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## **Non-linear Waves**

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# Non-linear Waves

Michael Brorsen

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Department of Civil Engineering



Aalborg University  
Department of Civil Engineering  
Section of Water and Soil

**DCE Lecture Notes No. 9**

# **Non-linear Waves**

by

Michael Brorsen

March 2007

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# Chapter 1

## Introduction to waves with finite wave height

In practice many waves have a steepness,  $H/L$ , so large that calculations done with 1. order theory are too inaccurate.

The mathematical solution of the 1. order problem is based on an *exact* solution of the Laplace equation with *approximate (linearised)* boundary conditions. Boundary conditions are linearized by assessing the order of magnitude of all terms. If the order of magnitude of a term is  $H/L$  times the order of magnitude of the largest term, the actual term is discarded.

At the free surface the boundary conditions (BC) are linearized as follows:

$$\text{kinematic BC : } \quad \frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial \eta}{\partial x} \quad \text{at } z = \eta \quad (1.1)$$

$$\text{linearised kin. BC : } \quad \frac{\partial \varphi}{\partial z} \simeq \frac{\partial \eta}{\partial t} \quad \text{at } z = 0 \quad (1.2)$$

$$\text{dynamic BC : } \quad g \eta + \frac{1}{2} \left( \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right) + \frac{\partial \varphi}{\partial t} = 0 \quad \text{at } z = \eta \quad (1.3)$$

$$\text{linearised dyn. BC: } \quad g \eta + \frac{\partial \varphi}{\partial t} \simeq 0 \quad \text{at } z = 0 \quad (1.4)$$

The linear BC's fulfilled at  $z = 0$  makes it easy to solve the Laplace equation

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0} \quad (1.5)$$



yielding the following expressions (see eg. Svendsen & Jonsson (1980)):

$$\eta = \frac{H}{2} \cos(\omega t - kx) \quad (1.6)$$

$$L = \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi h}{L}\right) \quad (1.7)$$

$$\varphi = -\frac{Hc}{2} \frac{\cosh k(z+h)}{\sinh kh} \sin(\omega t - kx) \quad (1.8)$$

$$u = \frac{\pi H}{T} \frac{\cosh k(z+h)}{\sinh kh} \cos(\omega t - kx) \quad (1.9)$$

and

$$w = -\frac{\pi H}{T} \frac{\sinh k(z+h)}{\sinh kh} \sin(\omega t - kx) \quad (1.10)$$

where:

$\eta$	surface elevation
$T$	wave period
$L$	wavelength
$H$	wave height
$c = L/T$	wave celerity or phase speed
$h$	water depth
$k = 2\pi/L$	wave number
$\omega = 2\pi/T$	cyclic frequency
$t$	time
$x$	horizontal coordinate
$z$	vertical coordinate
$\varphi$	velocity potential
$u = \partial\varphi/\partial x$	horizontal particle velocity
$w = \partial\varphi/\partial z$	vertical particle velocity

Even though we often have  $H/L < 0.08$ , measurements shows that the linearized BC's in 1.order theory often leads to unacceptable results. See for example Figure 1.1, where a cosine wave is compared to a measured surface profile.

It is seen that both the wave crest (the stretch where  $\eta > 0$ ) and wave trough ( $\eta < 0$ ) of the real wave are lifted compared to the cosine wave. The real wave crest is therefore shorter and steeper than the wave crest of the cosine wave, and the wave trough is longer and less steep than the wave trough of the cosine wave.

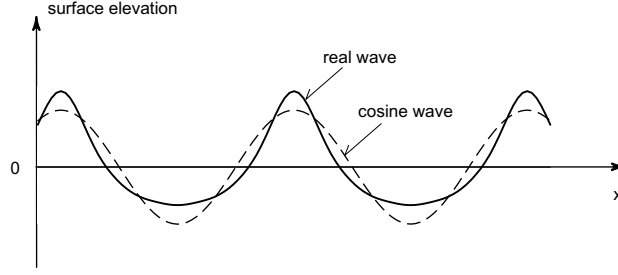


Figure 1.1: Comparison of real and 1. order (cosine) waves

If we want to describe real waves better than 1. order theory allows, we must discard fewer terms, when the BC's are linearized.

Furthermore it is necessary to introduce an extra BC, if we want to fulfill the BC's at  $z = \eta$  instead of at  $z = 0$ . It is a bit surprising that this problem can be illustrated by use of the expressions from 1. order theory. A current meter is placed at a level *below* the wave trough level, i.e. at  $z < \eta_{min}$ , and according to 1. order theory,  $u$  is in phase with  $\eta$ , see equations (1.6) and (1.10). The variation of  $u(t)$  is shown on Figure 1.2.

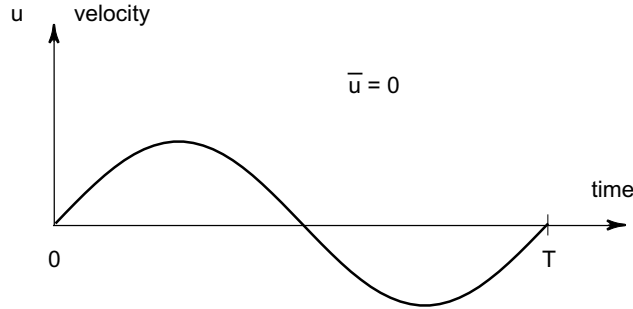


Figure 1.2: Horizontal velocity measured at level  $z < \eta_{min}$ .

This variation gives  $\bar{u} = 0$ , where  $\bar{u}$  is the average velocity at the current meter, i.e.

$$\bar{u}(z) = \frac{1}{T} \int_0^T u(z, t) dt = 0 \quad (1.11)$$

If the current meter is placed at a level  $z > \eta_{min}$ , one finds  $\bar{u} > 0$ , again based on 1. order theory. The average velocity is positive, because  $u = 0$  during the time interval, where the current meter is situated above the free surface. See Figure 1.3.

The instantaneous value of the discharge through a vertical plane reads:

$$q_{wave}(t) = \int_{-h}^{\eta(t)} u(z, t) dz = \int_{-h}^{\eta_{max}} u(z, t) dz$$

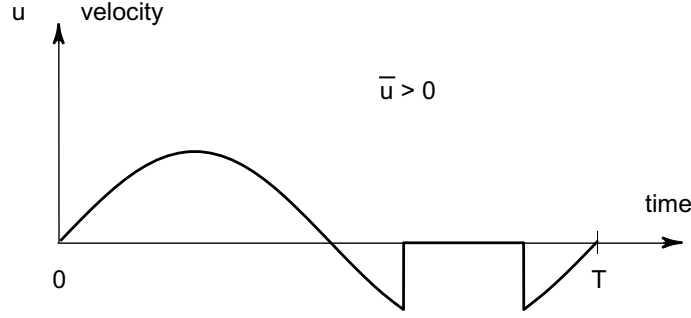


Figure 1.3: Horizontal velocity measured at the level  $z$ , where  $0 > z > \eta_{min}$ .

as  $u = 0$  for  $\eta(t) < z < \eta_{max}$ . Taking the average of  $q_{wave}(t)$  over one wave period gives:

$$\bar{q}_{wave} = \frac{1}{T} \int_0^T \left( \int_{-h}^{\eta_{max}} u(z, t) dz \right) dt = \int_{-h}^{\eta_{max}} \left( \frac{1}{T} \int_0^T u(z, t) dt \right) dz \quad (1.12)$$

or

$$\bar{q}_{wave} = \int_{-h}^{\eta_{max}} \bar{u} \cdot dz = \int_{\eta_{min}}^{\eta_{max}} \bar{u} \cdot dz > 0 ! \quad (1.13)$$

This discharge is only important for non-linear waves. Often it is named “Stokes drift”, but this name is sometimes also used for the corresponding average velocity defined as:

$$U_{Stokes} = \frac{\bar{q}_{wave}}{h} \quad (1.14)$$

If the conditions in a closed wave basin are considered, it is obvious that the average discharge must fulfill  $\bar{q} = 0$ . If not, all the water would end up in one end of the basin. One way to obtain  $\bar{q} = 0$  is the generation of a current  $U$  (constant velocity along a vertical, i.e. a potential flow), with  $q_{current} = Uh$ . As  $\bar{q} = \bar{q}_{wave} + q_{current} = 0$ , it is found that

$$U = -\frac{\bar{q}_{wave}}{h} = -U_{Stokes} \quad (1.15)$$

This means  $U < 0$ , and therefore current and waves are propagating in opposite directions. Notice that  $\bar{u} = U$  for  $z \leq \eta_{min}$ . Therefore  $U$  can be measured by a current meter placed below  $\eta_{min}$ .

According to 1. order theory it is found

$$u = u^{(1)} + U \quad (1.16)$$

where  $u^{(1)}$  is the 1. order horizontal velocity (equation (1.9)). As it can be shown that  $U = o\left(u^{(1)}\left(\frac{H}{L}\right)\right)$ , it is seen that this term should be discarded according to 1. order theory. For non-linear waves this term may be important and a more stringent derivation can be found in Svendsen (1985).

# Chapter 2

## Stokes Theory

In order to solve the non-linear flow problem a perturbation method is applied. It is here assumed that all variables may be expressed as a series expansion:

$$\varphi = \varphi^{(1)} + \varphi^{(2)} + \dots + \varphi^{(i)} + \dots \quad (2.1)$$

where

$$o(\varphi^{(i+1)}) = o(\varphi^{(i)} \cdot \frac{H}{L}) = o(\varphi^{(i-1)} \cdot (\frac{H}{L})^2) = \dots = o(\varphi^{(1)} \cdot (\frac{H}{L})^i) \quad (2.2)$$

and  $o(\quad)$  means the order of magnitude of the expression in the parenthesis. The expressions for surface elevation and dynamic pressure reads:

$$\eta = \eta^{(1)} + \eta^{(2)} + \dots + \eta^{(i)} + \dots \quad (2.3)$$

$$p^+ = p^{+(1)} + p^{+(2)} + \dots + p^{+(i)} + \dots \quad (2.4)$$

and similar expressions exists for the rest of the variables.

In a Stokes theory of order “ $i$ ” we first substitute equation (2.1) and (2.3) into the partial differential equation (PDE) and into the boundary conditions (BC). Then we discard all terms having a factor  $(H/L)^n$  (where  $n \geq i$ ) on their order of magnitude. According to equation (2.2) “ $i$ ” terms are taken into account in equation (2.1) in a Stokes theory of order “ $i$ ”.

Notice that  $\varphi^{(1)}$ ,  $\eta^{(1)}$  and  $p^{+(1)}$  are the known expressions from 1. order theory corresponding to the actual values of  $T$ ,  $H$  and  $h$ .

Because the free surface is no longer symmetrical about  $z = 0$ , it is necessary to define the wave height as:

$$H = \eta_{max} - \eta_{min} \quad (2.5)$$

## 2.1 Second Order Stokes Waves

In this theory the expressions for  $\varphi$  and  $\eta$  reads:

$$\varphi = \varphi^{(1)} + \varphi^{(2)} \quad (2.6)$$

$$\eta = \eta^{(1)} + \eta^{(2)} \quad (2.7)$$

where  $\varphi^{(1)}$  and  $\eta^{(1)}$  are known from 1. order theory, see equations (1.8) and (1.6), respectively. For the sake of clarity the PDE and BC's for  $\varphi^{(1)}$  are shown in figure 2.1.

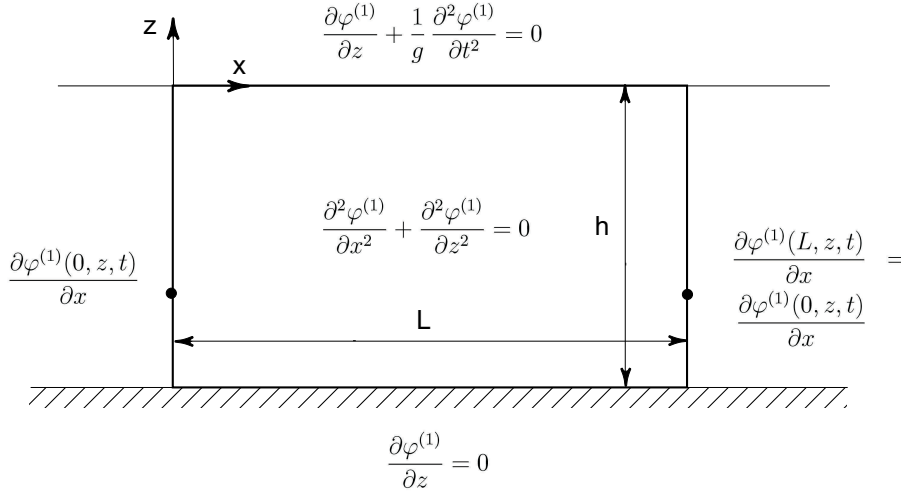


Figure 2.1: PDE, solution domain and BC's for  $\varphi^{(1)}$

Because  $\varphi^{(1)}$  fulfills the Laplace equation, i.e.

$$\frac{\partial^2 \varphi^{(1)}}{\partial x^2} + \frac{\partial^2 \varphi^{(1)}}{\partial z^2} = 0 \quad (2.8)$$

substitution of equation (2.6) into the Laplace equation leads to:

$$\frac{\partial^2 \varphi^{(2)}}{\partial x^2} + \frac{\partial^2 \varphi^{(2)}}{\partial z^2} = 0 \quad (2.9)$$

In order to find  $\varphi^{(2)}$  it is necessary to solve this PDE with the corresponding BC's. Just as in the 1. order theory, the problem with the unknown position of the free surface is solved by a Taylor series expansion from  $z = 0$  of the dynamic and kinematic BC, respectively. However, this time we only discard terms with an order of magnitude  $o((H/L)^2)$  times the leading terms.

The Taylor expansion of the kinematic BC, equation (1.1), may be expressed as:

$$\left( \frac{\partial \varphi}{\partial z} - \frac{\partial \eta}{\partial t} - \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} \right)_{z=0} + \eta \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial z} - \frac{\partial \eta}{\partial t} - \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} \right)_{z=0} + \dots = 0$$

and a similar expression exists for the dynamic BC, equation (1.3).

If the equations (2.6) and (2.7) are substituted into the Taylor expansions of the kinematic and dynamic BC's, and we subsequently discard the small terms, it is possible to eliminate  $\eta^{(2)}$  from the two BC's and combine them into one equation. After some non-trivial algebra, see e.g. Svendsen (1985), the BC at  $z = 0$  reads:

$$\frac{\partial \varphi^{(2)}}{\partial z} + \frac{1}{g} \frac{\partial^2 \varphi^{(2)}}{\partial t^2} = -\frac{3\omega}{4k} (kH)^2 \frac{\sin 2(\omega t - kx)}{\sinh 2kh}$$

In figure 2.2 is also shown the BC's at the rest of the border of the solution domain.

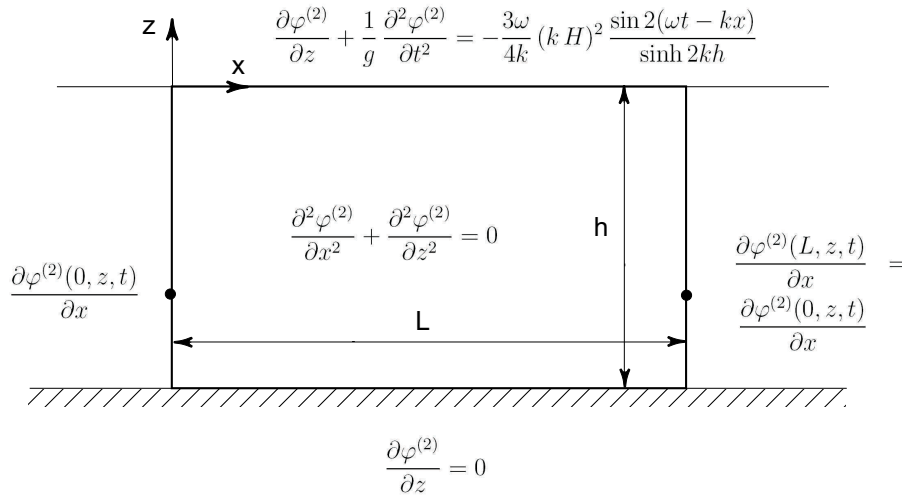


Figure 2.2: PDE, solution domain and BC's for  $\varphi^{(2)}$ .

Having found  $\varphi^{(2)}$ , the 2. order velocity components are derived by differentiation of  $\varphi^{(2)}$ , i.e.

$$u^{(2)} = \frac{\partial \varphi^{(2)}}{\partial x}$$

and

$$w^{(2)} = \frac{\partial \varphi^{(2)}}{\partial z}$$

Finally  $c^{(2)}$ ,  $\eta^{(2)}$  and  $p^{(2)}$  are derived the same way as in the 1. order theory.

In 2. order theory under the assumption  $\bar{q} = 0$  Svendsen & Jonsson (1980) derived this expression for the horizontal velocity:

$$u = u^{(1)} + \frac{3}{16} c (kH)^2 \frac{\cosh 2k(z+h)}{\sinh^4 kh} \cos 2(\omega t - kx) - \frac{1}{8} \frac{gH^2}{ch} \quad (2.10)$$

From (2.10) is seen that for  $z < \eta_{min}$ , the average value of the two first terms are zero, giving

$$\bar{u} = -\frac{1}{8} \frac{gH^2}{ch} \quad \text{for } z < \eta_{min} \quad (2.11)$$

Thus, according to 2. order theory the compensation current  $U$  is

$$U = -\frac{1}{8} \frac{gH^2}{ch} \quad (2.12)$$

It is furthermore found that

$$\eta^{(2)} = \Delta\eta \cdot \cos 2(\omega t - kx) \quad (2.13)$$

where

$$\Delta\eta = \frac{1}{16} kH^2 (3 \coth^3 kh - \coth kh) \quad (2.14)$$

and  $\eta^{(2)}$  is therefore a term oscillating twice as fast as the 1. order term. See Figure 2.3.

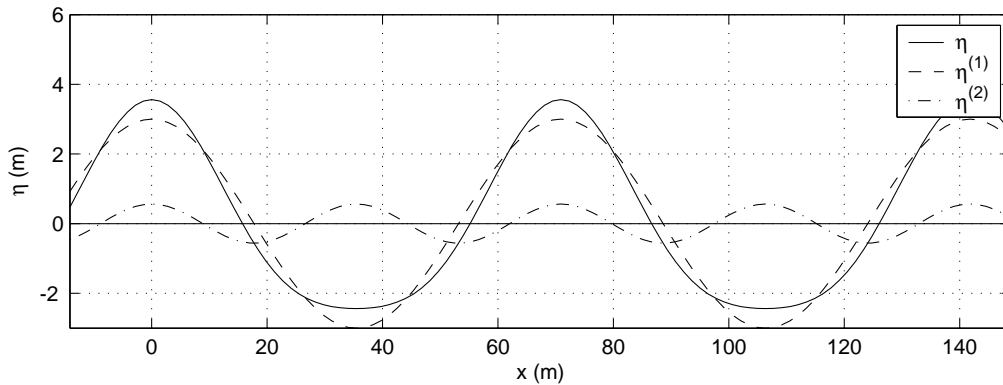


Figure 2.3: 2. order Stokes wave,  $H = 6$  m,  $T = 8$  secs and  $h = 10$  m

It is seen that  $\eta_{max} = \frac{H}{2} + \Delta\eta$  and  $\eta_{min} = -\frac{H}{2} + \Delta\eta$ , i.e. both wave crest and wave trough are raised  $\Delta\eta$ . Hereby the crest becomes shorter than the trough, and

$$\eta_{max} - \eta_{min} = \left(\frac{H}{2} + \Delta\eta\right) - \left(-\frac{H}{2} + \Delta\eta\right) = H$$

Finally it is surprisingly found that  $c^{(2)} = 0$  giving

$$c = c^{(1)} + c^{(2)} = c^{(1)} = \sqrt{\frac{g}{k} \cdot \tanh kh} \quad (2.15)$$

The propagation velocity is thus unchanged when compared to the 1. order theory. If we are not looking at a closed wave basin, the actual value of  $U$  ( $= \bar{u}$  for  $z < \eta_{min}$ ) must be prescribed.

In many books  $U = 0$  is assumed, but not mentioned. Watch out!

## 2.2 Third Order Stokes Waves

The most important result from the 3. order theory is  $c^{(3)} \neq 0$ , which gives:

$$c = c^{(1)} + c^{(2)} + c^{(3)} = c^{(1)} + c^{(3)} = \sqrt{\frac{g}{k} \cdot \tanh kh} + c^{(3)} \quad (2.16)$$

as  $c^{(2)} = 0$ .

The general expression for  $c^{(3)}$  is given in Svendsen & Jonsson (1980). Here we shall look at deep water only, where the expression reads:

$$c = \sqrt{\frac{g}{k}} \cdot \sqrt{1 + \left(\frac{kH}{2}\right)^2} = c^{(1)} \cdot \sqrt{1 + \left(\frac{kH}{2}\right)^2} \quad (2.17)$$

It is seen that the propagation velocity depends on  $H$ .

Notice that equation (2.17) is based on the assumption  $\bar{q} = 0$ .

## 2.3 Fifth Order Stokes Waves

The extent of calculations increases dramatically each time an extra term is included in each series for the variables. see e.g. the series for  $\varphi$  in equation (1.11). Despite this a 5. order theory was published in 1960, see Skjelbreia & al. (1960).

Notice that Skjelbreia assumed  $U = 0 \Leftrightarrow \bar{q} > 0$ , but in chapter 4 Skjelbreia's theory



is modified to handle  $U \neq 0$ .

The calculations in Skjelbreia's solution go like this:

- Given :  $T, h$  and  $H$   
 Calculate : 1)  $L$   
               2)  $\varphi$   
               3)  $u, w$  and  $\eta$

The wavelength  $L$  and the coefficient  $\lambda$  are found by iteration of the equations:

$$L = \frac{gT^2}{2\pi} \tanh(kh) \cdot (1 + \lambda^2 C_1 + \lambda^4 C_2) \quad (2.18)$$

and

$$\pi H = L \left( \lambda + \lambda^3 B_{33} + \lambda^5 (B_{35} + B_{55}) \right) \quad (2.19)$$

The coefficients  $B_{lm} = B_{lm}(kh)$  and  $C_n = C_n(kh)$  depend on  $kh$  (where  $k = 2\pi/L$  as usual). The expressions are given in Skjelbreia (1960), but notice that in Skjelbreia's original expression for  $C_2$ , the factor +2592 must be replaced by -2592, see Nishimura & al. (1977).

In practice values of  $L$  and  $\lambda$  are obtained by an iterative solution of the equations (2.18) and (2.19) rewritten to:

$$F(k, \lambda) = \omega^2 - gk \tanh(kh) \cdot (1 + \lambda^2 C_1 + \lambda^4 C_2) = 0 \quad (2.20)$$

and

$$f(\lambda, k) = \frac{kH}{2} - \left( \lambda + \lambda^3 B_{33} + \lambda^5 (B_{35} + B_{55}) \right) = 0 \quad (2.21)$$

To keep things simple *the Bisection Method* is recommended to solve  $F(k, \lambda) = 0$  and  $f(k, \lambda) = 0$  in each iteration.

The iteration is initiated by guessing  $\lambda = 0$ , because this value corresponds to  $L = L^{(1)}$ , see equation (2.18). This first  $\lambda$ -value is denoted  $\lambda_1$ , i.e. we have  $\lambda_1 = 0$ .

The first iteration:

- Solve  $F(k, \lambda_1) = 0$  and denote the solution  $k_1$
- Solve  $f(\lambda, k_1) = 0$  and denote the solution  $\lambda_2$

The second iteration:

- Solve  $F(k, \lambda_2) = 0$  and denote the solution  $k_2$

- Solve  $f(\lambda, k_2) = 0$  and denote the solution  $\lambda_3$

and so on ....

Iteration is repeated until convergence is obtained, e.g.  $|k_i - k_{i-1}| < 0.0001 k_i$ .

After calculation of  $L$  and  $\lambda$ , the velocity potential  $\varphi$  is calculated from:

$$\varphi = -\frac{c}{k} \sum_{j=1}^5 D_j \cosh jk(z+h) \sin j\theta \quad (2.22)$$

where

$$\begin{aligned} c &= L/T \\ \theta &= \omega t - kx \\ D_1 &= \lambda A_{11} + \lambda^3 A_{13} + \lambda^5 A_{15} \\ D_2 &= \lambda^2 A_{22} + \lambda^4 A_{24} \\ D_3 &= \lambda^3 A_{33} + \lambda^5 A_{35} \\ D_4 &= \lambda^4 A_{44} \\ D_5 &= \lambda^5 A_{55} \end{aligned}$$

and

$$A_{lm} = A_{lm}(kh) \text{ are known functions of } kh, \text{ see Skjelbreia (1960).}$$

Now the velocity components are found by differentiation of the velocity potential  $\varphi$ , giving:

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial \varphi}{\partial \theta} \cdot (-k) \quad (2.23)$$

or

$$u = c \sum_{j=1}^5 j D_j \cosh jk(z+h) \cos j\theta \quad (2.24)$$

and

$$w = \frac{\partial \varphi}{\partial z} \quad (2.25)$$

or

$$\boxed{w = -c \sum_{j=1}^5 j \cdot D_j \sinh jk(z+h) \sin j\theta} \quad (2.26)$$

Accelerations are found from:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \quad (2.27)$$

where

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} = \frac{\partial u}{\partial \theta} \cdot \omega = -c\omega \sum_{j=1}^5 j^2 \cdot D_j \cosh jk(z+h) \sin j\theta \quad (2.28)$$

$$\frac{\partial u}{\partial x} = ck \sum_{j=1}^5 j^2 \cdot D_j \cosh jk(z+h) \sin j\theta \quad (2.29)$$

$$\frac{\partial u}{\partial z} = ck \sum_{j=1}^5 j^2 \cdot D_j \sinh jk(z+h) \cos j\theta \quad (2.30)$$

The equations (2.28) and (2.29) shows (because  $\omega = ck$ ) that

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} \quad (2.31)$$

This result is obtained directly, if it is taken into account that the wave is propagating without changing form.

Finally  $\eta$  is calculated from:

$$\boxed{\eta = \frac{1}{k} \sum_{j=1}^5 E_j \cos j\theta} \quad (2.32)$$

where

$$\begin{aligned} E_1 &= \lambda \\ E_2 &= \lambda^2 B_{22} + \lambda^4 B_{24} \\ E_3 &= \lambda^3 B_{33} + \lambda^5 B_{35} \\ E_4 &= \lambda^4 B_{44} \\ E_5 &= \lambda^5 B_{55} \end{aligned}$$

and

$B_{lm} = B_{lm}(kh)$  are the same functions, applied under the calculation of  $L$ , see Skjelbreia (1960).

## 2.4 General comments to Stokes Waves

In the assessment of the order of magnitude of the individual terms it is assumed that  $o(\frac{h}{L}) = 1$ . Therefore problems occur for all orders of Stokes waves (except 1. order waves) on shallow water where  $\frac{h}{L} \ll 1$ . The problems results in secondary crests in the general wave trough. These secondary crests are not present in nature.

In practice it is necessary to have  $h/L > 0.10 - 0.15$ , to avoid secondary crests for Stokes 5. order waves. For waves of lower order the secondary crests appears for larger values of  $h/L$ .

Notice that all equations are derived under the assumption of potential flow,  $\text{curl } \vec{v} = \text{rot } \vec{v} = \vec{0}$ , and notice finally that *all* Stokes waves are *symmetrical* about the wave crest.

# Chapter 3

## Stream Function Theory

In order to get accurate values of wavelength and wave kinematics on shallow water,  $h/L < 0.10$ , one has to apply the so called stream function theory. This theory is based on an *approximate* numerical solution of the governing PDE together with the *exact* BC's fulfilled at  $z = \eta$ .

Because it is unnecessary to make assessments of terms and discard the small ones, the theory makes no demands to  $H/L$  or  $h/L$ .

It is assumed that we have potential flow and  $\bar{u} = U = 0$ , i.e. the wave is propagating on stagnant water. In chapter 4 it is shown how the problem can be solved if  $q = 0$  or  $U \neq 0$  is specified.

For the sake of convenience the wave is described in the  $(x_r, z_r)$ -system following the wave. See Figure 3.1. Seen from this coordinate system the wave profile is not moving, and the corresponding flow is consequently *steady*, which makes life easier!

The  $(x_r, z_r)$ -system has the velocity  $c_r$  compared to the stagnant water and the  $(x, z)$ -system fixed to the stagnant water body. So far this propagation velocity has simply been denoted  $c$ . However, in chapter 4 we will look at waves propagating on a body of water moving with the velocity  $U$  compared to the sea bottom. In this case the wave will also propagate with the velocity  $c_r$  compared to the water body, but the velocity compared to the bottom will be  $c_a = c_r + U$ .

Notice that the bottom and the  $(x, z)$ -system are moving to the left with the velocity  $c_r$  seen from the  $(x_r, z_r)$ -system. See Figure 3.1. Notice that  $z_r = z$ , and in the following we will not distinguish between  $z$  and  $z_r$ .

By introduction of the stream function  $\psi$  defined by the equations:

$$u = -\frac{\partial\psi}{\partial z} \tag{3.1}$$

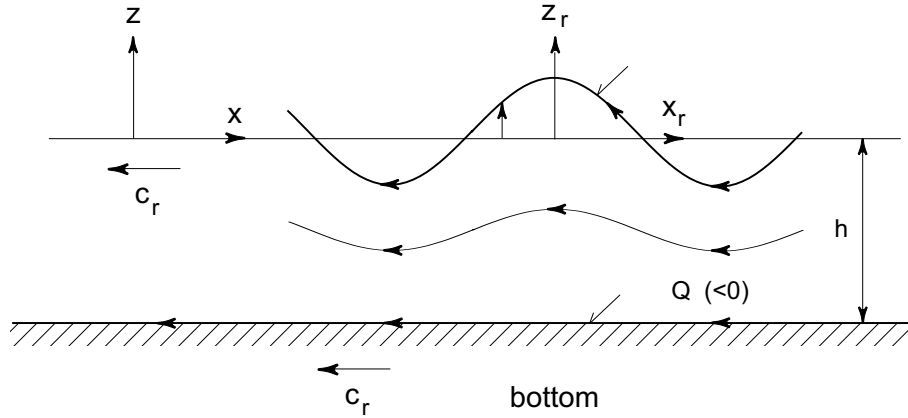


Figure 3.1: Sketch of the flow observed from a coordinate system following the wave.

and

$$w = \frac{\partial \psi}{\partial x_r} \quad (3.2)$$

the continuity equation for an incompressible liquid

$$\frac{\partial u}{\partial x_r} + \frac{\partial w}{\partial z} = 0 \quad (3.3)$$

is automatically fulfilled. The assumption of irrotational flow  $\Rightarrow \text{curl } \vec{v} = \text{rot } \vec{v} = \vec{0}$ , which for plane flow reads:

$$\frac{\partial w}{\partial x_r} - \frac{\partial u}{\partial z} = 0 \quad (3.4)$$

If equation (3.1) and equation (3.2) are substituted into (3.4), the result reads:

$$\boxed{\frac{\partial^2 \psi}{\partial x_r^2} + \frac{\partial^2 \psi}{\partial z^2} = 0} \quad (3.5)$$

i.e.  $\psi$  must fulfill the Laplace equation. The flow is sketched in Figure 3.1. The kinematic boundary conditions (no flow across a stream line) reads:

$$\boxed{\psi = Q} \quad \text{for } z = -h \quad (3.6)$$

and

$$\boxed{\psi = 0} \quad \text{for } z = \eta \quad (3.7)$$

Here the discharge  $Q$  through a vertical section is given by

$$Q = \int_{-h}^{\eta} u \, dz \quad (3.8)$$

The dynamic boundary condition at the free surface reads:

$$p = \text{constant}$$

and substitution into the generalized Bernoulli equation gives:

$$\boxed{g\eta + \frac{1}{2}(u^2 + w^2) + \frac{\partial\varphi}{\partial t} = R} \quad \text{for } z = \eta \quad (3.9)$$

where  $\frac{\partial\varphi}{\partial t} = 0$  due to steady flow.  $R$  is named the Bernoulli constant.

The main idea in the theory is the assumption that the stream function may be approximated by:

$$\boxed{\psi(x_r, z) = c_r(z + h) + \sum_{j=1}^N B_j \frac{\sinh jk(z + h)}{\cosh jkh} \cos jkx_r + Q} \quad (3.10)$$

The right hand side of equation (3.10) may be interpreted as a *truncated* Fourier-series of an *even* function. For waves symmetrical about the wave crest the stream function must be an even function, and we may therefore expect that the equation (3.10) will approximate  $\psi$  arbitrarily well if  $N$  is chosen big enough.

It is also seen that the expression for  $\psi$  fulfills both the bottom condition (3.6) and the Laplace equation, because the individual terms all fulfills the two equations. Notice also that an assumption of periodicity is hidden in equation (3.10), because  $\psi(x_r, z) = \psi(x_r + L, z)$ .

To calculate the stream function we therefore need to determine the  $N$  unknown coefficients  $B_j$ ,  $c_r$ ,  $k$  (or  $L$ ) and  $Q$  ( in total  $N + 3$  unknown).

This is done by *exact* fulfillment of the free surface conditions at  $N + 1$  points.

The kinematic BC (3.7) reads:

$$\psi(x_r, \eta) = 0 = c_r(\eta + h) + \sum_{j=1}^N B_j \frac{\sinh jk(\eta + h)}{\cosh jkh} \cos jkx_r + Q \quad (3.11)$$

and the dynamic BC (3.9) reads

$$g\eta + \frac{1}{2} \left( \left( -\frac{\partial\psi}{\partial z} \right)^2 + \left( \frac{\partial\psi}{\partial x_r} \right)^2 \right) = R$$

or

$$\begin{aligned} g\eta + \frac{1}{2} \left[ -c_r - k \sum_{j=1}^N jB_j \frac{\cosh jk(\eta + h)}{\cosh jkh} \cos jkx_r \right]^2 \\ + \frac{1}{2} \left[ -k \sum_{j=1}^N jB_j \frac{\sinh jk(\eta + h)}{\cosh jkh} \sin jkx_r \right]^2 = R \end{aligned} \quad (3.12)$$

In this way  $2N + 2$  equations are set up and apparently the system of equations seems to be over-determined. However, the  $\eta$ -values at the  $N + 1$  point are also unknown, giving:  $\eta_j$  ( $N + 1$  values),  $B_j$  ( $N$  values) plus  $c_r$ ,  $k$ ,  $Q$  and  $R$ , in total  $2N + 5$  unknown.

Therefore we must set up 3 extra equations in order to solve the system of equations.

Incompressible fluid corresponds to:

$$\bar{\eta} = \frac{1}{L} \int_0^L \eta \cdot dx_r = 0 \quad (3.13)$$

and the two definitions

$$H = \eta_{max} - \eta_{min} \quad (3.14)$$

$$L = c_r T \quad (3.15)$$

gives the two last equations necessary to solve the system. Notice that the equations are *non-linear*, but practice has shown that the non-linear equations can be solved by use of a generalized Newton-Raphson iteration, see Appendix A.

After the solution of the system, the velocity potential  $\psi$  is calculated by (3.10) and  $(u, w)$  by use of (3.1) og (3.2). It should be remembered that  $u$  in equation (3.1) has to be adjusted with the velocity  $c_r$ , when particle velocities with respect to the bottom are calculated.



At Appendix A expressions for  $u$  og  $w$  and corresponding accelerations are derived. Notice that in non-linear wave theory accelerations must be total accelerations (see e.g.  $du/dt$ , equation (2.27)), because the convective terms cannot be discarded as it is done in 1. order theory.

The surface elevation  $\eta(x)$  at arbitrary  $x$ -values are found by use of a finite Fourier-series based on the  $N + 1$ -values of  $\eta$ , giving:

$$\eta(x_r) = 2 \sum_{j=1}^{N-1} a_j \cos jkx_r + a_N \cos Nkx_r \quad (3.16)$$

Notice that the wave crest is situated at  $x_r = 0$  and that the missing factor 2 in the last term is *not* a misprint, see Appendix A.

# Chapter 4

## Waves propagating on a uniform current ( $U \neq 0$ )

### 4.1 Introduction

We will look at waves propagating on a body of water moving with uniform velocity  $U$  with respect to the sea bottom.

Only the mathematical description is given here. The effect from the current on wave breaking, refraction and wave forces on structures will not be considered.

Three different coordinate systems are used to describe the flow:

- 1) the  $(x_a, z_a)$ -system fixed to the sea bottom
- 2) the  $(x, z)$ -system fixed to the water body
- 3) the  $(x_r, z_r)$ -system following the wave

On Figure 4.1 is sketched the flow situation seen from the system fixed to the sea bottom.

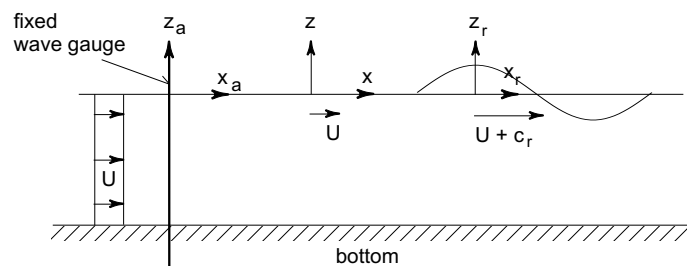


Figure 4.1: Sketch of the flow observed from the  $(x_a, z_a)$ -coordinate system fixed to the bottom

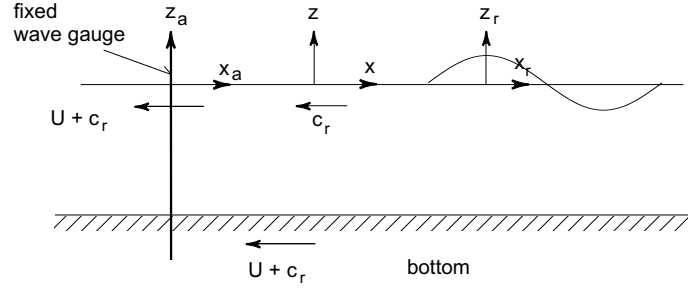


Figure 4.2: Sketch of the flow observed from the  $(x_r, z_r)$ -coordinate system following the wave

If the wave propagation velocity with respect to the fixed bottom is named  $c_a$ , it follows directly from Figure 4.1 that

$$c_a = c_r + U \quad (4.1)$$

where  $c_r$  is the wave propagation velocity with respect to the water body. Because a wave with a given length and height cannot "feel" if it is propagating on a fixed or a moving water body, the description of the flow in the following  $(x_r, z_r)$ -system will be the same whether  $U = 0$  or  $U \neq 0$ . All equations derived earlier in this system are consequently also valid when  $U \neq 0$ .

However, the wave period is normally measured at a fixed wave gauge. The equation relating wavelength and wave period must therefore be rewritten to

$$L = c_a \cdot T \quad (4.2)$$

where  $T$  is the wave period measured at a fixed point and  $c_a$  is the propagation velocity of the wave with respect to the sea bottom.

It is also clear that different wave periods will be observed from the the fixed  $(x, z)$ -system and the following  $(x_r, z_r)$ -system. The period observed from the  $(x_r, z_r)$ -system is here denoted  $T_r$ . However, we will observe the same wavelength (and the wave number) from both coordinate-systems, which leads the equations:

$$c_a = \frac{L}{T} = \frac{2\pi}{kT} = \frac{\omega_a}{k} \quad (4.3)$$

$$c_r = \frac{L}{T_r} = \frac{2\pi}{kT_r} = \frac{\omega_r}{k} \quad (4.4)$$

Substitution of equation (4.3) and (4.4) into equation (4.1) gives

$$\omega_a = \omega_r + kU \quad (4.5)$$

or

$$\omega_r = \omega_a - kU \quad (4.6)$$

The period  $T_r$  can be found from this equation, and the shift in frequency due to the current is named a Doppler shift.

## 4.2 Stokes Theory

In order to find the wavelength the dispersion equation is applied. The dispersion equation reads:

1. and 2. order theory :

$$\omega_r^2 = g k \tanh kh \quad (4.7)$$

5. order theory :

$$\omega_r^2 = g k \tanh kh \cdot (1 + \lambda^2 C_2 + \lambda^4 C_4) \quad (4.8)$$

The two versions of the dispersion equation are seen to be very similar. Therefore only the solution in case of 5. order theory is described in the following.

If  $\omega_r = \omega_a - kU$  is substituted into equation (4.8), the dispersion equation for 5. order waves reads:

$$(\omega_a - kU)^2 = g k \tanh kh \cdot (1 + \lambda^2 C_2 + \lambda^4 C_4) \quad (4.9)$$

which must be solved by iteration together with equation (2.19). Equation (4.9) is rewritten to:

$$FU(k, \lambda) = (\omega_a - kU)^2 - gk \tanh(kh) \cdot (1 + \lambda^2 C_1 + \lambda^4 C_2) = 0 \quad (4.10)$$

and equation (2.19) to:

$$f(\lambda, k) = \frac{kH}{2} - (\lambda + \lambda^3 B_{33} + \lambda^5 (B_{35} + B_{55})) = 0 \quad (4.11)$$

Hereafter the iterative solution of  $FU(k, \lambda) = 0$  and  $f(\lambda, k) = 0$  proceeds as described in chapter 2 to obtain values of  $k$  and  $\lambda$ .

Velocities and accelerations are calculated as described in chapter 2, but remember to add  $U$  to the horizontal velocity calculated by equation (2.24)!

In order to simulate the conditions in a wave basin, the condition  $\bar{q} = 0$  must be fulfilled. In the following it is described how a solution with an arbitrary value of  $\bar{q}$

is obtained.

The average discharge is given, i.e.  $\bar{q} = \bar{q}^*$ , where  $\bar{q}^*$  is a known value. No equation describes directly the relation between  $\bar{q}$  and  $U$  for Stokes' waves, but the problem may be solved by iteration this way :

- 1) guess  $U = \bar{q}^*/h$
  - 2) find  $u(z, t)$  (remember  $U!$ ) and  $\eta(t)$  for  $0 < t < T$
  - 3) calculate  $q(t) = \int_h^{\eta(t)} u(z, t) dz$  for  $0 < t < T$
  - 4) calculate  $\bar{q} = 1/T \int_0^T q(t) dt$
  - 5) if  $\bar{q} \neq \bar{q}^*$  re-calculate from step 2) with  $U_{new} = U_{old} - h(\bar{q} - \bar{q}^*)$
- until e.g.  $|\bar{q} - \bar{q}^*| < 0.001\bar{q}^*$ .

The integrals in step 3) and 4) are calculated numerically.

### 4.3 Stream Function Theory

We have introduced the new variabel,  $c_a$ , but also set up the new equation (4.1), which makes it possible to solve the flow problem when  $U \neq 0$ . The system of  $2N + 6$  equations are again solved by iteration.

In order to simulate the conditions in a wave basin, the condition  $\bar{q} = 0$  must be fulfilled. In the following it is described how a solution with an arbitrary value of  $\bar{q}$  is obtained.

First we note that  $\bar{q}_{wave}$ , which is the part of  $\bar{q}$  caused by the presence of waves, must be

$$\bar{q}_{wave} = Q - Q_{H=0} = Q - (-c_r h) = Q + c_r h \quad (4.12)$$

if observed from the  $(x_r, z_r)$ -system. It is so, because one would observe a discharge of  $-c_r h$  in this system, when  $H$  is very small. The propagation velocity  $c_r$  is nearly independent of  $H$ .

As the  $(x, z)$ -system has the velocity  $U$  with respect to the fixed  $(x_a, z_a)$ -system, we would in the latter system observe the average discharge

$$\bar{q} = q_{current} + \bar{q}_{wave} = Uh + Q + c_r h \quad (4.13)$$

Substitution of equation (4.1) into equation (4.13) gives:

$$\bar{q} = Uh + Q + (c_a - U) h$$

or

$$\bar{q} = Q + c_a h \tag{4.14}$$

Equation (4.14) is the extra equation necessary due to the extra unknown  $c_a$  in case  $\bar{q}$  is specified. Again iteration is used to solve the system of  $2N + 6$  equations.

Notice that the actual value of  $U$  can be found by equation (4.1) because both  $c_r$  and  $c_a$  are known after the solution of the system of equations.

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# Appendix A

## A.1 Solution of a system of non-linear equations by means of Newton-Raphson's method

The extended Newton-Raphson's method is first explained by means of this simple system of equations:

$$F(x, y) = 0 \tag{A.1}$$

og

$$G(x, y) = 0 \tag{A.2}$$

It is assumed that  $F(x, y)$  and  $G(x, y)$  are known functions of  $x$  and  $y$ . We also assume that  $(x_0, y_0)$  is an approximate solution to equation (A.1) and (A.2). This means that

$$F(x_0 + dx, y_0 + dy) = 0 \tag{A.3}$$

and

$$G(x_0 + dx, y_0 + dy) = 0 \tag{A.4}$$

where  $dx$  og  $dy$  are "small" quantities.

If  $F$  and  $G$  are expanded into Taylor series, where only 1. order terms are kept, equation (A.3) and (A.4) can be rewritten to

$$F(x_0, y_0) + dx\left(\frac{\partial F}{\partial x}\right)_0 + dy\left(\frac{\partial F}{\partial y}\right)_0 = 0 \tag{A.5}$$



and

$$G(x_0, y_0) + dx\left(\frac{\partial G}{\partial x}\right)_0 + dy\left(\frac{\partial G}{\partial y}\right)_0 = 0 \quad (\text{A.6})$$

Equation (A.5) and (A.6) can be solved with respect to  $dx$  og  $dy$  by means of a standard method, e.g. Gauss-elimination, because the equations are linear with respect to  $(dx, dy)$ . The matrix

$$\begin{bmatrix} \left(\frac{\partial F}{\partial x}\right)_0 & \left(\frac{\partial F}{\partial y}\right)_0 \\ \left(\frac{\partial G}{\partial x}\right)_0 & \left(\frac{\partial G}{\partial y}\right)_0 \end{bmatrix} \quad (\text{A.7})$$

is named the Jacobi matrix and its elements may be calculated either analytically or numerically. When  $(dx, dy)$  are known, an improved solution,  $(x_1, y_1)$ , reads:

$$x_1 = x_0 + dx \quad (\text{A.8})$$

and

$$y_1 = y_0 + dy \quad (\text{A.9})$$

If the "correction"  $(dx, dy)$  is small enough, i.e. if

$$\Delta = \sqrt{(dx)^2 + (dy)^2} < \varepsilon \quad (\text{A.10})$$

where  $\varepsilon$  is an appropriate small number, we accept  $(x_1, y_1)$  as the solution to the system of equations. Else the calculations are repeated  $(x_1, y_1)$  as the approximate solution.

## A.2 Example: Stream Function Theory. Set up and solution of the system of equations

In this example it is shown how the equations are set up and solved for  $N = 2$ , i.e. the BC's are fulfilled at 3 points on the free surface. Notice that  $N = 2$  is chosen only to reduce the amount of writing. Normally that few points will not provide a solution with proper accuracy.

Because it assumed that the wave is symmetrical about the wave crest, it is sufficient to place the 3 points on half a wave length only. See Figure A.1.

The points are normally placed equidistant yielding

$$\Delta x = \frac{0.5 \cdot L}{N} \quad (\text{A.11})$$

One might believe that the accuracy would be improved, if the concentration of points was largest near the crest, but test calculations has shown that this is not the case, see Fenton (1980).

Given :  $h, T, H, \bar{q}$

Calculate :  $c_r, c_a, B_1, B_2, \eta_1, \eta_2, \eta_3, Q, R, L$  ( $= 2N + 6 = 10$  unknown) (A.12)

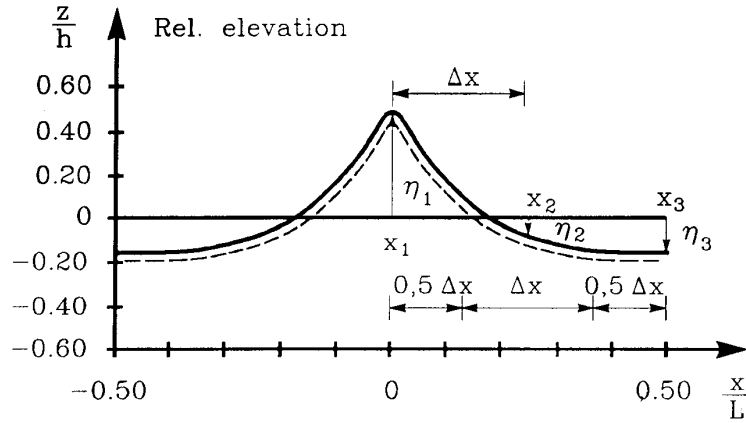


Figure A.1: Definition sketch,  $N = 2$ .

The kinematic BC, equation (3.11), at point  $(x_1, \eta_1)$  reads:

$$\begin{aligned} c_r(\eta_1 + h) + B_1 \frac{\sinh k(\eta_1 + h)}{\cosh kh} \cos kx_1 \\ + B_2 \frac{\sinh 2k(\eta_1 + h)}{\cosh 2kh} \cos 2kx_1 + Q = 0 \end{aligned} \quad (\text{A.13})$$

Similar expressions are set up at  $(x_2, \eta_2)$  and  $(x_3, \eta_3)$ .

The dynamic BC, equation (3.12), at point  $(x_1, \eta_1)$  reads:

$$\begin{aligned}
& g\eta_1 \\
& + \frac{1}{2} \left( -c_r - kB_1 \frac{\cosh k(\eta_1 + h)}{\cosh kh} \cos kx_1 - 2kB_2 \frac{\cosh 2k(\eta_1 + h)}{\cosh 2kh} \cos 2kx_1 \right)^2 \\
& + \frac{1}{2} \left( -kB_1 \frac{\sinh k(\eta_1 + h)}{\cosh kh} \sin kx_1 - 2kB_2 \frac{\sinh 2k(\eta_1 + h)}{\cosh 2kh} \sin 2kx_1 \right)^2 \\
& - R = 0
\end{aligned} \tag{A.14}$$

Similar expressions are set up at  $(x_2, \eta_2)$  and  $(x_3, \eta_3)$ .

Equation (3.13), i.e.  $\bar{\eta} = 0$ , is approximated by:

$$\sum_{i=1}^{N+1} \eta_i \Delta x_i = 0$$

where  $\Delta x_i = \Delta x$  for  $i = 2, \dots, N$  and  $\Delta x_i = 0.5\Delta x$  for  $i = 1$  and  $i = N + 1$ . In this case  $\Delta x = \frac{L}{4}$  yields:

$$\frac{L}{8} \eta_1 + \frac{L}{4} \eta_2 + \frac{L}{8} \eta_3 = 0 \tag{A.15}$$

The condition  $H = \eta_{max} - \eta_{min}$ , i.e. equation (3.14), reads

$$H - (\eta_1 - \eta_3) = 0 \tag{A.16}$$

$L = c_a \cdot T$ , i.e. equation (4.2), is rewritten to

$$L - c_a \cdot T = 0 \tag{A.17}$$

and finally  $\bar{q} = Q + c_a h$ , i.e. equation (4.14), is rewritten to

$$Q + c_a h - \bar{q} = 0 \tag{A.18}$$

In this way we have 10 equations, which are solved by means of Newton-Raphson's method.

The Jacobi matrix (A.7) is calculated by means of a straightforward numerical differentiation.

Initial guesses in the iteration reads:

$$\begin{aligned}
L &= L^{(1)}, \text{ i.e. the 1. order value} \\
c_r &= L/T
\end{aligned}$$

$$\begin{aligned}
c_a &= c_r \\
Q &= -c_r h \\
R &= 0.5 c_r^2 \\
\eta_i &= \frac{H}{2} \cos(kx_i) \quad \text{for } i = 1, 2 \text{ and } 3
\end{aligned}$$

plus

$$B_1 = -\frac{\pi H}{kT \tanh(kh)}$$

and  $B_2 = 0$

The guess of  $B_1$  is found by setting  $w^{(1)}$ , equation (1.10), equal to  $w = \frac{\partial \psi}{\partial x}$ , equation (3.2), and  $B_2 = 0$  is assumed because the size of Fourier coefficients normally are decreasing when the frequency is increasing.

In case of large values of  $H/h$  a step wise calculation is necessary to obtain convergence of the iteration. The reason is that the initial guesses corresponds to 1. order values. In each step the actual wave height is increased until the correct wave height is reached. The wave height is normally increased by  $\Delta H = \frac{H}{8}$  or less, and in each step the start guesses in the iteration are the values obtained in the previous step. Notice that only one calculation of the Jacobi matrix is necessary in each step.

### A.3 Calculation of $\eta(x)$ by means of Stream Function Theory

When the flow problem is solved, we know

$$(x_i, \eta_i) \quad i = 1, 2, \dots, N + 1$$

However, we often want to calculate an  $\eta(x)$ -value at an arbitrary  $x$ -value. This can be done by either interpolation of  $\eta$ -values or by an expansion of  $\eta$ -values in a truncated Fourier series. Here the Fourier series is applied, because this method also gives information about the accuracy of the calculation.

In general we know that the Fourier series of an even function, i.e.  $\eta(-x) = \eta(x)$ , with the period  $L$  reads:

$$\begin{aligned}
\eta(x) &= a_0 + 2 \sum_{j=1}^{\infty} (a_j \cos jkx + b_j \sin jkx) \\
\text{hvor } k &= \frac{2\pi}{L} \\
a_j &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \eta(x) \cos(jkx) dx = \frac{2}{L} \int_0^{\frac{L}{2}} \eta(x) \cos jkx dx \\
\text{og } b_j &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \eta(x) \sin jkx dx = 0
\end{aligned}$$

The coefficients  $a_j$  are calculated by

$$\begin{aligned} a_0 &= 0 \quad \text{see equation (3.13) or (A.15)} \\ \text{og } a_j &= \frac{2}{L} \sum_{i=1}^{N+1} \eta_i \cos jkx_i \Delta x_i \quad j = 1, 2, \dots, N \end{aligned}$$

In the series for  $\eta$  the maximum value of  $j$  is  $N$ . This is so because only  $2N$  points are available to describe the entire wave length. Consequently the shortest wave length in the Fourier series corresponds to  $\cos(Nkx)$ . See e.g. Newland (1975), where it is shown that at least two data points per oscillation are necessary to describe an oscillation properly.

Hereby the expression for  $\eta(x)$  reads

$$\eta(x) = 2 \sum_{j=1}^{N-1} a_j \cos jkx + a_N \cos Nkx \quad (\text{A.19})$$

Notice that the factor 2 is missing on the last term. This is necessary to obtain that (A.19) correctly gives  $\eta(x_i) = \eta_i$ , i.e. the graph of the Fourier series matches exactly the  $N + 1$  values of  $\eta$  on which it is based. The accuracy of the results from the Stream Function Theory is considered adequate if  $a_N \ll a_1$ , because this gives a free surface without tendency to secondary crests. If this requirement is not met, all calculations must be repeated with an increased value of  $N$ .

## A.4 Calculation of particle velocity and acceleration by means of Stream Function Theory

The expression for the stream function described in the  $(x_r, z_r)$ -system, equation (3.10), reads

$$\psi(x_r, z) = c_r(z + h) + \sum_{j=1}^N B_j \frac{\sinh jk(z + h)}{\cosh jkh} \cos jkx_r + Q$$

Using  $u = -\frac{\partial \psi}{\partial z}$  yields

$$u(x_r, z) = -c_r - \sum_{j=1}^N jkB_j \frac{\cosh jk(z + h)}{\cosh jkh} \cos jkx_r \quad (\text{A.20})$$

As  $c_r$  is the propagation velocity of the wave with respect to the water body, the particle velocity with respect to the water body ( or described in the  $(x, z)$ -system)

reads:

$$u(x, z) = - \sum_{j=1}^N jk B_j \frac{\cosh jk(z+h)}{\cosh jkh} \cos jk(x - c_r t) \quad (\text{A.21})$$

In equation (A.21) it is used that

$$x_r = x - c_r t \quad (\text{A.22})$$

If the water body has the velocity  $U$  with respect to the bottom the expression for  $u$  described in the fixed  $(x_a, z_a)$ -system reads

$$u(x_a, z) = U - \sum_{j=1}^N jk B_j \frac{\cosh jk(z+h)}{\cosh jkh} \cos jk(x_a - c_a t) \quad (\text{A.23})$$

because

$$x_r = x_a - c_a t \quad (\text{A.24})$$

In a similar way it is found that

$$w(x_a, z) = - \sum_{j=1}^N jk B_j \frac{\sinh jk(z+h)}{\cosh jkh} \sin jk(x_a - c_a t) \quad (\text{A.25})$$

The expression for horizontal particle accelerations reads

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \quad (\text{A.26})$$

As  $\frac{\partial u}{\partial t} = (-c_a) \cdot \frac{\partial u}{\partial x}$  because the wave propagates with constant form, equation (A.26) may be rewritten to

$$\frac{du}{dt} = (-c_a + u) \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \quad (\text{A.27})$$

Here  $u$  and  $w$  are given by equation (A.23) and (A.25). The derivatives  $\frac{\partial u}{\partial x}$  and  $\frac{\partial w}{\partial z}$  reads:

$$\frac{\partial u}{\partial x} = \sum_{j=1}^N (jk)^2 B_j \frac{\cosh jk(z+h)}{\cosh jkh} \sin jk(x_a - c_a t) \quad (\text{A.28})$$

and

$$\frac{\partial u}{\partial z} = - \sum_{j=1}^N (jk)^2 B_j \frac{\sinh jk(z+h)}{\cosh jkh} \cos jk(x_a - c_a t) \quad (\text{A.29})$$

The vertical particle acceleration reads

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \quad (\text{A.30})$$

Insertion of  $\frac{\partial w}{\partial t} = (-c_a) \cdot \frac{\partial w}{\partial x}$  (the wave propagates with constant form) gives

$$\frac{dw}{dt} = (-c_a + u) \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \quad (\text{A.31})$$

Here  $u$  and  $w$  are given by equation (A.23) and (A.25). The derivatives  $\frac{\partial w}{\partial x}$  and  $\frac{\partial w}{\partial z}$  reads:

$$\frac{\partial w}{\partial x} = - \sum_{j=1}^N (jk)^2 B_j \frac{\sinh jk(z+h)}{\cosh jkh} \cos jk(x_a - c_a t) \quad (\text{A.32})$$

$$\frac{\partial w}{\partial z} = - \sum_{j=1}^N (jk)^2 B_j \frac{\cosh jk(z+h)}{\cosh jkh} \sin jk(x_a - c_a t) \quad (\text{A.33})$$





