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SCHUR MULTIPLIER OPERATOR AND MATRIX INEQUALITIES

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Abstract. In this note, we obtain a reverse version of the Haagerup Theorem. In particular, if $A \in \mathbb{M}_n$ has a $2 \times 2-$ principal submatrix as $\begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix}$ with $\beta \neq \bar{\alpha}$, then $\|S_A\| > 1$ where the operator $S_A : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is defined by $S_A(B) := A \circ B$ where " $\circ$ " stands for Schur product.

Keywords: Inequalities, Schur multiplier operator, spectral norm, numerical radius.
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1. Introduction

Let $\mathbb{M}_n$ denote the C^*-algebra of all $n \times n$ complex matrices. A Hermitian matrix $A \in \mathbb{M}_n$ is called positive if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ (we write $A \geq 0$) and is called strictly positive if $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$ (we write $A > 0$).
For Hermitian matrices $A, B \in \mathbb{M}_n$ a partial order is defined as $A \geq B$ if $A - B \geq 0$.
Let $\|A\|$ and $\omega(A)$ denote the spectral norm (or operator norm) and the numerical radius of A, respectively. Recall that the numerical radius is defined as follows:

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

It is well-known that $\omega(.)$ defines a norm on $\mathbb{M}_n$, which is equivalent to the spectral norm $\|.\|$. In fact, for every $A \in \mathbb{M}_n$, the following inequality holds:

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

Also, if A is normal, then $\|A\| = \omega(A)$.
The Schur or entrywise product of $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}_n$ is defined by $A \circ B = [a_{ij}b_{ij}]$. With this multiplication, $\mathbb{M}_n$ becomes a commutative algebra for which the matrix with all entries equal to one is the unit. Given $A \in \mathbb{M}_n$, the Schur multiplier operator or for brevity the Schur map $S_A : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is defined by $S_A(B) := A \circ B$. We say that S_A is unital if $S_A(I) = I$ for the

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identity matrix $I \in \mathbb{M}_n$. In [2], the induced norms of S_A with respect to the spectral norm and numerical radius were defined respectively, by

$$\|S_A\| = \sup_{B \neq 0} \frac{\|A \circ B\|}{\|B\|}, \tag{1}$$

$$\|S_A\|_\omega = \sup_{B \neq 0} \frac{\omega(A \circ B)}{\omega(B)}. \tag{2}$$

It is well known that

$$\|S_A\| \leq \|S_A\|_\omega. \tag{3}$$

The study of the norm of the Schur map has been interesting for some researchers. One of the best research in this field was done by Ando and Okubo in [2]. They showed that $\|S_A\|_\omega \leq 1$ if and only if there exists positive semidefinite matrix $X \in \mathbb{M}_n$ such that $\begin{bmatrix} X & A \\ A^* & X \end{bmatrix} \geq 0$, where $X \circ I \leq I$ and they give other equivalent characterizations and derive similar results for $\|S_A\|$. For more information about the norm of the Schur multiplier operator and its applications see [1,5,6,9,10]. Let $A \in \mathbb{M}_n$. For index sets $\lambda, \mu \subseteq \{1, , \dots, n\}$, we denote by $A[\lambda, \mu]$ the (sub)matrix of entries that lie in the rows of A indexed by λ and the columns indexed by μ. If $\lambda = \mu$, the submatrix $A[\lambda, \lambda]$ is denoted by $A[\lambda]$ and it is called a principal submatrix of A.

2. Main results

In 1991, Ando and Okubo proved the following theorem [2, Theorem 1 and Corollary 3] which is well known as the Haagerup Theorem:

Theorem 2.1. *Let $A \in \mathbb{M}_n$. The following assertions are equivalent.*
(i) $\|S_A\| \leq 1$.
(ii) There exist $0 \leq X, Y \in \mathbb{M}_n$ *such that*

$$\begin{bmatrix} X & A \\ A^* & Y \end{bmatrix} \geq 0, \qquad X \circ I \leq I \qquad \text{and} \qquad Y \circ I \leq I.$$

In addition, if A is Hermitian,
(iii) $\|S_A\| = \|S_A\|_\omega$.

Also, they proved a similar theorem [2, Theorem 2 and Corollary 4] for $\|S_A\|_\omega$ as follows.

Theorem 2.2. *Let $A \in \mathbb{M}_n$. The following assertions are equivalent.*
(i) $\|S_A\|_\omega \leq 1$.
(ii)There exists $0 \leq X \in \mathbb{M}_n$ *such that*

$$\begin{bmatrix} X & A \\ A^* & X \end{bmatrix} \geq 0 \qquad \text{and} \qquad X \circ I \leq I.$$

Moreover, if $A = [a_{ij}] \geq 0$,
(iii) $\|S_A\|_\omega = \max\{a_{ii} : 1 \leq i \leq n\}$.

To prove the main results, we need the following lemma, which is known as the Schur complement theorem.

Lemma 2.3 ([4], Theorem 1.3.3)**.** *Let A, B be strictly positive matrices. Then the block matrix* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *is positive if and only if $B \geq X^* A^{-1} X$.*

Now, we state one of the main results of this section in the following theorem.

Theorem 2.4. *Let $A = [a_{ij}] \in \mathbb{M}_n$ such that $\|S_A\| = 1$. If $a_{ss} = a_{tt} = 1$ for some $1 \leq s < t \leq n$, then $a_{st} = \bar{a_{ts}}$.*

Proof. By the use of Theorem 2.1, there exist positive $n \times n$ matrices $X = [x_{ij}], Y = [y_{ij}]$ with $0 \leq x_{ii}, y_{ii} \leq 1, (1 \leq i \leq n)$, such that

$$\begin{bmatrix} X & A \\ A^* & Y \end{bmatrix} \geq 0.$$

Letting $\tilde{X} := [\tilde{x_{ij}}]$ such that $\tilde{x_{ij}} = x_{ij}$ if $i \neq j$ and $\tilde{x_{ii}} = 1$, and $\tilde{Y} := [\tilde{y_{ij}}]$ such that $\tilde{y_{ij}} = y_{ij}$ if $i \neq j$ and $\tilde{y_{ii}} = 1$, we have

$$\begin{bmatrix} \tilde{X} & A \\ A^* & \tilde{Y} \end{bmatrix} \geq \begin{bmatrix} X & A \\ A^* & Y \end{bmatrix} \geq 0.$$

It is known that any principal submatrix of a positive matrix is positive, so it follows that

$$C = \begin{bmatrix} 1 & x & 1 & a_{st} \\ \bar{x} & 1 & a_{ts} & 1 \\ 1 & \bar{a_{ts}} & 1 & y \\ \bar{a_{st}} & 1 & \bar{y} & 1 \end{bmatrix} \geq 0 \qquad \textit{where } x := \tilde{x_{st}} = x_{st}, y := \tilde{y_{st}} = y_{st}.$$

In fact, $C = \begin{bmatrix} \tilde{X} & A \\ A^* & \tilde{Y} \end{bmatrix} [\lambda]$, where $\lambda = \{s, t, n+s, n+t\}$.

So, in view of Lemma 2.3, we get

$$\begin{bmatrix} 1 & a_{ts} & 1 \\ \bar{a_{ts}} & 1 & y \\ 1 & \bar{y} & 1 \end{bmatrix} - \begin{bmatrix} \bar{x} \\ 1 \\ \bar{a_{st}} \end{bmatrix} [x \;\; 1 \;\; a_{st}] = \begin{bmatrix} 1-|x|^2 & a_{ts} - \bar{x} & 1 - \bar{x} a_{st} \\ \bar{a_{ts}} - x & 0 & y - a_{st} \\ 1 - x a_{st} & \bar{y} - \bar{a_{st}} & 1 - |a_{st}|^2 \end{bmatrix} \geq 0.$$

Since the determinant of principal submatrices of the above matrix is positive, we obtain $\bar{a_{ts}} - x = y - a_{st} = 0$ and hence

$$C = \begin{bmatrix} 1 & \bar{a_{ts}} & 1 & a_{st} \\ a_{ts} & 1 & a_{ts} & 1 \\ 1 & \bar{a_{ts}} & 1 & a_{st} \\ \bar{a_{st}} & 1 & \bar{a_{st}} & 1 \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of C is as follows:

$$f(\lambda) = \lambda^4 - 4\lambda^3 + (4 - 2|a_{st}|^2 - 2|a_{ts}|^2)\lambda^2 + 2(|a_{st}|^2 + |a_{ts}|^2 - 2\Re(a_{st}a_{ts}))\lambda,$$

where $\Re(a_{st}a_{ts})$ is the real part of $a_{st}a_{ts}$. Now, if $a_{st} \neq \bar{a_{ts}}$, we obtain that the coefficient of λ is positive and then $f(\lambda)$ has a negative root, which is in contradiction with $C \geq 0$, and hence $a_{st} = \bar{a_{ts}}$. □

The next corollary is easily deduced from Theorem 2.4.

Corollary 2.5. *If $S_A : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is an unital map with $\|S_A\| = 1$, then A is Hermitian.*

The following corollary is convenient to be as a reverse of the Haagerup theorem.

Corollary 2.6. *If $A \in \mathbb{M}_n$ has a $2\times 2-$principal submatrix as $\begin{bmatrix} 1 & \alpha \\ \beta & 1 \end{bmatrix}$ with $\beta \neq \bar{\alpha}$, then $\|S_A\| > 1$.*

Employing a strategy similar to the proof of Theorem 2.4, we obtain the following result.

Theorem 2.7. *Let $A = [a_{ij}] \in \mathbb{M}_n$ such that $\|S_A\|_\omega = 1$. If $a_{ss} = 1$, then $a_{sj} = \bar{a_{js}}$ for all $1 \leq j \leq n$.*

Proof. From Theorem 2.2, it follows that there exists a positive matrix $X = [x_{ij}] \in \mathbb{M}_n$ with $0 \leq x_{ii} \leq 1\,(1 \leq i \leq n)$ such that

$$\begin{bmatrix} X & A \\ A^* & X \end{bmatrix} \geq 0.$$

Setting $\tilde{X} := [\tilde{x_{ij}}]$ such that $\tilde{x_{ij}} = x_{ij}$ for $i \neq j$ and $\tilde{x_{ii}} = 1$, we have

$$\begin{bmatrix} \tilde{X} & A \\ A^* & \tilde{X} \end{bmatrix} \geq \begin{bmatrix} X & A \\ A^* & X \end{bmatrix} \geq 0.$$

Since every principal submatrix of the above matrix is positive, it follows that for all integer $j \in \{1, 2, \ldots, n\}$ such that $j \neq s$,

$$B = \begin{bmatrix} 1 & x & 1 & a_{sj} \\ \bar{x} & 1 & a_{js} & a_{jj} \\ 1 & \bar{a_{js}} & 1 & x \\ \bar{a_{sj}} & \bar{a_{jj}} & \bar{x} & 1 \end{bmatrix} \geq 0, \qquad \text{where } x := \tilde{x_{sj}} = x_{sj}.$$

In fact, $B = \begin{bmatrix} \tilde{X} & A \\ A^* & \tilde{X} \end{bmatrix}[\lambda]$, where $\lambda = \{j, s, n+j, n+s\}$.

Hence, using Lemma 2.3, we obtain that

$$\begin{bmatrix} 1 & a_{js} & a_{jj} \\ \bar{a_{js}} & 1 & x \\ \bar{a_{jj}} & \bar{x} & 1 \end{bmatrix} - \begin{bmatrix} \bar{x} \\ 1 \\ \bar{a_{sj}} \end{bmatrix} [x \;\; 1 \;\; a_{sj}] = \begin{bmatrix} 1-|x|^2 & a_{js}-\bar{x} & a_{jj}-\bar{x}a_{sj} \\ \bar{a_{js}}-x & 0 & x-a_{sj} \\ \bar{a_{jj}}-x\bar{a_{sj}} & \bar{x}-\bar{a_{sj}} & 1-|a_{sj}|^2 \end{bmatrix} \geq 0.$$

Since the determinant of principal submatrices of the above matrix is nonnegative, we have $\bar{a_{js}} - x = x - a_{sj} = 0$ and hence $a_{sj} = \bar{a_{js}}$. □

The following corollary is readily obtained.

Corollary 2.8. *If $S_A : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is an unital map with $\|S_A\|_\omega = 1$, then A is Hermitian.*

The following straightforward result can be regarded as a reverse version of Theorem 2.2.

Corollary 2.9. *If $A \in \mathbb{M}_n$ has a $2 \times 2-$principal submatrix as $\begin{bmatrix} \alpha & \beta \\ \gamma & \theta \end{bmatrix}$ with $\alpha = 1$ or $\theta = 1$, and $\beta \neq \bar{\gamma}$, then $\|S_A\|_\omega > 1$.*

3. Applications

In the following remark, we explain the way how to use Theorem 2.4, to refuse some matrix inequalities.

Remark 3.1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Introduce $F = [f_{ij}]$, such that $F \circ I = I$ and $f_{ij} = \dfrac{M_1(a_i, a_j)}{M_2(a_i, a_j)}$, where M_1, M_2 are functions of a_i, a_j possibly they are means. If F is not symmetric, then by Corollary 2.6, $\|S_F\| > 1$, which means the inequality $\|(M_1(a_i, a_j)) \circ X\| \leq \|(M_2(a_i, a_j)) \circ X\|$ does not hold in general or there exists $X \in \mathbb{M}_n$ such that $\|(M_1(a_i, a_j)) \circ X\| > \|(M_2(a_i, a_j)) \circ X\|$. Also we remark that if F is not symmetric, then by Corollary 2.8, $\|S_F\|_\omega > 1$. By the same way as before, we can refuse some numerical radius inequalities.

As an application of Corollary 2.6, we have the following theorem.

Theorem 3.2 ([8], Theorem 2.3)**.** *Let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix and $0 < \nu < 1$ be a real number such that $\nu \neq \frac{1}{2}$. Then there exists $X \in \mathbb{M}_n$ such that*

$$\| A^\nu X A^{1-\nu} \| > \| \nu AX + (1-\nu) XA \| . \tag{4}$$

Proof. Without loss of generality, we can assume that $A = \mathbf{d}iag(a_1, a_2, \ldots, a_n)$. Using Lemma 2.2 in [8], the $n \times n$ matrix $F = \left[\frac{a_i^\nu a_j^{1-\nu}}{\nu a_i + (1-\nu) a_j}\right]$ is not symmetric, so by Corollary 2.6, $\|S_F\| > 1$. Hence by the argument in Remark 3.1, we conclude that there exists $X \in \mathbb{M}_n$ such that $\| A^\nu X A^{1-\nu} \| > \| \nu AX + (1-\nu) XA \|$. □

By Theorem 3.2, we conclude that for $A, B, X \in \mathbb{M}_n$ where $A, B \geq 0$, the inequality $\| A^\nu X B^{1-\nu} \| \leq \| \nu AX + (1-\nu) XB \|$ is not true in general.
By the same way as in the proof of the above theorem and Corollary 2.8, we get the following result:

Corollary 3.3. *Let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix and $0 < \nu < 1$ be a real number such that $\nu \neq \frac{1}{2}$. Then there exists $X \in \mathbb{M}_n$ such that*

$$\omega(A^\nu X A^{1-\nu}) > \omega(\nu AX + (1-\nu) XA). \tag{5}$$

Theorem 3.4. *Let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix. Then there exists $X \in \mathbb{M}_n$ such that $\|AXA^{-1}\| > \|X\|$.*

Proof. Without loss of generality, we assume that $A = \mathbf{d}iag(a_1, a_2, \dots, a_n)$. Applying Corollary 2.6, for non symmetric matrix $F = [a_i a_j^{-1}] \in \mathbb{M}_n$, the results is obtained. □

In view of Theorem 3.4, we conclude that the inequality $\|AXA^{-1}\| \leq \|X\|$ for all $A, X \in \mathbb{M}_n$ where $A > 0$ does not hold.
Using the proof of Theorem 3.4 and Corollary 2.8, we have the following corollary.

Corollary 3.5. *Let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix. Then there exists $X \in \mathbb{M}_n$ such that $\omega(AXA^{-1}) > \omega(X)$.*

One can use Corollary 2.9 to show that for $A, B, X \in \mathbb{M}_n$ where $A, B \geq 0$, the inequality $\omega(AXB) \leq \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q)$ is not true in general.

Theorem 3.6 ([7], Theorem 2)**.** *Let $p > q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $A \in \mathbb{M}_n$ be a non scalar strictly positive matrix such that $1 \in \sigma(A)$. Then there exists $X \in \mathbb{M}_n$ such that*

$$\omega(AXA) > \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q). \tag{6}$$

Proof. Without loss of generality, we assume that $A = \mathbf{d}iag(a_1, a_2, \dots, a_n)$, such that $a_1 = 1, a_2 \neq 1$. It is not difficult to show that

$$\frac{a_2^p}{p} + \frac{1}{q} \neq \frac{a_2^q}{q} + \frac{1}{p}. \tag{7}$$

If $F = \left[\frac{a_i a_i}{\frac{a_i^p}{p} + \frac{a_j^q}{q}}\right]$, then $f_{11} = 1$ but by inequality (7), $f_{12} \neq f_{21}$. Hence by Corollary 2.9, we conclude that $\|S_F\|_\omega > 1$. So, by the same argument in Remark 3.1, there exists $X \in \mathbb{M}_n$ such that $\omega(AXA) > \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q)$. □

The reverse of the classical Young inequality says that:

$$\nu a + (1-\nu)b \leq a^\nu b^{1-\nu}, \tag{8}$$

when $a, b \geq 0$ and $\nu \leq 0$ or $\nu \geq 1$.
In [3] a matrix version of the above inequality for Hilbert -Schmidt norm by Bakherad et al. is given as follows:

Theorem 3.7 ([3], Theorem 2.3)**.** *Let* $A, B, X \in \mathbb{M}_n$ *and let* m *and* m' *be positive scalars. If* $A \geq mI \geq B > 0$, *and* $\nu \geq 1$, *or* $B \geq m'I \geq A > 0$, *and* $\nu \leq 0$, *then*

$$\| \nu AX + (1-\nu)XB\|_2 \leq \| A^{\nu}XB^{1-\nu}\|_2. \tag{9}$$

Here we show that the conclusion of Theorem 3.7 becomes false for the numerical radius and operator norm instead of Hilbert Schmidt norm.

Theorem 3.8. *Let* $A \in \mathbb{M}_n$ *be a non scalar strictly positive matrix and* $\nu \geq 1$ *or* $\nu \leq 0$. *Then there exists* $X \in \mathbb{M}_n$ *such that*

$$\omega(\nu AX + (1-\nu)XA) > \omega(A^{\nu}XA^{1-\nu}). \tag{10}$$

Proof. Without loss of generality, we assume that $A = \mathbf{d}iag(a_1, a_2, \ldots, a_n)$ such that $a_1 = 1$ and $a_2 \neq 1$. It is straightforward to prove that

$$\frac{\nu a_2 + (1-\nu)}{a_2^{\nu}} \neq \frac{(1-\nu)a_2 + \nu}{a_2^{1-\nu}}. \tag{11}$$

If $F = \left[\dfrac{a_i^{\nu}a_j^{1-\nu}}{\nu a_i + (1-\nu)a_j}\right]$, then $f_{11} = 1$ but by inequality (11), $f_{12} \neq f_{21}$. Taking the same approach as in the proof of Theorem 3.7, the result holds. □

In the proof of Theorem 3.8 since $f_{ii} = 1$ for all $1 \leq i \leq n$, then by Corollary 2.6, a similar result holds for operator norm too.

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