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## **Linear Wave Models**

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# Linear Wave Models

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**AALBORG UNIVERSITY**  
Department of Civil Engineering



Aalborg University  
Department of Civil Engineering  
Section of Water and Soil

**DCE Lecture Notes No. 10**

## **Linear Wave Models**

by

Michael Brorsen

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# Chapter 1

## Introduction

These lecture notes consider the governing partial differential equation (PDE) and boundary conditions (BC's) for linear waves propagating in areas, where the variation of the water depth is *small*. On the other hand, no restrictions are set up with respect to the horizontal geometry of the area, and in general the wave fronts are *curved*. It is therefore possible at the same time to handle problems with diffraction and refraction. The local distance between two succeeding wave fronts is denoted  $L$  and the local propagation velocity of the front is denoted  $c$ , but notice that both quantities normally vary in the horizontal plane  $(x, y)$ , see Figure 1.1.

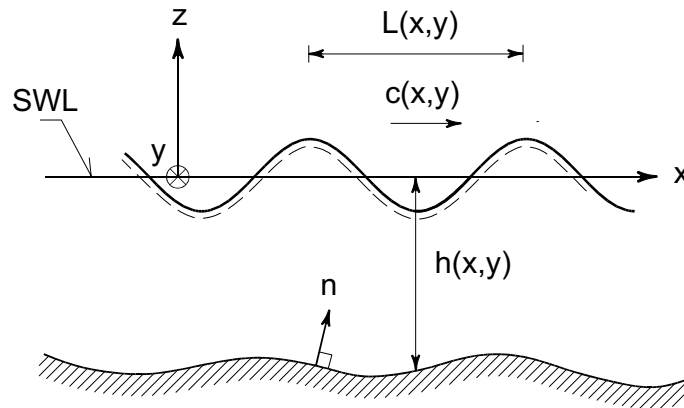


Figure 1.1: *Definition sketch*

In chapter 2 and 3 the governing PDE is integrated in the vertical direction. This removes the  $z$ -coordinate from the PDE, and the problem is reduced from 3D to 2D. The new PDE in two dimensions is rather easy to solve numerically. However, in order to integrate in vertical direction, it is necessary to assume an approximate vertical variation of the variables. This assumption limits the new PDE to be valid for mild bottom slopes only.



## 1.1 Governing differential equation and boundary conditions

It is assumed that we have ideal fluid and potential flow. Therefore the equation for particle velocity,  $\vec{v} = (u, v, w)$  reads:

$$\vec{v} = \text{grad } \varphi \quad (1.1)$$

where  $\varphi$  is the velocity potential and

$$\text{grad} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Substitution of equation (1.1) into the continuity equation

$$\text{div } \vec{v} = 0 \quad (1.2)$$

yields the Laplace equation

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0} \quad (1.3)$$

At the free surface the dynamic BC reads  $p = 0$ . Substitution into the linearized Bernoulli equation yields:

$$\eta = -\frac{1}{g} \frac{\partial \varphi}{\partial t} \quad \text{for } z = 0 \quad (1.4)$$

The kinematic BC corresponding to a particle on the free surface stays there reads:

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial \eta}{\partial y} \frac{\partial \varphi}{\partial y} \quad \text{for } z = \eta \quad (1.5)$$

After linearization we have the same BC as we had for linear waves with *straight* fronts:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} \quad \text{for } z = 0 \quad (1.6)$$

If equation (1.4) is derived with respect to  $t$  and equation (1.6) is substituted, we find:

$$\boxed{\frac{\partial \varphi}{\partial z} + \frac{1}{g} \frac{\partial^2 \varphi}{\partial t^2} = 0} \quad \text{for } z = 0 \quad (1.7)$$

At the bottom the BC reads

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{for } z = -h(x, y) \quad (1.8)$$

or

$$\text{grad } \varphi \cdot \vec{n} = 0 \quad \text{for } z = -h(x, y) \quad (1.9)$$

after use of the expression  $\partial(\ )/\partial n = \text{grad}(\ ) \cdot \vec{n}/n$  where  $\vec{n}$  is the normal to the bottom.

Mathematics tells us that a surface in space may be described by the equation

$$G(x, y, z) = 0$$

In this case the function  $G(x, y, z)$  is easy to determine, because the bottom is situated at  $z = -h(x, y)$ . Therefore the equation for  $G(x, y, z)$  reads

$$G(x, y, z) = z + h(x, y)$$

As  $\text{grad } G$  is perpendicular to a surface, where  $G$  is a constant, the normal at the bottom is  $\text{grad } G$ , or

$$\vec{n} = \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, 1 \right) \quad (1.10)$$

Now equation (1.9) is rewritten to

$$\boxed{\frac{\partial \varphi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial h}{\partial y} + \frac{\partial \varphi}{\partial z} = 0} \quad \text{for } z = -h(x, y) \quad (1.11)$$

At vertical walls we assume full reflection, yielding the BC

$$\frac{\partial \varphi}{\partial n} = 0 \quad (1.12)$$

# Chapter 2

## Regular Waves

### 2.1 Constant water depth

In case of constant water depth it is possible to find the correct vertical variation of the variables and to rewrite the 3D Laplace equation to a 2D Helmholtz equation by means of the Method of Separation. In this way we can express the velocity potential for regular (=harmonic) waves as

$$\begin{aligned}\varphi(x, y, z, t) &= \phi(x, y, t) \cdot f(z) \\ &= \phi^*(x, y) \cdot \sin(\omega t + \delta(x, y)) \cdot f(z) \\ &= \phi^*(x, y) \cdot \cos \delta(x, y) \cdot \sin \omega t \cdot f(z) \\ &\quad + \phi^*(x, y) \cdot \sin \delta(x, y) \cdot \cos \omega t \cdot f(z) \\ &= \phi_1^*(x, y) \cdot \sin \omega t \cdot f(z) + \phi_2^*(x, y) \cdot \cos \omega t \cdot f(z)\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ \sin(a + b) &= \sin a \cdot \cos b + \cos a \cdot \sin b \\ \phi_1^* &= \phi^* \cdot \cos \delta \\ \phi_2^* &= \phi^* \cdot \sin \delta\end{aligned}$$

At a given point  $(x, y, z)$  the potential is thus a harmonic function with amplitude  $\phi^*(x, y) \cdot f(z)$  and phase angle  $\delta(x, y)$ .

In order to find the potential along a vertical we must therefore determine the 2 variables depending  $x$  and  $y$ , i.e.  $\phi^*(x, y)$  and  $\delta(x, y)$  (or  $\phi_1^*(x, y)$  and  $\phi_2^*(x, y)$ ) together with the function describing the vertical variation,  $f(z)$ .

The surface elevation is found by means of equation (1.4), and in principle the

expression for  $\eta$  can be written:

$$\eta(x, y, t) = a(x, y) \cdot \sin(\omega t + \delta_\eta(x, y)) \quad (2.2)$$

where  $a(x, y)$  is the amplitude and  $\delta_\eta(x, y)$  is the phase angle. As shown above equation (2.2) may be rewritten to

$$\begin{aligned} \eta(x, y, t) &= a(x, y) \cdot \cos \delta_\eta(x, y) \cdot \sin \omega t + a(x, y) \cdot \sin \delta_\eta(x, y) \cdot \cos \omega t \\ &= a_1(x, y) \cdot \sin \omega t + a_2(x, y) \cdot \cos \omega t \end{aligned} \quad (2.3)$$

It is seen that the surface elevation is determined when either  $a(x, y)$  and  $\delta_\eta(x, y)$  or  $a_1(x, y)$  and  $a_2(x, y)$  are determined.

The expression for the velocity potential

$$\varphi = \phi(x, y, t) \cdot f(z) \quad (2.4)$$

is substituted into equation (1.3) rewritten to

$$\nabla^2 \varphi + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad , \quad \text{where } \vec{\nabla} = \left( \frac{\partial}{\partial x} , \frac{\partial}{\partial y} \right)$$

This yields

$$\nabla^2 \phi f + f'' \phi = 0$$

which after division by  $\phi f$  reads:

$$-\frac{\nabla^2 \phi}{\phi} = \frac{f''}{f} = K \quad (2.5)$$

This equation can only be satisfied for all values of  $(x, y, z)$ , if  $K = \text{constant}$ . This yields:

$$f''(z) - K f(z) = 0 \quad (2.6)$$

and

$$\nabla^2 \phi + K \phi = 0 \quad (2.7)$$

A negative value of  $K$  corresponds to a periodic variation  $f$  with  $z$  (sine or cosine). That option must be rejected, because all particle motions created by traveling waves are decreasing downwards.

A positive value,  $K = \lambda^2 > 0$ , yields:

$$f(z) = B \cosh \lambda z + C \sinh \lambda z$$

where  $B$  and  $C$  are constants.

The bottom BC may be written  $f'(-h) = 0$  due to constant water depth giving:

$$B = C \frac{\cosh \lambda h}{\sinh \lambda h}$$

If this expression is substituted into the expression for  $f(z)$  one finds

$$f(z) = C \frac{\cosh \lambda(z+h)}{\sinh \lambda h} \quad (2.8)$$

where the identity

$$\cosh \lambda(z+h) = \cosh \lambda h \cdot \cosh \lambda z + \sinh \lambda h \cdot \sinh \lambda z$$

has been applied. Now equation (2.8) is rewritten to

$$\begin{aligned} f(z) &= \frac{C}{\tanh \lambda h} \frac{\cosh \lambda(z+h)}{\cosh \lambda h} \\ &= C_1 \frac{\cosh \lambda(z+h)}{\cosh \lambda h} \end{aligned}$$

where  $C_1 = C / \tanh \lambda h$ . This leads to this expression for the potential:

$$\varphi = \frac{\cosh \lambda(z+h)}{\cosh \lambda h} C_1 \cdot \phi^*(x, y) \sin(\omega t + \delta(x, y)) \quad (2.9)$$

Because  $\phi^*(x, y)$  still is undetermined, no loss of information occurs by choosing  $C_1 = 1$ , and the expression for the vertical variation of the potential reads:

$$\boxed{f(z) = \frac{\cosh \lambda(z+h)}{\cosh \lambda h}} \quad (2.10)$$

Notice that this expression is valid for both straight and curved wave fronts. The only thing left is the determination of the coefficient  $\lambda$ .

Equation (2.9) is substituted into the surface condition

$$\frac{\partial \varphi}{\partial z} + \frac{1}{g} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad \text{for } z = 0$$

yielding

$$\phi^* \cdot \lambda \frac{\sinh \lambda h}{\cosh \lambda h} \cdot \sin(\omega t + \delta) + \frac{1}{g} \phi^* \cdot \frac{\cosh \lambda h}{\cosh \lambda h} (-\omega^2) \sin(\omega t + \delta) = 0$$

and consequently

$$\omega^2 = g\lambda \tanh \lambda h \quad (2.11)$$

The constant  $\lambda$  can be found from this equation, when  $\omega = (2\pi)/T$  is given. The equation is named the dispersion equation for linear waves with curved fronts.

Equation (2.11) is now compared to the dispersion equation for 1. order waves with straight fronts, i.e.

$$\omega^2 = gk \tanh kh \quad (2.12)$$

where  $k$  is the ordinary wave number defined as

$$k = \frac{2\pi}{L} \quad (2.13)$$

and  $L$  is the wave length for waves with straight fronts.

For given values of water depth and wave period, the two dispersion equations are identical and their solutions has to be equal, i.e.

$$\lambda = k \quad (2.14)$$

Consequently the vertical variation of the potential does not depend of the curvature of the wave fronts, and the coefficient  $\lambda$  is equal to the ordinary wave number for waves with straight fronts.

After the determination of the vertical variation of the potential, the horizontal variation is found from equation (2.7). Substitution of  $\lambda = k$  yields:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0 \quad (2.15)$$

or

$$\nabla^2 \phi + k^2 \phi = 0 \quad (2.16)$$

Equation (2.16) is named the *Helmholz* equation. It is necessary to determine 2 unknown variables at all points because

$$\phi(x, y, t) = \phi_1^*(x, y) \sin \omega t + \phi_2^*(x, y) \cos \omega t$$

If equation (2.16) is applied at two different moments, e.g.  $t = 0$  and  $t = T/4$ , this yields these two Helmholtz equations:

$$\nabla^2 \phi_2^* + k^2 \phi_2^* = 0 \quad (2.17)$$

$$\nabla^2 \phi_1^* + k^2 \phi_1^* = 0 \quad (2.18)$$

With proper BC's, the solution of the two equations provides  $\phi_1^*$  and  $\phi_2^*$ .

Only very few analytical solutions exist. The most well-known is *Sommerfeld's* solution for an infinitely long, straight breakwater exposed to regular waves with straight fronts far away from the breakwater, see Figure 2.1.

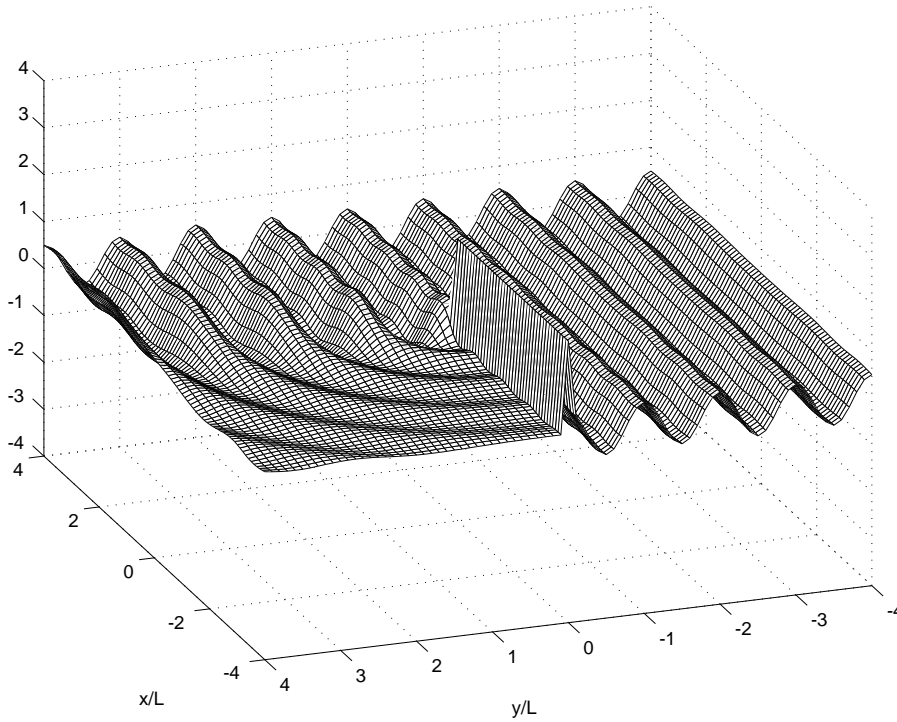


Figure 2.1: *Elevations around a fully absorbing breakwater parallel to the incoming fronts*

It is seen that waves are able to propagate around corners, a phenomenon called *diffraction*. The wave height behind the breakwater is reduced, which is quantified by the *diffraction coefficient*  $K_d$ . The definition reads

$$K_d = \frac{H}{H_i} \quad (2.19)$$

where  $H$  is the local wave height, and  $H_i$  is the wave height if the incoming waves. Figure 2.2 shows the placement of wave fronts and *iso-diffraction* curves. The lat-





velocities are determined by

$$\vec{v} = \text{grad}\varphi$$

and elevations of the free surface are determined by use of the dynamic BC that reads:

$$\eta = -\frac{1}{g} \frac{\partial\varphi}{\partial t} \quad \text{for } z = 0$$

Substitution of equation (2.9) with  $\lambda = k$  yields:

$$\eta = -\frac{1}{g} \phi^* \frac{\cosh kh}{\cosh kh} \omega \cos(\omega t + \delta) \quad (2.20)$$

and thus the expression for elevation reads

$$\boxed{\eta = -\frac{\phi^*}{g} \omega \cos(\omega t + \delta)} \quad (2.21)$$

If we define  $p^+$  as the deviation in pressure from hydrostatic pressure, the linearized Bernoulli equation may be rewritten to

$$p^+ = -\rho \frac{\partial\varphi}{\partial t} \quad (2.22)$$

Substitution of equation (2.9) yields:

$$p^+ = -\rho \phi^* \frac{\cosh k(z+h)}{\cosh kh} \omega \cos(\omega t + \delta)$$

Use of equation (2.21) finally yields:

$$\boxed{p^+ = \rho g \eta \cdot \frac{\cosh k(z+h)}{\cosh kh}} \quad (2.23)$$

Notice that exactly the same relation between  $p^+$  and  $\eta$  was found for waves with straight fronts!

## 2.2 Slowly varying water depth (mild bottom slopes)

We would like to express the velocity potential as

$$\varphi = \phi(x, y, t) \cdot f(z) \quad (2.24)$$

i.e. as we did in case of constant water depth. Because we only consider mild bottom slopes, it seems natural to approximate the vertical variation of the potential by

$$f \approx \frac{\cosh k(z + h)}{\cosh kh} \quad (2.25)$$

This corresponds to assume a locally constant water depth. If we also assume the dispersion equation (2.11) to be valid for waves on mild bottom slopes, we find  $\lambda = k = k(x, y)$ , as  $h = h(x, y)$ . Consequently the  $f$ -function indirectly depends on  $x$  og  $y$ , but hopefully only weakly!

Notice that due to clarity an index  $x, y, z$  or  $t$  in the following means the partial derivative with respect to  $x, y, z$  or  $t$ . Derivation with respect to  $x$  of

$$\varphi = \phi(x, y, t) \cdot f \quad (2.26)$$

where

$$\phi(x, y, t) = \phi_1^*(x, y) \sin \omega t + \phi_2^*(x, y) \cos(\omega t) \quad (2.27)$$

yields:

$$\varphi_x = \phi_x f + \phi f_x$$

where

$$f_x = f_h h_x + f_k k_h h_x$$

Notice that  $f_x = 0$  for constant water depth. Differentiation one more time gives:

$$\begin{aligned} \varphi_{xx} &= \phi_{xx} f + \phi_x f_x + \phi_x f_x + \phi f_{xx} \\ &= \phi_{xx} f + 2 \phi_x f_x + \phi f_{xx} \\ &= \phi_{xx} f + f_1(x, y, z) \end{aligned}$$

where

$$f_1(x, y, z) = 2 \phi_x f_x + \phi f_{xx}$$

As  $f = f(h(x, y), k(x, y), z)$  we find  $f_1 = f_1(x, y, z)$ , but we have  $f_1(x, y, z) = 0$  for constant water depth.

Similarly one finds

$$\varphi_{yy} = \phi_{yy} f + f_2(x, y, z)$$

where

$$f_2(x, y, z) = 2\phi_y f_y + \phi f_{yy}$$

and  $f_2(x, y, z) = 0$  for constant water depth. Finally we find

$$\varphi_{zz} = \phi f_{zz} = \phi k^2 f$$

Substitution of the expressions for  $\varphi_{xx}$ ,  $\varphi_{yy}$  and  $\varphi_{zz}$  into the Laplaces equation yields:

$$\nabla^2 \phi f + k^2 \phi f + f_3(x, y, z) = 0$$

where  $f_3$  is a known function. Division by  $f$  gives:

$$\nabla^2 \phi + k^2 \phi + f_4(x, y, z) = 0 \tag{2.28}$$

where  $f_4 = f_3/f$  is a known function, and we know that  $f_4 = 0$  for constant water depth.

From the PDE, equation (2.28), it is clearly seen that  $\phi$  must depend on  $z$ , i.e.  $\phi = \phi(x, y, z, t)$ . Along a vertical  $\phi = \phi(x, y, z, t)$  will vary due to the dependence of  $z$ , but hopefully the dependence is weak.

Anyway, this dependence is in contradiction to the original definition of  $\phi = \phi(x, y, t)$  in equation (2.24). We must therefore conclude that in principle the assumption of locally constant water depth or

$$f = \frac{\cosh h(z + h)}{\cosh kh}$$

makes it impossible to find a unique function  $\phi = \phi(x, y, t)$ .

Therefore we will adopt an approximating function,  $\phi = \phi(x, y, t)$ , which "in average" fulfills the Laplace equation over the vertical. In general this potential yields:

$$\Delta \varphi = \Delta(\phi \cdot f) = \frac{\partial^2(\phi \cdot f)}{\partial x^2} + \frac{\partial^2(\phi \cdot f)}{\partial y^2} + \frac{\partial^2(\phi \cdot f)}{\partial z^2} \neq 0$$

but it fulfills the equation

$$\int_{-h}^0 \Delta \varphi dz = 0 \tag{2.29}$$

At some places along the vertical we have  $\Delta\varphi < 0$ , but this is compensated for at the places, where  $\Delta\varphi > 0$ .

One could instead try to achieve  $\Delta\varphi \simeq 0$  along the parts of the vertical where the wave motion is greatest, i.e. near the surface. Berkhoff (1972) chose the weight factor:

$$w(z) = \cosh k(z + h) \quad (2.30)$$

which weights all points equal in case of shallow water waves and weights the upper points most in case of deep water waves. He solved the equation

$$\int_{-h}^0 \Delta\varphi \cdot w(z) dz = 0 \quad (2.31)$$

and after having discarded small terms, the result was this PDE for  $\phi$ :

$$\vec{\nabla} \cdot \left( A(x, y) \vec{\nabla} \phi(x, y) \right) + k^2 A(x, y) \phi(x, y) = 0 \quad (2.32)$$

where

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \quad (2.33)$$

and

$$A(x, y) = \frac{1}{g} \left( \frac{\omega}{k} \right)^2 \cdot \frac{1}{2} \left( 1 + \frac{2kh}{\sinh 2kh} \right) \quad (2.34)$$

By means of the dispersion equation (2.11) the expression for  $A(x, y)$  is rewritten to:

$$\boxed{A(x, y) = \frac{1}{2k} \tanh(kh) \left( 1 + \frac{2kh}{\sinh 2kh} \right)} \quad (2.35)$$

Substitution of the expressions for propagation velocity  $c$  and group velocity  $c_g$  for linear waves with straight fronts, i.e.

$$c = \frac{L}{T} = \frac{\omega}{k} \quad (2.36)$$

$$c_g = c \cdot \frac{1}{2} \left( 1 + \frac{2kh}{\sinh 2kh} \right) \quad (2.37)$$

leads to the expressions

$$\boxed{A(x, y) = \frac{c c_g}{g}} \quad (2.38)$$

and

$$k^2 A(x, y) = \frac{1}{g} \omega^2 \frac{c_g}{c} \quad (2.39)$$

After substitution of these expressions equation (2.32) reads

$$\vec{\nabla} \cdot \left( c c_g \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) \right) + \omega^2 \frac{c_g}{c} \phi = 0$$

or

$$\boxed{\frac{\partial}{\partial x} \left( c c_g \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( c c_g \frac{\partial \phi}{\partial y} \right) + \omega^2 \frac{c_g}{c} \phi = 0} \quad (2.40)$$

This equation is named The Mild Slope Equation on elliptic form, because the type of the PDE is elliptic. The solution of this type of PDE includes a numerical solution of very large systems of equations. Furthermore,  $\phi(x, y, t)$  must be found at two different moments in order to determine  $\phi_1^*(x, y)$  and  $\phi_2^*(x, y)$  and thereby  $\phi^*(x, y)$  and  $\delta(x, y)$ . The elliptic version of The Mild Slope Equation is therefore normally avoided, and instead solutions are based on the equations (3.32) and (3.33).

# Chapter 3

## Irregular waves

Hamilton's principle, see Dingemans (1997) or Yourgran (1979), is applied to derive an equation, which can describe the propagation of linear, irregular waves in an area with arbitrary geometry in the horizontal plane.

### 3.1 Hamilton's principle

If the kinetic energy of a system is denoted  $E_{kin}$  and the potential energy denoted  $E_{pot}$ , the Lagrangian (or Lagrange's function),  $F$ , is defined as

$$F = E_{kin} - E_{pot} \quad (3.1)$$

Under the motion from one position to another during the interval  $\Delta t$ , *Hamilton's principle* states that the motion will take in such a way that the integral

$$\int_0^{\Delta t} F dt \quad (3.2)$$

is stationary (or insensitive) to small variations of the motion between the two positions. Within the theory of Calculus of Variations, see e.g. Hansen (1969), this is normally expressed as:

$$\delta \int_0^{\Delta t} F dt = 0 \quad (3.3)$$

If a wave motion is considered the Lagrangian is found by integration of kinetic and potential energy of the actual volume of water. The potential energy over the volume with the horizontal cross section area  $A^*$  reads:

$$\begin{aligned} E_{pot} &= \int_{A^*} \left( \int_{-h}^{\eta} \rho g z dz \right) dA^* - \int_{A^*} \left( \int_{-h}^0 \rho g z dz \right) dA^* \\ &= \int_{A^*} \left( \int_0^{\eta} \rho g z dz \right) dA^* \end{aligned}$$

or

$$E_{pot} = \int_{A^*} \frac{1}{2} \rho g \eta^2 dA^* \quad (3.4)$$

The kinetic energy reads:

$$E_{kin} = \int_{A^*} \left( \int_{-h}^{\eta} \frac{1}{2} \rho g v^2 dz \right) dA^* \quad (3.5)$$

Notice that these expressions in principle are valid for both linear and non-linear waves.

## 3.2 Slowly varying water depth (mild bottom slopes)

Due to the assumptions of potential flow, the the expression for local velocity  $v$  reads:

$$v^2 = \varphi_x^2 + \varphi_y^2 + \varphi_z^2 \quad (3.6)$$

where  $\varphi = \varphi(x, y, z, t)$  is the velocity potential.

To simplify the calculations it is assumed that the vertical variation of  $\varphi$  can be approximated by the variation corresponding to linear waves on constant water depth. Therefore the expression for  $\varphi$  reads:

$$\varphi(x, y, z, t) \approx f \cdot \phi(x, y, t) \quad (3.7)$$

where

$$f = f(z, h, k) = \frac{\cosh k(z + h)}{\cosh kh} \quad (3.8)$$

and the 'wave number'  $k$  is calculated by the dispersion equation (2.12),

$$\omega^2 = gk \tanh kh \quad (3.9)$$

It is seen directly that  $\phi$  is the velocity potential at  $z = 0$ , named the Still Water Level (SWL), because

$$f(0, h, k) = 1 \quad (3.10)$$

Application of the horizontal gradient operator  $\vec{\nabla}$ ,

$$\vec{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \quad (3.11)$$

gives that the expression for  $v^2$  reads:

$$v^2 = \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi + (\varphi_z)^2 = |\vec{\nabla}\varphi|^2 + (\varphi_z)^2 \quad (3.12)$$

$$= |f \vec{\nabla}\phi + \phi \vec{\nabla}f|^2 + (\phi f_z)^2 \quad (3.13)$$

$$= f^2 |\vec{\nabla}\phi|^2 + \phi^2 |\vec{\nabla}f|^2 + 2f\phi \vec{\nabla}\phi \cdot \vec{\nabla}f + \phi^2 (f_z)^2 \quad (3.14)$$

This expression is substituted into the expression for  $E_{kin}$ , equation (3.5). However, we are not able to solve the integral with respect to  $z$  directly, because the actual



placement of the free surface is unknown. An approximate expression of  $E_{kin}$  can be derived under the assumptions *mild bottom slopes* and *small wave heights*, i.e.

$$\frac{h}{L^*} \ll 1 \quad \text{og} \quad \frac{H}{h} \ll 1 \quad (3.15)$$

where  $L^*$  is a characteristic length of the variations of the bathymetry. Dingemans (1997) shows that these assumptions, after a vertical integration where small terms are discarded, yields:

$$E_{kin} = \int_{A^*} \frac{1}{2} \rho \left( |\vec{\nabla} \phi|^2 A(x, y) + \phi^2 B(x, y) \right) dA^* \quad (3.16)$$

where the coefficients  $A(x, y)$  and  $B(x, y)$  are given by

$$A(x, y) = \int_{-h}^0 f^2 dz = \frac{1}{2k} \tanh(kh) \left[ 1 + \frac{2kh}{\sinh 2kh} \right] \quad (3.17)$$

$$B(x, y) = \int_{-h}^0 f_z^2 dz = \frac{k}{2} \tanh(kh) \left[ 1 - \frac{2kh}{\sinh 2kh} \right] \quad (3.18)$$

It is seen that  $A(x, y)$  is the same coefficient appearing in the Mild Slope equations for regular waves.

Substitution of the dynamic BC:

$$\eta = -\frac{1}{g} \varphi_t = -\frac{1}{g} \phi_t \quad \text{for} \quad z = 0 \quad (3.19)$$

into the expression for  $E_{pot}$  gives

$$E_{pot} = \int_{A^*} \frac{1}{2} \rho \frac{1}{g} \phi_t^2 dA^* \quad (3.20)$$

We can now rewrite Hamilton's principle to

$$\delta \int_0^{\Delta t} \int_{A^*} \left( \frac{1}{2} \rho [A |\vec{\nabla} \phi|^2 + B \phi^2] - \frac{1}{2} \rho \frac{1}{g} \phi_t^2 \right) dA^* dt = 0 \quad (3.21)$$

As this equation must be fulfilled for an arbitrary choice of the area  $A^*$ , Hamilton's principle may therefore also written as:

$$\delta \int_0^{\Delta t} \left( A |\vec{\nabla} \phi|^2 + B \phi^2 - \frac{1}{g} \phi_t^2 \right) dt = 0 \quad (3.22)$$

The integrand, named the modified Lagrange's function, reads:

$$\boxed{G = A |\vec{\nabla}\phi|^2 + B \phi^2 - \frac{1}{g} \phi_t^2} \quad (3.23)$$

and it is seen that  $G$  depends only on  $\phi$  and derivatives of  $\phi$ . From Calculus of Variations, see e.g. Hansen (1969), is known that equation (3.22) is fulfilled, if  $G$  is the solution of the so called *Euler* equation, which in this case reads:

$$G_\phi - (G_{\phi_x})_x - (G_{\phi_y})_y - (G_{\phi_t})_t = 0 \quad (3.24)$$

Substitution of the expression for  $G$  yields:

$$2B\phi - (A2\phi_x)_x - (A2\phi_y)_y - \left(-\frac{2}{g}\phi_t\right)_t = 0 \quad (3.25)$$

or

$$\boxed{B\phi - (A\phi_x)_x - (A\phi_y)_y + \frac{1}{g}\phi_{tt} = 0} \quad (3.26)$$

If equation (3.26) is derived with respect to  $t$ , we get

$$B\phi_t - (A\phi_{tx})_x - (A\phi_{ty})_y + \frac{1}{g}\phi_{ttt} = 0 \quad (3.27)$$

Substitution of the dynamic surface BC, equation (3.19), rewritten to

$$\phi_t = -g\eta \quad (3.28)$$

yields

$$B(-g\eta) - (A(-g\eta)_x)_x - (A(-g\eta)_y)_y + \frac{1}{g}(-g\eta_{tt}) = 0 \quad (3.29)$$

or

$$\boxed{B\eta - (A\eta_x)_x - (A\eta_y)_y + \frac{1}{g}\eta_{tt} = 0} \quad (3.30)$$

Equation (3.26) and equation (3.30) are often named the *time dependent* Mild Slope equation for  $\phi$  and  $\eta$ , respectively.

Notice that both equations are derived without a direct assumption about regular waves. Only the adopted variation of  $\varphi$  in vertical direction corresponds to the

variation in regular waves, but in principle we could have adopted any other  $f(z)$ -function. The time dependent Mild Slope equation may therefore be applied to calculate the propagation of irregular waves.

However, the vertical variation of the potential,  $f(z, h, k)$ , depends on the frequency of the waves. In irregular waves many frequencies are present, but we are forced to use only one variation of  $f$ , when  $A$  and  $B$  are calculated.

One might e.g. use  $f(z, h, k)$  corresponding to the peak frequency of the variance spectrum,  $\omega_p$ , dvs.

$$f(z, h, k_p) = \frac{\cosh k_p(z + h)}{\cosh k_p h} \quad (3.31)$$

where  $k_p$  is the wave number corresponding to  $\omega_p$ . However, it is clear that errors will be present, when this variation is used for a frequency  $\omega \neq \omega_p$ .

Reliable results are normally obtained if we have a "narrow" variance spectrum as e.g. a JONSWAP spectrum truncated at the upper frequency  $f = 2 f_p$ .

In practice we do not use equation (3.30) to obtain a numerical solution of the wave field. In order to make an effective generation of the incoming waves, experience has shown that it is more effective to use equation (3.26) in combination with equation (3.19). The latter equation is used to rewrite  $\phi_{tt}$  in equation (3.26) and this gives the following coupled PDE's:

$$\boxed{\eta = -\frac{1}{g} \phi_t} \quad (3.32)$$

and

$$\boxed{\eta_t + (A \phi_x)_x + (A \phi_y)_y - B \phi = 0} \quad (3.33)$$

where the coefficients  $A$  and  $B$  corresponds to  $k_p$ , i.e.

$$A = \frac{1}{2k_p} \tanh(k_p h) \left[ 1 + \frac{2k_p h}{\sinh 2k_p h} \right] \quad (3.34)$$

and

$$B = \frac{k_p}{2} \tanh(k_p h) \left[ 1 - \frac{2k_p h}{\sinh 2k_p h} \right] \quad (3.35)$$

These coupled PDE's are easily solved numerically in the time domain by use of an explicit finite difference method. See e.g. Brorsen and Helm-Petersen (1998).

All types of Mild Slope equations in 2 dimensions are based on discarding of "small terms", but the assessment of the order of magnitude of these terms is to some extent based on intuition. The upper limit of the allowable bottom slopes is therefore found by experiments, and Booij (1983) showed that in general reliable results are obtained for bottom slopes less than 1 : 3.

### 3.3 Varying water depth (steep bottom slopes)

Suh et al. (1997) included terms having an order of magnitude  $(\vec{\nabla}h)^2 = h_x^2 + h_y^2$  and  $\vec{\nabla}^2h = h_{xx} + h_{yy}$  in the derivation of the PDE's. The only effect is that  $B$ -coefficient in equation (3.33) is modified to

$$B = \frac{k}{2} \tanh(kh) \left[ 1 - \frac{2kh}{\sinh 2kh} \right] + \frac{\omega^2}{g} (R_1 (\vec{\nabla}h)^2 + R_2 \vec{\nabla}^2h) \quad (3.36)$$

where the expressions for the coefficients  $R_1 = R_1(k, h)$  and  $R_2 = R_2(k, h)$  are given in Suh et al. (1997). The inclusion of these terms makes it possible in practice to handle bottom slopes less than 1 : 1.

## 3.4 Dissipation

Even though the Mild Slope equations are based on the assumption of ideal fluid and potential flow, Dingemans (1997) showed that it is possible to include the effect of dissipation in the boundary layer at the bottom and from wave breaking.

The general expression for dissipation (unit:  $Watt/m^2$ ) reads:

$$D = W E = W \rho g m_o = W \rho g \sigma_\eta^2 \quad (3.37)$$

after substitution of

$$E = \rho g \sigma_\eta^2 \quad (3.38)$$

which is the average energy per  $m^2$  of the plane at still water level (unit:  $Joule/m^2$ ). When  $D$  has been determined,  $W$  (unit:  $s^{-1}$ ) is calculated as

$$W = \frac{D}{E} \quad (3.39)$$

Energy losses are included in the PDE's by adding of the term  $-W \eta$  at the right hand side of equation (3.33). The coupled PDE's then reads:

$$\boxed{\eta = -\frac{1}{g} \phi_t} \quad (3.40)$$

and

$$\boxed{\eta_t + (A \phi_x)_x + (A \phi_y)_y - B \phi = -W \eta} \quad (3.41)$$

Normally dissipation due to wave breaking is much more important than dissipation due to bottom friction.

### 3.4.1 Wave breaking

The expression for dissipation due to wave breaking reads:

$$D_b = W_b E = W_b \rho g m_o = W_b \rho g \sigma_\eta^2 \quad (3.42)$$

First  $D_b$  is determined. Then  $W_b$  (unit:  $s^{-1}$ ) is determined and finally substituted into equation (3.41).

For regular waves Battjes (1978) presented an expression based on arguments of analogy between a breaking wave and an advancing hydraulic jump. This expression reads:

$$D_b = \frac{\alpha \rho g f}{4} \frac{H_m^3}{h} \quad (3.43)$$

where  $\alpha \approx 1$ ,  $f = 1/T$  is the wave frequency and  $H_m$  is the maximum wave height possible at that place. Battjes (1978) uses the breaking criterion:

$$H_m = \frac{0.88}{k} \tanh\left(\frac{\gamma}{0.88} kh\right) \quad (3.44)$$

where  $\gamma \approx 0.8$ .

For irregular waves expression for  $D_b$  is modified to

$$\boxed{D_b = \frac{\alpha \rho g f_p}{4} \frac{H_m^3}{h} Q_b} \quad (3.45)$$

where  $f_p$  is the peak frequency, and  $Q_b$  is the probability of wave breaking at that point. If the distribution of non-broken wave heights is assumed to be a Rayleigh distribution, the expression for  $Q_b$  reads:

$$\frac{Q_b - 1}{\ln Q_b} = \frac{8 \sigma_\eta^2}{H_m^2} \quad (3.46)$$

Notice that Battjes model only describes statistically how much energy is dissipated by breaking. However, in practice this is often sufficient, when wave disturbance or sediment transport has to be assessed.

For irregular waves the term describing dissipation due to wave breaking therefore reads:

$$\boxed{W_b = \frac{D_b}{E} = \frac{\alpha f_p H_m^3 Q_b}{4 h \sigma_\eta^2}} \quad (3.47)$$

### 3.4.2 Bottom friction

The expression for dissipation due to bottom friction reads:

$$D_f = W_f E = W_f \rho g m_o = W_f \rho g \sigma_\eta^2 \quad (3.48)$$

First  $D_f$  and subsequently  $W_f$  (unit:  $s^{-1}$ ) are determined. Hereafter  $W_f$  is substituted into equation (3.41).

The shear stress at the bottom is usually assessed by the expression

$$\tau_o(t) = f_w \frac{1}{2} \rho |u_o(t)| u_o(t) \quad (3.49)$$

where  $u_o(t)$  is the velocity of the potential flow just above the bottom boundary layer. According to 1.order theory we have

$$u_o(t) = a_u \sin \omega t \quad (3.50)$$

where

$$a_u = \frac{\pi H}{T} \frac{1}{\sinh kh} \quad (3.51)$$

Nielsen (1992) put forward this expression for the friction factor  $f_w$ :

$$\ln f_w = 5.5 \left( \frac{k_N}{a} \right)^{0.2} - 6.3 \quad (3.52)$$

where  $a$  is the amplitude of the particle path at the bottom and  $k_N$  is the Nikuradse sand roughness or equivalent sand roughness, i.e. the diameter of grains of sand causing the same shear stress. From 1. order theory we have

$$a = \frac{a_u}{\omega} \quad (3.53)$$

According to Dingemans (1997) the instantaneous value of the dissipation due to bottom friction reads:

$$D_f(t) = \tau_o(t) u_o(t) \quad (3.54)$$

The average value over one wave period reads:

$$D_f = \frac{1}{T} \int_0^T \tau_o(t) u_o(t) dt = f_w \frac{1}{2} \rho a_u^3 \frac{1}{T} \int_0^T |\sin \omega t|^3 dt \quad (3.55)$$

As

$$\frac{1}{T} \int_0^T |\sin \omega t|^3 dt = \frac{4}{3\pi} \quad (3.56)$$

the expression for  $D_f$  reads:

$$\boxed{D_f = \frac{2}{3\pi} f_w \rho a_u^3} \quad (3.57)$$

Substitution of  $E = 1/8 \rho g H^2$  valid for regular waves yields:

$$W_f = \frac{D_f}{E} = \frac{\frac{2}{3\pi} f_w \rho a_u^3}{\frac{1}{8} \rho g H^2} = \frac{16}{3\pi} f_w \frac{a_u^3}{g H^2} \quad (3.58)$$

Finally substitution of  $a_u$  gives:

$$\boxed{W_f = \frac{D_f}{E} = \frac{16 \pi^2}{3 g} f_w \frac{H}{(T \sinh kh)^3}} \quad \text{for regular waves} \quad (3.59)$$

For irregular waves it is not so easy to determine the dissipation due to bottom friction. The energy in the irregular waves is  $E = 1/8 \rho g H_{rms}^2$ ,  $H_{rms}$  is the root-mean-square wave height. One might assume that the irregular wave are equivalent to regular waves having the same content of energy. This leads to regular waves with period  $T = T_p$  and wave height  $H = H_{rms}$ , and the dissipation due to bottom friction reads:

$$W_f = \frac{D_f}{E} = \frac{16 \pi^2}{3 g} f_w \frac{H_{rms}}{(T_p \sinh kh)^3} \quad (3.60)$$

If the distribution of wave heights is a Rayleigh distribution, we have  $H_{rms} = \sqrt{8} \sigma_\eta$ , which substituted into equation (3.60) yields:

$$\boxed{W_f = \frac{D_f}{E} = \frac{16 \pi^2}{3 g} f_w \frac{\sqrt{8} \sigma_\eta}{(T_p \sinh kh)^3}} \quad \text{for irregular waves} \quad (3.61)$$

### 3.5 Fields of application for Mild Slope equations

In practice results obtained by Mild Slope equations are reliable as long as non-linear effects are insignificant. This means that harbour disturbance due to short periodic waves normally is modelled quite well, but the long periodic waves bound to the wave groups are not modelled automatically. If long periodic waves are suspected to be important (harbour resonance, motions of moored ships), it is necessary to generate these long period waves together with the short periodic waves. Also wave forces from non-breaking waves on large structures are modelled accurately due to a very accurate modelling of diffraction phenomena.



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