider a collection of BFNs denoted as $\alpha_{t_k} = \left(\mu_{t_k}^{+}, \nu_{t_k}^{-}\right)$ for $k = 1, 2, \dots, \eta$, where all members in this collection are identical, i.e., for all values of k, $\alpha_{t_k} = \alpha_{t_j}$ for some $j \in \{1,2,\dots,\eta\}$, where, $\alpha_{t_j} = \left(\mu_{t_j}^{+}, \nu_{t_j}^{-}\right)$. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_\eta}\right]^T$ represents the weight vector linked with t_k, for $k = 1,2,\dots,\eta$, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$. Then,

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \alpha_{t_j}.$$

Proof: Given that $\alpha_{t_k} = \left(\mu_{t_k}^{+}, \nu_{t_k}^{-}\right)$, for $k = 1,2,\dots,\eta$, and $\alpha_{t_k} = \alpha_{t_j}$ for some j. Then, we have

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \otimes_{k=1}^{p} {\alpha_{t_k}}^{\lambda_{t_k}}$$

$$= \left(\prod_{k=1}^{\eta}\left(\mu_{t_k}^{+}\right)^{\lambda_{t_k}}, -1 + \prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\right)$$

$$= \left((\mu_{t_j}^{+})^{\sum_{k=1}^{\eta}\lambda_{t_k}}, -1 + \left(1+\nu_{t_j}^{-}\right)^{\sum_{k=1}^{\eta}\lambda_{t_k}}\right)$$

$$= \left(\mu_{t_j}^{+}, -1 + \left(1+\nu_{t_j}^{-}\right)\right) = \left(\mu_{t_j}^{+}, \nu_{t_j}^{-}\right).$$

Consequently,

$$BFDWG\left(\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_\eta}\right) = \alpha_{t_j}.$$

Theorem 7. (Boundedness property) Consider a collection of BFNs denoted as $\alpha_{t_k}=\left(\mu_{t_k}^{+},\nu_{t_k}^{-}\right)$ for $k=1,2,\ldots,\eta$. Let $\alpha_t^{-}=\min\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right)=(\mu_t^{-},\nu_t^{-})$, $\alpha_t^{+}=\max\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right)=(\mu_t^{+},\nu_t^{+})$ and $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2},\ldots,\lambda_{t_\eta}\right]^T$ represents the weight vector linked with t_k, for $k=1,2,\ldots,\eta$, such that $\lambda_{t_k}\in[0,1]$ and $\sum_{k=1}^{\eta}\lambda_{t_k}=1$, then $\alpha_t^{-}\leq BFDWG\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right)\leq\alpha_t^{+}$.

Proof: Let $\alpha_{t_k}=\left(\mu_{t_k}^{+},\nu_{t_k}^{-}\right)$ be collection of BFNs. We define $\acute{\mu}_t^{-}=\min\left\{\mu_{t_k}^{+}\right\}$, $\acute{\nu}_t^{-}=\max\left\{\nu_{t_k}^{-}\right\}$, $\acute{\mu}_t^{+}=\max\left\{\mu_{t_k}^{+}\right\}$ and $\acute{\nu}_t^{+}=\min\left\{\nu_{t_k}^{-}\right\}$.

Consequently, we can derive the following inequalities:

$$(\acute{\mu}_t^{-})^{\lambda_{t_k}}\leq\mu_{t_k}^{+}\leq(\acute{\mu}_t^{+})^{\lambda_{t_k}}$$

$$\Rightarrow\prod_{k=1}^{\eta}(\acute{\mu}_t^{-})^{\lambda_{t_k}}\leq\prod_{k=1}^{\eta}\left(\mu_{t_k}^{+}\right)^{\lambda_{t_k}}\leq\prod_{k=1}^{\eta}(\acute{\mu}_t^{+})^{\lambda_{t_k}}$$

$$\Rightarrow(\acute{\mu}_t^{-})^{\sum_{k=1}^{\eta}\lambda_{t_k}}\leq\prod_{k=1}^{\eta}\left(\mu_{t_k}^{+}\right)^{\lambda_{t_k}}\leq(\acute{\mu}_t^{+})^{\sum_{k=1}^{\eta}\lambda_{t_k}}.$$

Hence,

$$(\acute{\mu}_t^{-})\leq\prod_{k=1}^{\eta}\left(\mu_{t_k}^{+}\right)^{\lambda_{t_k}}\leq(\acute{\mu}_t^{+}).\tag{3.3}$$

Moreover,

$$\acute{\nu}_t^{+}\leq\nu_{t_k}^{-}\leq\acute{\nu}_t^{-}$$

$$\Rightarrow 1+\acute{\nu}_t^{+}\leq 1+\nu_{t_k}^{-}\leq 1+\acute{\nu}_t^{-}$$

$$\Rightarrow\prod_{k=1}^{\eta}(1+\acute{\nu}_t^{+})^{\lambda_{t_k}}\leq\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\leq\prod_{k=1}^{\eta}(1+\acute{\nu}_t^{-})^{\lambda_{t_k}}$$

$$\Rightarrow\prod_{k=1}^{\eta}(1+\acute{\nu}_t^{+})^{\lambda_{t_k}}\leq+\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\leq\prod_{k=1}^{\eta}(1+\acute{\nu}_t^{-})^{\lambda_{t_k}}$$

$$\Rightarrow(1+\acute{\nu}_t^{+})^{\sum_{k=1}^{\eta}\lambda_{t_k}}\leq\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\leq(1+\acute{\nu}_t^{-})^{\sum_{k=1}^{\eta}\lambda_{t_k}}$$

$$\Rightarrow 1+\acute{\nu}_t^{+}\leq\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\leq 1+\acute{\nu}_t^{-}$$

$$\Rightarrow\acute{\nu}_t^{-}\leq-1+\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}}\leq\acute{\nu}_t^{+}.\tag{3.4}$$

Inequalities (3.3) and (3.4) provide a conclusion

$$\alpha_t^{-}\leq BFDWG\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right)\leq\alpha_t^{+}.$$

Theorem 8. (Monotonicity property) consider α_{t_k}, for $k=1,2,\ldots,\eta$, and $\acute{\alpha}_{t_k}$ for $k=1,2,\ldots,\eta$, be two sets of BFNs. Let $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2},\ldots,\lambda_{t_\eta}\right]^T$ represents the weight vector linked with t_k, for $k=1,2,\ldots,\eta$, such that $\lambda_{t_k}\in[0,1]$ and $\sum_{k=1}^{\eta}\lambda_{t_k}=1$. If $\mu_{t_k}^{+}\leq\acute{\mu}_{t_k}^{+}$ and $\nu_{t_k}^{-}\geq\acute{\nu}_{t_k}^{-}$ for all k, then

$$BFDWG\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right)\leq BFDWG\left(\acute{\alpha}_{t_1},\acute{\alpha}_{t_2},\ldots,\acute{\alpha}_{t_\eta}\right).$$

Proof: Since $\mu_{t_k}^{+}\leq\acute{\mu}_{t_k}^{+}$ for all k, then

$$\prod_{k=1}^{\eta}\left(\mu_{t_k}^{+}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{\eta}(\acute{\mu}_{t_k}^{+})^{\lambda_{t_k}}.$$

Next, assuming that $\nu_{t_k}^{-} \geq \acute{\nu}_{t_k}^{-}$, we derive:

$$1+\nu_{t_k}^{-} \geq 1+\acute{\nu}_{t_k}^{-}$$

$$\Rightarrow \prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}} \geq \prod_{k=1}^{\eta}\left(1+\acute{\nu}_{t_k}^{-}\right)^{\lambda_{t_k}}$$

$$\Rightarrow -1+\prod_{k=1}^{\eta}\left(1+\nu_{t_k}^{-}\right)^{\lambda_{t_k}} \leq -1+\prod_{k=1}^{p}\left(1+\acute{\nu}_{t_k}^{-}\right)^{\lambda_{t_k}}.$$

Hence,

$$BFDWG\left(\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_\eta}\right) \leq BFDWG\left(\acute{\alpha}_{t_1},\acute{\alpha}_{t_2},\ldots,\acute{\alpha}_{t_\eta}\right).$$

4. Mathematical mechanisms for solving MADM problems in bipolar fuzzy dynamic environment

Here, we introduce an innovative MADM method for bipolar fuzzy information. This method uses bipolar fuzzy dynamic weighted aggregation operators. The following steps outline the suggested approach's framework:

Step 1: Consider a discrete set of alternatives $\tilde{A} = \{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m\}$.

Step 2: Consider a set $H = \{H_1, H_2, \ldots, H_n\}$ consisting of attributes and $w = [w_1, w_2, \ldots, w_n]^T$ is weight vectors where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Step 3: There are η different periods t_k, for $k = 1,2,\ldots,\eta$, whose weight vector is $\lambda_{t_k} = \left[\lambda_{t_1},\lambda_{t_2},\ldots,\lambda_{t_\eta}\right]^T$, where $\lambda_{t_k} \in [0,1]$, and $\sum_{k=1}^{\eta} \lambda_{t_k} = 1$.

Step 4: Suppose that $R(t_k) = [r_{ij(t_k)}]_{m\times n} = \left\langle\left(\mu_{ij(t_k)}^{+},\nu_{ij(t_k)}^{-}\right)\right\rangle_{m\times n}$ is the bipolar fuzzy decision matrix at periods t_k where $\mu_{ij(t_k)}^{+}$ indicates the degree that the alternative $\tilde{A}_i$ satisfies the attribute H_i at periods t_k and $\nu_{ij(t_k)}^{-}$ indicates the degree that the alternative $\tilde{A}_i$ doesn't satisfy the attribute H_i at periods t_k, such that

$$\mu_{ij(t_k)}^{+} \in [0,1], \nu_{ij(t_k)}^{-} \in [-1,0], \text{ where } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n.$$

We develop two procedures based on BFDWA and BFDWG operators to rank the alternatives based on the decision information.

4.1. Procedure for BFDWA

Step 1. The BFDWA operator is used to aggregate a set of bipolar fuzzy decision matrices into a collective decision matrix R.

$$r_{ij(t_k)} = \left(\mu_{ij(t_k)}^{+},\nu_{ij(t_k)}^{-}\right) = BFDWA\left(r_{ij(t_1)},r_{ij(t_2)},\ldots,r_{ij(t_\eta)}\right)$$
$$= \left(1-\prod_{k=1}^{\eta}\left(1-\mu_{ij(t_k)}^{+}\right)^{\lambda_{t_k}}, -\prod_{k=1}^{\eta}\left(\left|\nu_{ij(t_k)}^{-}\right|\right)^{\lambda_{t_k}}\right).$$

Step 2. The BFWA operator is implemented to calculate the cumulative preference values r_i for the alternative $\tilde{A}_i$, with i ranging from 1 to m. In this context, $w = [w_1, w_2, \ldots, w_n]^T$ represent the weight vectors of the attribute.

$$r_i = (\mu_i^+, \nu_i^-) = BFWA(r_{i1}, r_{i2}, \dots, r_{in^T})$$
$$= \left(1 - \prod_{j=1}^{n}(1-\mu_i^+)^{w_j}, -\prod_{j=1}^{n}(|\nu_i^-|)^{w_j}\right).$$

Step 3. Determine the scores $\check{S}(r_i)$ of the overall bipolar fuzzy preference values r_i in order to rank all alternatives $\tilde{\mathrm{A}}_i$ by using Definition 5.

Step 4. In accordance with $\check{S}(r_i)$ the alternatives $\tilde{\mathrm{A}}_i$ will be ranked and the best one will be selected. The step-by-step procedure for the suggested technique is visually depicted in Figure 2.

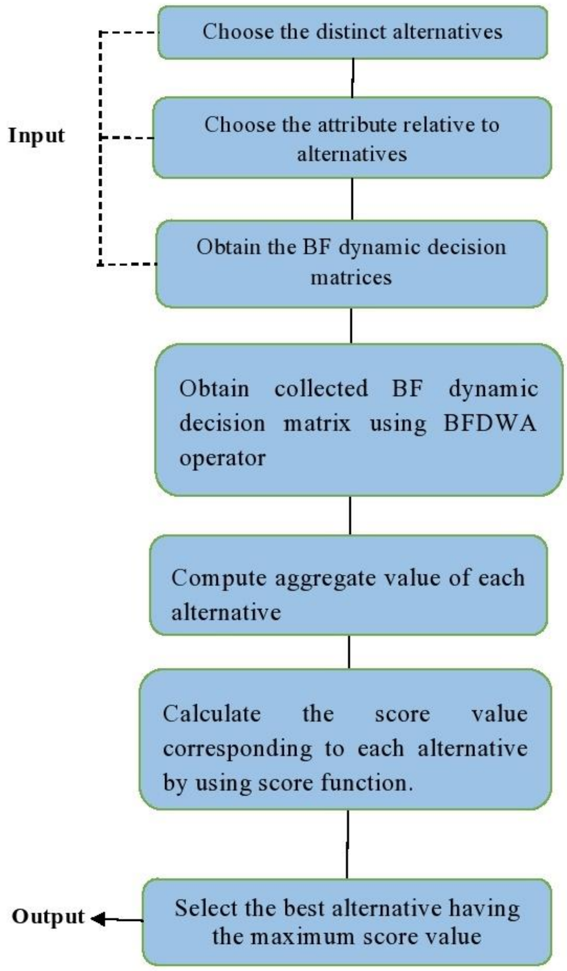

Figure 2. Flow chart of the algorithm for solving MADM problems using BFDWA operator.

4.2. Procedure for BFDWG

Step 1. The BFDWG operator is used to aggregate a set of bipolar fuzzy decision matrices into a collective decision matrix R.

$$r_{ij(t_k)} = \left(\mu_{ij(t_k)}^+, \nu_{ij(t_k)}^-\right) = BFDWG\left(r_{ij(t_1)}, r_{ij(t_2)}, \dots, r_{ij(t_\eta)}\right)$$
$$= \left(\prod_{k=1}^{\eta}\left(\mu_{ij(t_k)}^+\right)^{\lambda_{t_k}}, -1 + \prod_{k=1}^{\eta}\left(1 + \nu_{ij(t_k)}^-\right)^{\lambda_{t_k}}\right).$$

Step 2. The BFWG operator is implemented to calculate the cumulative preference values r_i for the alternative $\tilde{\tilde{A}}_i$, with i ranging from 1 to m. In this context, $w = (w_1, w_2, \dots, w_n)^T$ represent the weight vectors of the attribute.

$$r_i = \left(\mu_{ij}^{+}, \nu_{ij}^{-}\right) = BFWG(r_{i1}, r_{i2}, \dots, r_{in})$$

$$= \left(\prod_{j=1}^{n}(\mu_{ij}^{+})^{\lambda_k}, -1 + \prod_{j=1}^{n}\left(1 + \nu_{ij}^{-}\right)^{\lambda_k}\right).$$

Step 3. Determine the scores $\check{S}(r_i)$ of the overall bipolar fuzzy preference values r_i in order to rank all alternatives $\tilde{\tilde{A}}_i$ by using Definition 5.

Step 4. In accordance with $\check{S}(r_i)$ the alternatives $\tilde{\tilde{A}}_i$ will be ranked and the best one will be selected.

Figure 3 provides an illustrative flowchart of the suggested process.

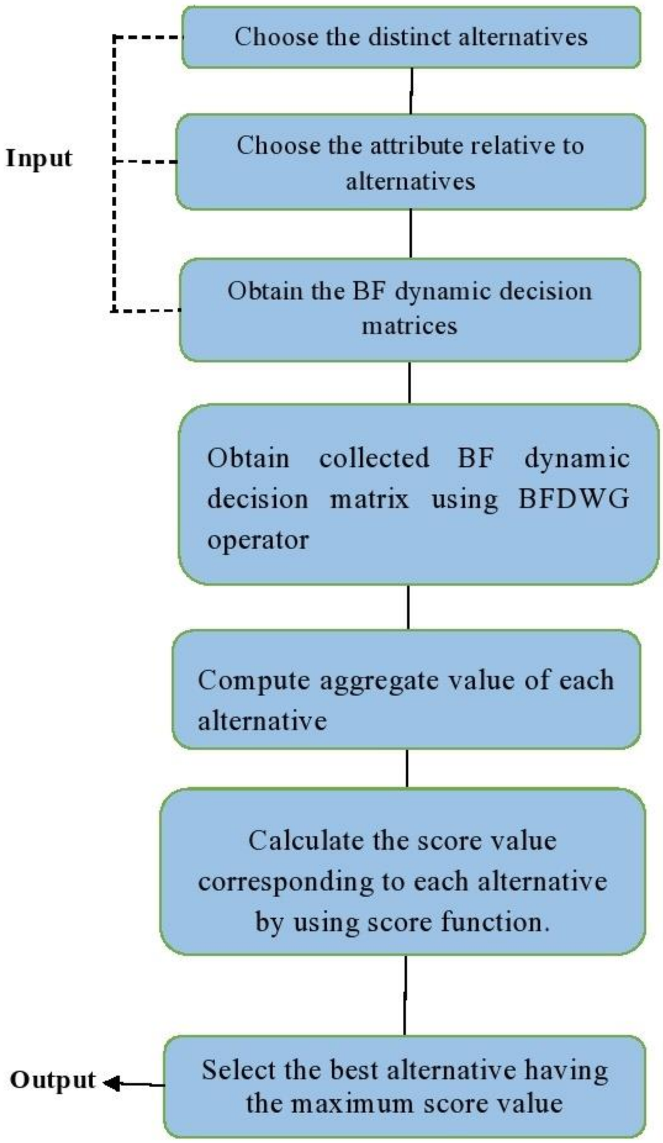

Figure 3. Flow chart of the algorithm for solving MADM problems using BFDWG operator.

5. A numerical example for selecting the optimal emerging technology enterprise under bipolar fuzzy dynamic weighted aggregation settings

In this section, we successfully implement the suggested approaches to select appropriated emerging technology enterprise under BF environment.

Since bipolar fuzzy sets explicitly handle positive and negative information, they are able to depict uncertainty more subtly than IFS, q-ROF and IVIFS. They exhibit remarkable competence in scenarios involving decision-making that necessitate meticulous evaluation of both favorable and unfavorable aspects, therefore enhancing the comprehensibility of the results. While IFS and IVIFS can handle uncertainty, they could not be as precise as is needed to represent bipolarity correctly. With its unique mathematical properties, q-ROF is an appropriate choice for particular applications.

5.1. Case study

Enterprise systems represent a confluence of software applications, modular extensions, and comprehensive databases, constituting the foundational framework underpinning an organization’s operational activities, procedural workflows, and strategic decision-making processes [31,32]. The widespread adoption of enterprise systems by enterprises is attributed to their innate capability to seamlessly integrate disparate functional domains and business processes, operating efficaciously at both intra-organizational and inter-organizational echelons. Enterprise systems consistently outperform organization-wide information systems (IS) due to advances in information technology, the growing prevalence of technological solutions, and the accessibility of diverse system paradigms, including people-centric and transaction-centric models.

The development of enterprise systems can be scrutinized through a five-stage framework, as articulated by Wang et al. [33] in 2013. In the initial stage, designated as the first phase, these systems assume an application-centric character. They comprise localized subsystems that facilitate data processing and decision-making by harnessing database technology. These systems offer support to organizational departments endeavoring to assimilate legacy systems. Notably, they serve as repositories for structured item data, notably evident in systems like Material Resource Planning (MRP), which are tailored for production planning. Subsequently, in their second lifecycle stage, enterprise systems pivot towards a data-centric orientation. Enterprise systems such as Enterprise Resource Planning, leverage client-server architecture to optimize efficiency by integrating data across the entire organization. In the third stage of development, these systems assume a process-centric approach, extending their reach to multiple sites. They enhance efficiency and support the supply chain by utilizing Internet technology and drawing upon integrated data from a spectrum of stakeholders, including colleagues, suppliers, clients, and horizontal collaborators within the supply chain.

Transitioning to the fourth stage, enterprise systems adopt a human-centric focus. In the fifth stage, they shift to object-centric configurations, emphasizing resilience through real-time data and cloud computing. At this juncture, things-centric systems, underpinned by smart objects and cloud-connected sensors, form the foundation for generating and analyzing vast volumes of data, as elaborated by Panetto et al. [34]. These technologies are purposefully designed for universal deployment within fifth-generation (5G) broadband cellular networks, characterized by significantly augmented capacity to facilitate concurrent communication across multiple devices. They function as plug-and-play devices, underscoring their paramount emphasis on internet connectivity and seamless interoperability. Firms employ these systems with the objective of “managing and leveraging all potential networked connections involving individuals, processes, data, objects, and services to attain strategic objectives". To achieve this, companies necessitate the acquisition of appropriate technological tools and resources. Enterprise systems have emerged as indispensable constituents of organizational infrastructure, serving dual roles as repositories for organizational data and collectors of pertinent information. In addition to their integrative functions, these systems are instrumental in archiving valuable organizational data. Previously subjected to rudimentary analysis, this data, although fundamental, held limited potential for managerial decision-making. With the advent of advanced technology, the processes of data collection and analysis have evolved, growing in both complexity and diversity. Consequently, there have been substantial improvements in the quality of information accessible to management for the purposes of strategic planning and decision-making. Furthermore, the influence of technology that augments human expertise has reshaped the manner in which managers harness and apply this information to further enhance decision-making processes.

5.2. Illustration

The significance of choosing an emerging technology enterprise has grown exponentially due to the swift evolution and ubiquitous integration of information technology. This section aims to identify optimal performance among various emerging technology business entities, thereby presenting a quantitative outcome that substantiates the prospective evaluation of technology commercialization using bipolar fuzzy information. A committee is tasked with the selection of five potential emerging technology enterprises, denoted as $\hat{\tilde{A}}_i$ for $i = 1, 2, \ldots, 5$, from a pool of available alternatives.

In this study, we employ a comprehensive framework to assess the performance of five prominent emerging technology enterprises. The evaluation is based on four pivotal criteria: H_1, "Technical Advancement", which scrutinizes their innovations and contributions to cutting-edge technologies; H_2, "Potential Market and Market Risk", evaluating market size, growth potential, and associated risks; H_3, "Financial Conditions, Industrialization Infrastructure and Human Resources" appraising their operational infrastructure, workforce, and financial stability; and H_4, "Employment Creation and Development of Science and Technology", gauging their impact on job creation and contributions to scientific and technological progress. These criteria collectively provide a comprehensive view of their performance in the dynamic landscape of emerging technologies.

Let $\lambda_{t_k} = (0.39, 0.28, 0.33)$ denote the weight vectors corresponding to three distinct time periods, denoted as t_1, t_2 and t_3, representing the years *2020, 2021* and *2022*, respectively. Additionally, let $(0.32, 0.24, 0.16, 0.28)^T$ be the associated weight vector assigned to the attributes.

The evaluation of five potential alternatives is to be conducted using bipolar fuzzy information, as provided by the decision maker, within the context of the four attributes outlined above, and across the aforementioned time periods. These assessments are documented in Tables 1, 2 and 3.

Table 1. Decision matrix $R(t_1)$.

$$\begin{bmatrix} (0.2,-0.1) & (0.4,-0.2) & (0.5,-0.3) & (0.6,-0.3) \\ (0.6,-0.4) & (0.4,-0.1) & (0.6,-0.4) & (0.4,-0.5) \\ (0.2,-0.3) & (0.5,-0.2) & (0.4,-0.1) & (0.2,-0.6) \\ (0.7,-0.4) & (0.3,-0.4) & (0.3,-0.4) & (0.7,-0.5) \\ (0.6,-0.2) & (0.4,-0.1) & (0.6,-0.5) & (0.3,-0.2) \end{bmatrix}$$

Table 2. Decision matrix $R(t_2)$.

$$\begin{bmatrix} (0.6,-0.2) & (0.5,-0.2) & (0.4,-0.2) & (0.4,-0.1) \\ (0.3,-0.1) & (0.3,-0.1) & (0.5,-0.2) & (0.6,-0.2) \\ (0.4,-0.2) & (0.2,-0.4) & (0.3,-0.4) & (0.3,-0.5) \\ (0.7,-0.3) & (0.7,-0.4) & (0.6,-0.3) & (0.4,-0.2) \\ (0.5,-0.2) & (0.4,-0.2) & (0.5,-0.4) & (0.4,-0.5) \end{bmatrix}$$

Table 3. Decision matrix $R(t_3)$.

$$\begin{bmatrix} (0.6,-0.2) & (0.7,-0.2) & (0.3,-0.5) & (0.4,-0.2) \\ (0.3,-0.1) & (0.4,-0.2) & (0.5,-0.2) & (0.7,-0.3) \\ (0.2,-0.4) & (0.2,-0.3) & (0.2,-0.6) & (0.3,-0.1) \\ (0.4,-0.5) & (0.2,-0.1) & (0.3,-0.2) & (0.3,-0.1) \\ (0.5,-0.2) & (0.5,-0.4) & (0.6,-0.1) & (0.4,-0.3) \end{bmatrix}$$

The MADM problem that has been presented is addressed in relation to the BFDWA and BFDWG operators. In order to tackle this complex decision problem, we provide an analysis of the results and methodologies of two distinct approaches.

5.3. Method I (BFDWA operator)

Step 1. The initial phase involves the application of the BFDWA operator to consolidate a set of bipolar fuzzy decision matrices into a unified decision matrix R_1, as displayed in Table 4.

Table 4. Unified decision matrix R_1 evolved through BFDWA operator

$$\begin{bmatrix} (0.4758,-0.1526) & (0.5464,-0.2000) & (0.4120,-0.3170) & (0.4878,-0.1929) \\ (0.4373,-0.1717) & (0.3735,-0.1257) & (0.5417,-0.2621) & (0.5739,-0.3268) \\ (0.2619,-0.2945) & (0.3339,-0.2776) & (0.3111,-0.4573) & (0.2626,-0.3156) \\ (0.6229,-0.3972) & (0.4230,-0.2532) & (0.4015,-0.2936) & (0.5182,-0.2275) \\ (0.5417,-0.2000) & (0.4350,-0.1919) & (0.5742,-0.2761) & (0.3628,-0.2955) \end{bmatrix}$$

Step 2. Subsequently, we employ BFWA operator to compute the cumulative preference values r_i for the alternative $\tilde{\text{Å}}_i$, with i ranging from 1 to 5.

$$\check{r}_1 = (0.4876,-0.1955), \check{r}_2 = (0.4831,-0.2041), \check{r}_3 = (0.2879,-0.2913)$$

$$\check{r}_4 = (0.5184,-0.2906), \check{r}_5 = (0.4777,-0.2326).$$

Step 3. Next, we determine the scores $\check{S}(r_i)$ of the overall bipolar fuzzy preference values r_i in order to rank $\tilde{\text{Å}}_i$.

$$\check{S}(\check{r}_1) = 0.6488, \check{S}(\check{r}_2) = 0.6395, \check{S}(\check{r}_3) = 0.5057$$

$$\check{S}(\check{r}_4) = 0.6139, \check{S}(\check{r}_5) = 0.6226.$$

Step 4. Finally, based on the scores $\check{S}(r_i)$, we rank the alternatives $\tilde{\text{Å}}_i$, with the best one being selected. The ranking of the emerging technology enterprises is as follows:

$$\H{A}_1 \succ \H{A}_2 \succ \H{A}_5 \succ \H{A}_4 \succ \H{A}_3.$$

The aforementioned process is visually illustrated in Figure 4, which displays the numerical values of the alternatives derived by the BFDWA operator.

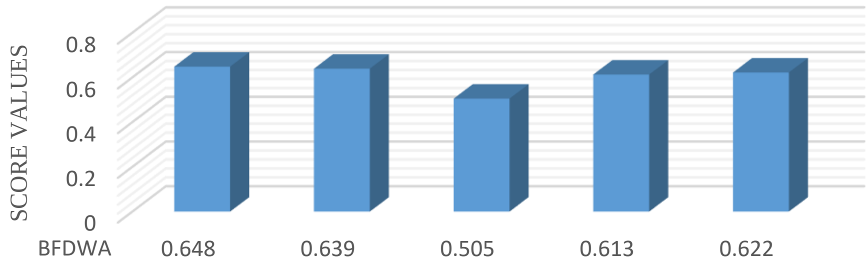

Figure 4. Ranking of alternatives using BFDWA.

5.4. Method II (BFDWG operator)

Step 1. The BFDWG operator is applied to amalgamate a set of bipolar fuzzy decision matrices into a unified decision matrix R_2, as displayed in Table 5.

Table 5. Unified decision matrix R_2 evolved through BFDWA operator

$$\begin{bmatrix} (0.3909, -0.1624) & (0.5122, -0.2000) & (0.3968, -0.3497) & (0.4685, -0.2151) \\ (0.3931, -0.2316) & (0.3690, -0.1343) & (0.5368, -0.2849) & (0.5390, -0.3627) \\ (0.2428, -0.3094) & (0.2859, -0.2937) & (0.2936, -0.3852) & (0.2561, -0.4436) \\ (0.5820, -0.4101) & (0.3327, -0.3141) & (0.3643, -0.3111) & (0.4525, -0.3076) \\ (0.5368, -0.2000) & (0.4306, -0.2383) & (0.5701, -0.3611) & (0.3575, -0.3289) \end{bmatrix}$$

Step 2. Subsequently, we employ BFWG operator to compute the cumulative preference values r_i for the alternative $\H{A}_i$.

$$\check{r}_1 = (0.4391, -0.2188), \check{r}_2 = (0.4446, -0.2582, \check{r}_2 = (0.2642, -0.3585),$$

$$\check{r}_4 = (0.4400, -0.3442), \check{r}_5 = (0.4588, -0.2739).$$

Step 3. Next, we determine the scores $\check{S}(r_i)$ of the overall bipolar fuzzy preference values r_i in order to rank $\H{A}_i$.

$$\check{S}(\check{r}_1) = 0.6102, \check{S}(\check{r}_2) = 0.5932, \check{S}(\check{r}_3) = 0.4528,$$

$$\check{S}(\check{r}_4) = 0.5479, \check{S}(\check{r}_5) = 0.5925.$$

Step 4. Finally, based on the scores $\check{S}(r_i)$, we rank the alternatives $\H{A}_i$, with the best one being selected. The ranking of the emerging technology enterprises is as follows:

$$\H{A}_1 \succ \H{A}_2 \succ \H{A}_5 \succ \H{A}_4 \succ \H{A}_3.$$

Consequently, the most favorable emerging technology enterprise identified is Microsoft Corporation.

The process described above is graphically illustrated in Figure 5, where the numerical values of the alternatives generated by the BFDWA operator are presented.

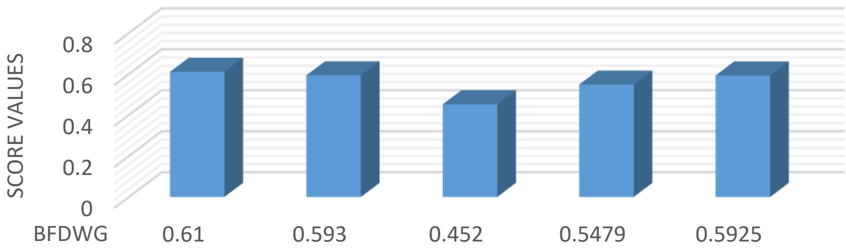

Figure 5. Ranking of alternatives using BFDWG.

5.5. *Comparative analysis*

Here, we compare the efficacy of our proposed approaches to MADM strategies presented in [27] and [35]. In [27] Wei et al. introduce BF Hamacher aggregation operators (BFHWA and BFHWG) for MADM problems. Similarly, in [35] Jana et al. introduce BF Dombi aggregation operators (BFDoWA and BFDoWG) for MADM problem. We apply BFDoWA and BFHWA to R_1 and BFDoWG and BFHWG to R_2 and calculate the aggregated values. Table 6 displays the aggregated values derived from the implementation of BFDoWA and BFDoWG operators, while Table 7 delineates the aggregated values obtained through the utilization of BFHWA and BFHWG operators. The rankings of the existing operators for BFNs are compared to those of the suggested operators in Table 8.

Table 6. Aggregated values derived from the implementation of BFDoWA and BFDoWG operators.

Alternative	BFDoWA	BFDoWG
$\tilde{\hat{A}}_1$	$(0.5051, -0.2830)$	$(0.4970, -0.6642)$
$\tilde{\hat{A}}_2$	$(0.5285, -0.2658)$	$(0.5014, -0.6391)$
$\tilde{\hat{A}}_3$	$(0.5442, -0.3914)$	$(0.3659, -0.5289)$
$\tilde{\hat{A}}_4$	$(0.4376, -0.3073)$	$(0.4247, -0.4172)$
$\tilde{\hat{A}}_5$	$(0.4538, -0.3147)$	$(0.4347, -0.6025)$

Table 7. Aggregated values derived from the implementation of BFHWA and BFHWG operators.

Alternative	BFHWA	BFHWG
$\tilde{\hat{A}}_1$	$(0.5023, -0.2264)$	$(0.5003, -0.4858)$
$\tilde{\hat{A}}_2$	$(0.5201, -0.2255)$	$(0.5133, -0.4811)$
$\tilde{\hat{A}}_3$	$(0.4639, -0.2663)$	$(0.4157, -0.4714)$
$\tilde{\hat{A}}_4$	$(0.4327, -0.2275)$	$(0.4294, -0.3729)$
$\tilde{\hat{A}}_5$	$(0.4464, -0.2390)$	$(0.4416, -0.4708)$

Table 8. Hierarchical ranking of alternatives utilizing various BF operators.

Methods	Ranking order
BFDoWA [35]	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_5 \succ \tilde{A}_4$
BFDoWG [35]	$\tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3 \succ \tilde{A}_1 \succ \tilde{A}_5$
BFHWA [27]	$\tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_5 \succ \tilde{A}_4 \succ \tilde{A}_3$
BFHWG [27]	$\tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_1 \succ \tilde{A}_5 \succ \tilde{A}_3$
Proposed BFDWA	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_5 \succ \tilde{A}_4 \succ \tilde{A}_3$
Proposed BFDWG	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_5 \succ \tilde{A}_4 \succ \tilde{A}_3$

Due to the absence of time intervals in the aggregation operators described in [27,35], a substantial amount of data is lost. The suggested aggregation operators are dynamic, and information is gathered from three separate time periods. The proposed operators possess the capacity to deal this sort of data whereas the operators introduced in [27,35] cannot tackle such situations. Therefore, this quality renders the proposed dynamic operators more adaptable than traditional time-independent aggregating operators.

5.6. Advantages and disadvantages of the work

BF dynamic aggregation operators are capable of processing data containing both favorable and unfavorable opinions. In addition, BF dynamic aggregation operators facilitate effective management of time-varying scenarios. However, the proposed operators unable to deal the information involving non-membership or neutral aspect of the situation.

6. Conclusions

In this study, we have delved into the realm of multi-attribute decision-making, harnessing the potential of BF dynamic information. We have developed two novel aggregation operators, the BFDWA and BFDWG, designed to address the complexities of MADM problems. We have also established fundamental properties, namely, Idempotency, monotonicity and boundedness of these operators. In addition, we have presented a step by step mechanism to solve MADM problems under BF dynamic aggregation operators. Moreover, we have demonstrated the practical utility of these operators through the formulation of strategies aimed at addressing real-world MADM challenges. As an illustrative example, we have considered the task of selecting an emerging technology for enterprise systems, highlighting how BFDWA and BFDWG can facilitate informed decision-making in such contexts.

Bipolar fuzzy sets can capture the uncertainty expressed in positive and negative membership values lying in [0, 1] and [-1, 0], respectively. The uncertain data, which require complex number positive and negative membership values for their complete description, lies outside the scope of bipolar fuzzy environments. Moreover, if the positive and negative pinions are described in the form of intervals rather than a single number, we need to use interval-valued bipolar fuzzy sets. These are some limitations of the proposed work.

The primary purpose of future study will be to rectify the above mention limitations by introducing these strategies to complex BF dynamic and interval values BF dynamic environment. Another objective of future work will be to develop a comprehensive decision-analysis aid using BF dynamic aggregation operators, with the aim of maximizing its significance and effectiveness. The proposed approaches in this article will be versatile and applicable to various scenarios, including the development of more adaptable financial strategies, real-time monitoring of online social media activities, dynamic assessment of military management, dynamic and confidential shortlisting processes, tackling the energy crisis in developing nations and resolving time-dependent MADM problems.

Use of AI tools declaration

We have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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