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A VIEW ON WEIGHTED EXPONENTIAL ENTROPY AND EXAMINING SOME OF ITS FEATURES

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Abstract. One of the alternative versions of Shannon entropy is a measure of information which is called exponential entropy. Shannon and exponential entropies depend only on the event probabilities. These measures have also been extended to incorporate a set of weights associated with the events. Such weights may reflect some additional characteristics of the events such as their relative importance. In this paper, Axiomatic derivations and properties of weighted exponential entropy parallel to those achieved for weighted entropy are investigated. The relation between weighted exponential entropy of X and a strictly monotone and nonnegative function of X has obtained. The generalized weighted entropy and the generalized weighted exponential entropy for continuous random variable have been presented.

Keywords: Shannon entropy; Exponential entropy; Weighted entropy; Weighted exponential entropy.
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1. Introduction

The first time, Shannon [22] considered the communication as a mathematical problem and revealed the method of determination the capacity of a channel, after the second World War. It is the key concept of information which provides a measure of uncertainty associated with the probability distributions. Its generalization has been widely used in various directions with their applications in many fields. For example, Rényi [24], Havrda Charvát [14] and Tsallis [24] are extensions of the Shannon entropy. Rényi generalized entropy to one parameter family of entropies and defined Rényi entropy. Tsallis [24] proposed the generalization of the entropy by postulating a non-extensive entropy, (i.e.,Tsallis entropy), which covers Shannon entropy in particular cases. This measure is non-logarithmic and is obtained through the joint generalization of the averaging procedures and the concept of information gain. Tsallis entropy is not extensive but generalizes the concept of the information gain

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and is obtained by the linear averaging procedure [24]. More content about the properties of Shannon entropy have been collected in Reza [20]. Campbell [4] and Pal and Pal [17, 18] introduced variant measure which is called exponential entropy. They considered a non-negative and decreasing function of p as a measure of uncertainty and introduced exponential entropy. Al-Nasser et al. [1] proposed a new class of weighted exponential distribution called Entropy-Based Exponential weighted distribution (EBEWD). Bhat and Baig [2] developed the concept of weighted generalized entropy and its dynamic residual (version). They have derived the general expressions of these two uncertainty measures corresponding to some well-known lifetime distributions. Mahdy [15] provided the new results of weighted entropies with some characterizations. Furthermore, he has presented some results for weighted entropy residual and weighted past residual of order statistics with some application of some reliability systems such as a series structure and a parallel structure. The exponential entropy is claimed to have certain advantages over the classical Shannon entropy. Shannon's function is based on the concept that information gain from an event is inversely related to its probability of occurrence. The logarithmic behavior of entropy is considered to incorporate the additive property of information unlike the logarithmic behavior of Shannon's entropy, the gain function considered here is of exponential nature so that the gain in information from an event i with probability of occurrence pi is defined at all points with bounds at both ends. All other properties except the additive property for independent event (which does not carry any extra weight for an image, as pixel intensities are normally dependent on each other) of Shannon's entropy are also proved. Weighted entropy, give as a quantitative weighted of possible events. The concept of weighted entropy takes into account values of different outcomes, i.e., makes entropy context-dependent, through the weight function many papers on weighted entropy are published, such as Belis and Guiasu [3], Dial and Taneja [9], De Cuhna et al. [8], Guiasu [10] and Kapur [13] which are discussing the axiomatic properties and characterizations related to weighted entropy. The definition and initial results on weighted entropy were introduced in [3,10]. The purpose was to introduce disparity between outcomes of the same probability. In the case of a standard entropy such outcomes contribute the same amount of information uncertainty, which is appropriate in context free situations. However, imagine two equally rare medical conditions, occurring with probability $\mathrm{p} < 1$, one of which carries a major health risk while the other is just a peculiarity. Nawrocki and Harding [16] explained the weighted entropy which measures the investment risk as a performance of weighted entropy. Also an axiomatic version on generalized weighted entropy published by Das [6]. Kvalseth [14] obtained aspects on weighted exponential entropy. He introduced two different families of weighted exponential entropies as immediate extensions of the one-parameter generalized entropies and proposed a weighted form of the exponential entropies. Some properties and examples of such weighted exponential entropies are discussed. Since the additivity hypothesis in thermodynamics, Shannon entropy neglects the correlations between the subsystems, whereas non-extensive processes are common at many physical levels in statistical mechanics and atomic physics. There are two ways to overcome these intrinsic drawbacks. The one way is to extend the additivity to non-additivity, such as Rényi entropy and Tsallis entropy. The other way is taking some prior statistical information into account (Yu and Huang [25]). While the entropy depends only on the event probabilities p_1, ..., p_n. This measure has also been extended to incorporate a set of weights $W = (w_l, , w_n)$ associated with the n events (outcomes). Such weights may reflect some additional characteristics of the events. The occurrence of an event removes a double uncertainty: the quantitative one, related to the probability with which it occurs, and the qualitative one, related to its utility for the attainment of the goal or to its significance with respect to a given qualitative characteristic. For instance, an event of small probability can have a great weight, likewise, an event of great probability can have a very small weight. Naturally, the ascription of a weight to every elementary event is not a thing just so easy to be done. These weights may be either of objective or subjective character. Thus, the weight of one event may express some qualitative objective characteristic, but it also may express the subjective utility of the respective event with respect to the experimenter's goal. The weight ascribed to an elementary event may be also related to the subjective probability with which respective events occur, and this does not always coincide with the objective probability. We shall suppose that these qualitative weights are non-negative, finite, real numbers. The aim of this paper is to show axiomatic notes how the weighted exponential entropy as a generalized version of the Shannon entropy has various properties parallel and some of them similar to weighted entropy. Like all other information measures, weighted exponential entropy is used in image processing. we shall give a formula for the exponential entropy as a measure of uncertainty or information supplied by a probabilistic experiment depending both on the probabilities of events and on qualitative (objective or subjective) weights of the possible events. This formula will be called the weighted exponential entropy. In Section 2, some preliminaries about entropy and some generalized entropies are presented. We describe weighted exponential entropy criteria in Section 3. The extension of weighted entropy and weighted exponential entropy will be introduced and by considering different weight functions, we will investigate its features. Finally, some examples of continuous statistical distribution will be presented.

2. PRELIMINARIES

Let X be a discrete random variable with support S and probability mass function$P(x) = P(X = x)$, $x \in S$. The following definitions are noticeable:

Definition 2.1. For a discrete random variable X, The Shannon entropy is defined

$$H(X) = -\sum_{x\in S} p(x)logp(x). \tag{1}$$

The log is base 2 and the entropy is expressed in bits. For example, the entropy of a fair coin toss is 1 bit. We will use the convention that $0log0 = 0$, which is easily justified by continuity since $xlogx \to 0asx \to 0$. Adding terms of zero probability does not change the entropy.

Similarly, for the continuous case the following definition is of our interest:

Definition 2.2. Let X be a continuous random variable with probability density function $f(x)$ and support S, then entropy of X with respect to density f is

$$H(X) = -\int_{x\in S} f(x)logf(x)dx \tag{2}$$

which is called differential entropy. Shannon entropy in continuous case is not the limit of discrete case. One of the main drawbacks of this measure is that it may not always be non-negative and if it is negative then $H(X)$ is no longer an uncertainty measure [4].

Exponential entropy was defined by Pal and Pal [17] with regard to features of Shannon entropy as below:

Definition 2.3. Let X be a discrete random variable with probability mass function $p(x)$ and support S then the exponential entropy of X is defined by

$$H_e(X) = \sum_{x\in S} p(x)e^{(1-p(x)}. \tag{3}$$

The above definition can be extended to the continuous case also.

For a continuous random variable X, with probability density function f with support S, the exponential entropy is

$$h_e(X) = \int_s f(x)e^{1-f(x)}dx. \tag{4}$$

As we see in continuous case, exponential entropy is always non-negative against Shannon entropy and this is an advantage. Tsallis [24] defined the entropy as

$$T_q(X) = \frac{1}{q-1}\int_s f(x)^q dx; q > 0, q \neq 1 \tag{5}$$

and Rényi [21] expressed extended version of the Shannon entropy via the following formula

$$R_q(X) = \frac{1}{q-1} ln \int_s f(x)^q dx; q > 0, q \neq 1, \tag{6}$$

which is connected to Tsallis as

$$T_q(X) = \frac{e^{(1-q)R_q(X)-1}}{1-q}. \tag{7}$$

The exponential entropy is connected to Tsallis entropy as seen below:

$$h_e(X) = e[1 - \sum_{j=1}^{n} jh_{T_{j+1}}(X) - 1)]. \tag{8}$$

Also it can be expressed by Rényi entropy based on the following formula

$$h_e(X) = e[1 + \sum_{j=1}^{n} e^{-jh_{R_{j+1}}(X)}]. \tag{9}$$

3. WEIGHTED EXPONENTIAL ENTROPY CRITERIA

The concept of weighted entropy was proposed by Belis and Guiasu [3]. They took the weight into consideration and expressed the weighted entropy for each of the events as follows:

$$H(w_1, ..., w_n; p_1, ..., p_n) = -\sum_{i=1}^{n} w_i p_i log p_i, \tag{10}$$

In which $w_i(i = 1..n)$ are the weights related to the events. These weights are non-negative and real valued and they reflect some of the extra features of the events such as relative importance or their desirability. Also, these weights may be independent from the objective probabilities which take place. For instance, an event with a low probability may have a large weight or vice versa Belis and Guiasu [10]. It is clear that weighted entropy is an extended form of Shannon's entropy, because if , then weighted entropy converts into Shannon's entropy. After the introducing of the weighted entropy, it has been attracted the attention of many researchers including Guiasu, Kannapaan and Sahoo [12], Parkash and Taneja [19] and Singh and Bhardwaj [23]. In correspondence with the definition of weighted entropy, weighted exponential entropy can be defined for discrete random variable through considering the weight of each event as belowThe following is an example of a theorem, proof, corollary, proposition and remark.

Definition 3.1. Suppose that the events in which have the probability of and the positive weight values of respectively; hence, the function of the weighted exponential entropy would be

$$H_e(w_1, ..., w_n; p_1, ..., p_n) = \sum_{i=1}^{n} w_i p_i e^{1-p_i}, \tag{11}$$

some properties of weighted entropy described in Guiasu [10] (Properties$1-6$ for weighted entropy). Now, we investigate some properties of weighted exponential entropy. Properties $1-6$, are easily proved. Proof of the Property

7 is presented here.
• PROPERTY1: $H_e(w_1, ..., w_n; p_1, ..., p_n) \geq 0$.
• PROPERTY2: If $w_1, \cdots, w_n = w$ then

$$H_e(w_1, ..., w_n; p_1, ..., p_n) = w \sum_{i=1}^{n} w_i p_i e^{1-p_i} = wH_e(X)$$

where$H_e(X)$ is the exponential entropy.
• PROPERTY3: let $p_{i_0} = 1, P_i = 0(i = 1, ..., n; i \neq i_0)$, then

$$H_e(w_1, ..., w_n; p_1, ..., p_n) = w_{i_0}$$

where the weights are $w_1, ..., w_n$.
• PROPERTY4: If $p_i = 0, W_i \neq 0$ for every $i \in I$, $p_j \neq 0, W_j = 0$ for every $j \in J$ where

$$I \cup J = \{1, \cdots, n\}, I \cap J = \emptyset$$

then $H_e(w_1, ..., w_n; p_1, ..., p_n) = 0$. This property shows that the direct fact of an experiment whose possible events are unprofitable or non-significant, and whose useful or significant events are impossible, supplies a total information equal to zero even if the corresponding exponential entropy $H_e(X)$is different from zero, provided the set J has at least two elements. In particular, when all events have zero weights, we get the total information $H_e(w_1, ..., w_n; p_1, ..., p_n) = 0$. even if the exponential entropy$H_e(X)$ is not null, i.e. if there exist $0 < p_i < 1$.
• PROPERTY5:

$$H_e(w_1, ..., w_n, w_{n+1}; p_1, ..., p_n, 0) = H_e(w_1, ..., w_n; p_1, ..., p_n)$$

where the weights are $w_1, ..., w_n, w_{n+1}$ and the complete system of probabilities is. $p_1, ..., p_n$.
• PROPERTY6: For every non-negative, real number λ we have

$$H_e(\lambda w_1, ..., \lambda w_n; p_1, ..., p_n) = \lambda H_e(w_1, ..., w_n; p_1, ..., p_n)$$

.Yet, we did not impose any limitation on the weights ascribed to the primary events of the physical experiment (except that they are non-negative real numbers). Let us suppose that weight of the union of two incompatible events is the mean value of the weights of the respective events, i.e.,

$$w(E \cup F) = \frac{p(E)w(E) + p(F)w(F)}{p(E) + p(F)} \quad (12)$$

For any incompatible events E, F where $W(E)$ is the weight of the event E and $p(E)$ is the probability of the same event$E$. In particular, if E and F are complementary events, then$w(E \cup F) = p(E)w(E) + (1 - p(E))w(F)$
• PROPERTY7: If the relation12 for the weights holds, then

$$\begin{aligned} H_e(w_1, ..., w_{n-1}, w^{'}, w^{"} \quad ; \quad & p_1, ..., p_{n-1}, p^{'}, p^{"}) = H_e(w_1, ..., w_n; p_1, ..., p_n) \\ + \quad & p_n[w^{'} \frac{p^{'}}{p_n}(e^{1-p^{'}} - e^{1-p_n}) + w^{"} \frac{p^{"}}{p_n}(e^{1-p^{"}} - e^{1-p_n})] \end{aligned}$$

where $p_n = p^{'} + p^{"}$, $w_n = \frac{p^{'}w^{'}+p^{"}w^{"}}{p^{'}+p^{"}}$.
Proof:Taking into account the definition of the weighted exponential entropy and writing $p_n = p^{'} + p^{"}$, $w_n = \frac{p^{'}w^{'}+p^{"}w^{"}}{p^{'}+p^{"}}$

$$\begin{aligned}
H_e(w_1,...,w_{n-1},w^{'},w^{"} \quad &; \quad p_1,...,p_{n-1},p^{'},p^{"}) = \\
&= \sum_{i=1}^{n-1} w_ip_ie^{1-p_i} + w^{'}p^{'}e^{1-p^{'}} + w^{"}p^{"}e^{1-p^{"}} \\
&= \sum_{i=1}^{n-1} w_ip_ie^{1-p_i} + w_np_ne^{1-p_n} - w_np_ne^{1-p_n} + w^{'}p^{'}e^{1-p^{'}} + w^{"}p^{"}e^{1-p^{"}} \\
&= \sum_{i=1}^{n} w_ip_ie^{1-p_i} - (p^{'}w^{'} + p^{"}w^{"})e^{1-p_n} + w^{'}p^{'}e^{1-p^{'}} + w^{"}p^{"}e^{1-p^{"}} \\
&= H_e(w_1,...,w_n;p_1,...,p_n) - p^{'}w^{'}e^{1-p_n} - p^{"}w^{"}e^{1-p_n} + p^{'}w^{'}e^{1-p^{'}} + p^{"}w^{"}e^{1-p^{"}} \\
&= H_e(w_1,...,w_n;p_1,...,p_n) + p^{'}w^{'}(e^{1-p^{'}} - e^{1-p_n}) + p^{"}w^{"}(e^{1-p^{"}} - e^{1-p_n}).
\end{aligned}$$

• PROPERTY8: $H_e(w_1,...,w_n;p_1,...,p_n)$ is a symmetric function with respect to all pairs of variables $(w_k,p_k), k = 1,...,n$ (In other words, its value should not change by replacing its protrusions).
• PROPERTY9:

$$H_e(w_1,...,w_n;\frac{1}{n},...,\frac{1}{n}) = A_{(n)}\frac{w_1 + ... + w_n}{n},$$

where $A_{(n)}$ being a positive number for every number $n > 1$. Also by setting$w_1 = w_2 = ...w_n = \frac{1}{n}$ and $w_1 = w_2 = ...w_n = 1$, we obtain $H_e(\frac{1}{n},...,\frac{1}{n}) = \frac{A_{(n)}}{n^2}$ and $H_e(1,...,1,\frac{1}{n},...,\frac{1}{n}) = A_{(n)}$ respectively.
• PROPERTY10: $H_e(w_1,...,w_n;p_1,...,p_n)$ is a concave function with respect to p.
Proof:
let $h = wpe^{1-p}, p \in [0,1]$ then

$$h^{'}(p) = w(1-p)e^{1-p}, h^{"}(p) = w(p-2)e^{1-p} < 0.$$

We know that a sum of concave functions is also a concave function. Hence is a concave function.

Theorem 3.2. *If any zero probability is changed to a non zero probability and $w_i > w_j$ such that $i > j$, then the exponential entropy increases, i.e.,* $H_e(w_1,...,w_n,\delta,p_2,p_3-\delta,...,p_n) - H_e(w_1,...,w_n,0,p_2,...,p_n) > 0$.
Proof:

Definition 3.1 implies

$$\begin{aligned} H_e(w_1,...,w_n,\delta,p_2,p_3-\delta,...,p_n) &- H_e(w_1,...,w_n,0,p_2,...,p_n)\\ &= w_1\delta e^{1-\delta}+w_2p_2e^{1-p_2}+w_3(p_3-\delta)e^{1-p_3+\delta}\\ &+ \sum_{i=4}^{n} w_ip_ie^{1-p_i}-\sum_{i=2}^{n} w_ip_ie^{1-p_i}\\ &= w_1\delta e^{1-\delta}+w_3(p_3-\delta)e^{1-p_3+\delta}-w_3p_3e^{1-p_3}\\ &= w_1\delta e^{1-\delta}+w_3(p_3-\delta)e^{1-p_3+\delta}\\ &- w_3(p_3-\delta)e^{1-p_3}-w_3\delta(e^{1-p_3})\\ &= \delta(w_1e^{1-\delta}-w_3e^{1-p_3})+w_3(p_3-\delta)e^{\delta-1}. \end{aligned}$$

Since $p_3-\delta>0$ *we have*

$$\delta<p_3\to e^{1-\delta}>e^{1-p_3}\to \delta w_1e^{1-\delta}>\delta w_2e^{1-p_3}\to \delta(w_1e^{1-\delta}-w_3e^{1-p_3})>0$$

Again $\delta>0\to(e^{\delta}-1)>0\to w_3(p_3-\delta)e^{\delta-1}$ *therefore*

$$H_e(w_1,...,w_n,\delta,p_2,p_3-\delta,...,p_n)-H_e(w_1,...,w_n,0,p_2,...,p_n)>0.$$

Remark 3.3. Note that in Theorem 3.2 if any zero probability is changed to a non zero probability by reducing last probability and $w_i>w_j$ so that $i>j$ then weighted exponential entropy increases. Di Crescenzo and Longobardi [9] have been defined weighted entropy as follow:

Definition 3.4. Let X be a non-negative random variable with probability density function $f(x)$, then the weighted entropy is defined (Di Crescenzo and Longobardi [9]) as: $H^w(X)=-\int_0^\infty xf_x(x)logf_X(x)dx$ Similarly, the weighted exponential entropy for continuous random variable is defined by

$$H_e^w(X)=\int_0^\infty xf(x)e^{1-f(x)}dx. \tag{13}$$

The following example shows that the exponential entropy of two random variables are possible to be identical although such a result is not possible for weighted expotential entropy. The following example answers the question of whethere in the case of equall exponential entropy for two random variables, their wheighted exponential entropy of them, takes the same value.

Example 3.5. *Let* X *and* Y *be random variables with pdf* $f(x)=2x, 0<x<1$ *and* $g(y)=2(1-y), 0<y<1$ *respectively. Then* $H_e(X)=H_e(Y)=(\frac{e}{2}-\frac{e^{-1}}{2})-e^{-1}$ *But* $H_e^w(X)=0.44, H_e^w(Y)=0.37$ *threfore* $H_e^w(X)>H_e^w(Y)$.

Theorem 3.6. *Let* $Y=\phi(X)$ *, where*ϕ *is a strictly monotone and non-negative function, and* $|\phi^{'}(X)|\leq 1$ *, in order that both* X *and* Y *are absolutely continuous random variables. Then* $H_e^w(\phi(X))\leq\int_{\phi^{-1}(0)}^{\phi^{-1}(\infty)}\phi(x)f(x)e^{1-f(x)}dx$ *and equality is hold when* $|\phi^{'}(X)|=1$.

Proof:
Via equation (13), we have

$$H_e^w(Y) = \int_0^\infty y f_X(\phi^{-1}(y)) \frac{1}{|\phi'(\phi^{-1}(y))|} e^{1-f_X(\phi^{-1}(y))\frac{1}{|\phi'(\phi^{-1}(y))|}} dy.$$

Let ϕ be strictly increasing. By setting $Y = \phi(x)$, we obtain

$$H_e^w(Y) = \int_{\phi^{-1}(0)}^{\phi^{-1}(\infty)} \phi(x) f(x) e^{1-\frac{1}{\phi'(x)} f_x(x)} dx \leq \int_{\phi^{-1}(0)}^{\phi^{-1}(\infty)} \phi(x) f(x) e^{1-f(x)} dx$$

If ϕ is strictly decreasing, we similarly obtain

$$H_e^w(Y) = \int_{\phi^{-1}(0)}^{\phi^{-1}(\infty)} \phi(x) f(x) e^{1-\frac{1}{-\phi'(x)} f_x(x)} dx \leq \int_{\phi^{-1}(0)}^{\phi^{-1}(\infty)} \phi(x) f(x) e^{1-f(x)} dx.$$

Results:
Let $Y = aX + b$ such that $|a| \leq 1$, $b > 0$ then $H_e^w(Y) \leq H_e^w(X) + bH_e(X)$
Also, in Theorem 3.6 the inequality is reversed if $|\phi^{'}(X)| \geq 1$.
As we know, the transformation for discrete random variable has not been changed. For weighted exponential entropy we have the same achievement with the different weight. That is obtained based on the transformation.

3.1. **Generalized of Weighted Exponential Entropy.** In this section, we will consider the weight function in a general manner. Moreover, the extension of weighted entropy and weighted exponential entropy will be introduced and by considering different weight functions, we will investigate its features. Finally, some examples of continuous statistical distribution will be presented.

Definition 3.7. Let$X$be a non-negative random variable with probability density function $f_X(x)$ and $w(x)$ be a non-negative and differentiable weight function, then the generalized weighted entropy is defined by

$$GH^w(x) = -\int_s w(x) f(x) log f(x) dx$$

Similarly the generalized weighted exponential entropy for continuous random variable defined by $GH_e^w(x) = \int_s w(x) f(x) e^{1-f(x)} dx$.

Example 3.8. *Let X and Y be random variables with $f(x) = 2x, 0 < x < 1$ and $g(y) = 2(1-y), 0 < y < 1$ respectively.Then Presented in Example 1, it is shown that: $H_e(X) = H_e(Y)$ While by putting $w(x) = x^2$ and simple calculations we obtain$GH_e^w(X) = 0.29, GH_e^w(Y) = 0.22$. Therefore $GH_e^w(X) > GH_e^w(Y)$.*

In countinue to investigate generalized of weighted entropy for exponential distribution, we show in the next example that in the exponential distribution with parameter λ , generalized weighted exponential entropy with weight function $w(x) = e^{-\lambda x}$, is decreasing function of the λ.

Example 3.9. *Let X be a random variable with exponential distribution and parameter $\lambda > 0, w(x) = e^{-\lambda x}$ therefore:*

$$\begin{aligned} GH_e^w(x) &= \int_0^\infty w(x)f(x)e^{1-f(x)}dx \\ &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\lambda^2 e^{-\lambda x} e^{1-\lambda e^{-\lambda x}}dx \\ &= \frac{1}{\lambda^2}(e - e^{1-\lambda}) - \frac{e^{1-\lambda}}{\lambda} \end{aligned}$$

Since $e^{-\lambda}[\frac{(\lambda+1)^2}{+}12] < 1$ *so we have:*

$$\begin{aligned} \frac{dGH_e^w(x)}{d\lambda} &= \frac{2\lambda^2 e^{1-\lambda} + 2\lambda e^{1-\lambda} + \lambda^3 e^{1-\lambda} - 2\lambda e}{\lambda^4} \\ &= \frac{2\lambda e(e^{-\lambda}[\frac{(\lambda+1)^2+1}{2}] - 1)}{\lambda^4} < 0. \end{aligned}$$

Hence, $GH_e^w(x)$ is a decreasing function of λ .

Remark 3.10. If f is a density (probability) function the exponential entropy of the wighted distribution$g(x) = \frac{w(x)}{E(w(x))}f(x)$ is:

$$\begin{aligned} H_e^w(x) &= \sum_D \frac{w(x)}{E(w(x))}f(x)e^{1-\frac{w(x)}{E(w(x))}f(x)} \\ &= \sum_{j=0}^\infty \frac{e(-1)^j}{j!}\sum_D (\frac{w(x)}{E(w(x))})^{j+1}(f(x))^{j+1}. \end{aligned}$$

On noting that $H_{eh}(X) = \sum_D h(x)f(x)e^{1-f(x)} = \sum_{j=0}^\infty \frac{e(-1)^j}{j!}\sum_D h(x)(f(x))^{j+1}$ so $H_{eh}(X) = H_e^w(x(w))$ where$h(x) = \frac{w(X)}{E(w(X))}^{j+1}$, with variant weights of $w(.)$ we can find $h(.)$ explicitly.

4. **Conclusion**

The properties and axiomaticly aspects of weighted exponential entropy are given in this short note. We show that the exponential entropy of two random variables are possible to be identical although such a result is not possible for weighted expotential entropy. If ϕ is a strictly monotone and non-negative function of x, a relation between weighted exponential entropy of x and ϕ have been derived. The generalized weighted entropy and the generalized weighted exponential entropy for continuous random variable have been presented. we showed that in the exponential distribution with parameter$\lambda > 0$ generalized weighted exponential entropy with weight function $w(x) = e^{-\lambda x}$ is a decreasing function of the λ. In continuation of this work, in the future, we can work on the issue of inequalities related to weighted exponential entropy. Tsallis entropy is a generalized form of Shannon entropy, considering this problem, we will generalize weighted entropy. We will also use weighted exponential entropy in economics and image processing. The discussion of maximum entropy itself is a complete and comprehensive topic that can be investigated for this, but this topic will be discussed in a separate work.

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