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Publication date:
2007

Document Version
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

Citation for published version (APA):

Fu, C-M. K., & Vestergaard, P. D. (2007). Distance domination in partitioned graphs. Aalborg: Department of Mathematical Sciences, Aalborg University. (Research Report Series; No. R-2007-05).

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by

Chin-Mei Kau Fu and Preben Dahl Vestergaard

R-2007-05

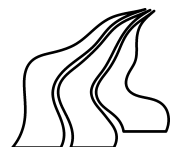
February 2007

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Distance domination in partitioned graphs

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To appear in **Congressus Numerantium**

Keywords: Domination, partition, tree.

AMS Mathematics Subject Classification: 05C75

Abstract For a graph G with its vertex set partitioned into, say two sets $V(G) = V_1 \cup V_2$, bounds for $\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2)$ have earlier been considered. This is generalized. We define a vertex set to distance d dominate all vertices at distance at most d from it. For partitioned graphs and any $d \geq 2$ we generalize theorems about ordinary distance one domination to distance d domination. Further, we give bounds for distance 2 domination of a graph partitioned into three sets and state a conjecture.

Definitions For $d \geq 1$ the vertex x in a graph is said to *distance d dominate* itself and all vertices at distance at most d away from x . A set D of vertices *distance d dominate* D and all vertices having distance at most d to D . The *distance d domination number* $\gamma_{\leq d}(G)$ of the graph G is the cardinality of a smallest set D which distance d dominates all vertices in G . For $d = 1$ we get the usual domination number $\gamma_{\leq 1}(G) = \gamma(G) = |D|$.

Let $k \geq 2$ be an integer and V_1, V_2, \dots, V_k a partition of $V(G)$. For $i, 1 \leq i \leq k$, we shall by $\gamma_{\leq d}(G, V_i)$ denote the order of a smallest set of vertices in G which distance d dominates V_i . I.e. there exists $D_i \subseteq V(G)$ such that every vertex of V_i either belongs to D_i or in G has distance at most d to a vertex in D_i , and $\gamma_{\leq d}(G, V_i) = |D_i|$ for a smallest such D_i . Let $f_{\leq d}(k, G)$ denote the maximum taken over all partitions V_1, \dots, V_k of $V(G)$ of the sum $\gamma_{\leq d}(G) + \sum_{i=1}^k \gamma_{\leq d}(G, V_i)$.

For $d = 1$ we write $\gamma(G, V_i)$ and $f(k, G)$. When no misunderstanding is possible we may write $\gamma_{\leq d}(V_i), f_{\leq d}(G)$ for short. Hartnell and Vestergaard gave upper bounds for $f_{\leq d}(k, G) = \gamma_{\leq d}(G) + \sum_{i=1}^k \gamma_{\leq d}(G, V_i)$, when $d = 1$. We shall generalize to $d \geq 1$.

For $d = 1$ and $k = 2$ we can slightly reformulate their result:

Theorem 1. [2] *Let G be a graph with at least 3 vertices in each component and let V_1, V_2 be any partition of $V(G)$. Then*

$$\gamma(G) + \gamma(G, V_1) + \gamma(G, V_2) \leq \frac{5}{4}|V(G)|, \text{ i.e. } f(2, G) \leq \frac{5}{4}|V(G)|.$$

Equality occurs if and only if each component of G satisfies

- (i) *the number of vertices is a multiple of four.*
- (ii) *Every vertex has degree one or is adjacent to exactly one degree one vertex.*
- (iii) *Every vertex of degree three or more is adjacent to exactly one degree two vertex having a degree one neighbour.*
- (iv) *All degree one vertices are in one class V_1 , all degree two vertices in the other class V_2 and vertices of degree ≥ 3 can be in either class.*

For $d \geq 2$ we have Theorem 2 below.

Theorem 2. *Let $d \geq 2$ and let G be a graph with at least $d + 2$ vertices in each component. For any partition V_1, V_2 of $V(G)$ we have*

$$\gamma_{\leq d}(G) + \gamma_{\leq d}(G, V_1) + \gamma_{\leq d}(G, V_2) \leq \frac{6}{2d+3}|V(G)|$$

and equality holds if and only if

- (i) *the order of each component of G is a multiple of $2d + 3$ and*
- (ii) *G can be constructed from a set of disjoint paths of lengths $2d + 2$ by arbitrarily adding edges between their central vertices.*

Proof of inequality.

It suffices to prove the inequality of Theorem 2 for trees. We shall use induction on $n = |V(G)|$.

The inequality is true for $n = d + 2$, for consider, in fact, any tree T on $n \geq d + 2$ vertices and with diameter at most $2d$; then $f_{\leq d}(2, T) \leq 3$, as we can place 3 dominators in the central vertex, when the diameter is an even number, and in an end vertex of the central edge when the diameter of T is an odd number. Obviously $3 \leq \frac{6}{2d+3}(d+2) \leq \frac{6}{2d+3}n$, so the inequality holds for small values of n .

Assume the inequality to be true for trees with fewer than n vertices. If T has diameter $\geq 2d + 3$ there is an edge e in T such that $T - e$ consists of two trees each having at least $d + 2$ vertices and the inequality holds. So we may assume T has diameter $2d + 1$ or $2d + 2$.

Case 1. $\text{Diam}(T) = 2d + 1$.

Let $P = v_1v_2 \dots v_{2d+2}$ be a diametrical path in T . If $T = P$, let $D = \{v_{d+1}, v_{2d+2}\}$, let D_1, D_2 both contain v_{d+1} and place v_{d+2} in D_i if $v_{d+2} \in V_i$, $i = 1, 2$.

Then D dominates $V(T)$, D_i dominates V_i for $i = 1, 2$, and $f_{\leq d}(T) \leq 5$. That satisfies the inequality as $d \geq 2$ implies $5 \leq \frac{6}{2d+3}(2d+2)$.

Otherwise, $n \geq 2d+3$ and with $D = D_1 = D_2 = \{v_{d+1}, v_{d+2}\}$ we obtain

$$f_{\leq d}(2, T) \leq 6 \leq \frac{6}{2d+3}n.$$

Case 2. $\text{Diam}(T) = 2d + 2$.

Let $P = v_1v_2 \dots v_{2d+3}$ be a diametrical path of T . If $\deg(v_i) \geq 3$ for any $i \neq d+2$ there is in T an edge e such that the two trees of $T - e$ both have $\geq d+2$ vertices and the inequality holds.

So we may assume that on P no other vertex than v_{d+2} has degree more than two. Assume $T - E(P)$ contains a path $v_{d+2}x_1x_2 \dots x_{d+1}$. If $\deg(x_j) \geq 3$ for any j , $1 \leq j \leq d$, the two trees in $T - v_{d+2}x_1$ both have $\geq d+2$ vertices and the inequality holds. So we may assume that $\deg(x_j) = 2$ for $1 \leq j \leq d$.

Thus T contains α paths, $\alpha \geq 2$, each of length $d+1$ and pendent from the central vertex v_{d+2} and possibly T also has other vertices, they all are within distance d from v_{d+2} .

Case 2A. Assume T consists of α paths of length $d+1$ pendent from v_{d+2} . Then $n = |V(T)| = 1 + \alpha(d+1)$ and we see that $f_{\leq d}(2, T) \leq 2\alpha + 2$ by placing α vertices adjacent to v_{d+2} in D , placing v_{d+2} in both D_1 and D_2 and placing the α vertices at distance $d+1$ from v_{d+2} in D_i when they belong to V_i , $i = 1, 2$. We certainly have $2\alpha + 2 \leq \frac{6}{2d+3}(\alpha d + \alpha + 1)$ as $\alpha \geq 2$.

Case 2B. Assume T consists of α paths of length $d+1$ pendent from v_{d+2} and also of vertices y_1, y_2, \dots, y_t , $1 \leq t$, such that for $1 \leq i \leq t$, y_i has distance $\leq d$ from v_{d+2} .

Note that those of y_1, y_2, \dots, y_t , $1 \leq t$ which are within distance $d-1$ from v_{d+2} are dominated by the D-dominators already chosen in Case 2A. For the remaining vertices y_i at distance d from v_{d+2} there exists in T a path $v_{d+2}y_1y_2 \dots y_d$ and we have $n \geq 1 + \alpha(d+1) + d$. Taking the dominators from case 2A together with v_{d+2} added to D we obtain

$$f_{\leq d}(2, T) \leq 2\alpha + 3 \leq \frac{6}{2d+3}(1 + \alpha(d+1) + d) \leq \frac{6}{2d+3}n.$$

This proves the inequality of Theorem 2. Finally, let $f_{\leq d}(2, G) = \frac{6}{2d+3}|V(G)|$. Then deletion of edges from G to obtain a tree and smaller trees in the process of proving the inequality of Theorem 2 must at every stage preserve equality,

therefore the final components are paths P_{2d+3} and if additional edges have ends at other vertices than centers of these paths, we get inequality. This proves Theorem 2. \blacksquare

Comment. *The bound of Theorem 2 is best possible, but only slightly better than the crude evaluation $f_{\leq d}(2, G) \leq 3 \cdot \gamma_{\leq d}(G) \leq 3 \frac{1}{d+1} |V(G)|$. (cf. [4])*

For partition into 3 classes, a best possible inequality is given by Hartnell and Vestergaard [2].

Theorem 3. [Hartnell, Vestergaard 2003] *Let $n \geq 3$ be an integer. Let T be a tree on n vertices such that $T \notin \{P_4, P_7\}$ and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then*

$$\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) + \gamma_T(V_3) \leq \frac{7n}{5}.$$

For distance 2 domination of a tree T with its vertex set partitioned into 3 sets we shall prove.

Theorem 4. *Let $n \geq 4$ be an integer. Let T be a tree on n vertices and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then*

$$\gamma_{\leq 2}(T) + \gamma_{\leq 2}(V_1) + \gamma_{\leq 2}(V_2) + \gamma_{\leq 2}(V_3) \leq n.$$

Proof. It is enough to prove the theorem for trees. By induction on n it is enough to prove the theorem for trees T with diameter ≤ 6 , since otherwise, T has an edge e such that both trees in $T - e$ have ≥ 4 vertices. If T has diameter 2 or 4 it suffices to place its central vertex in each of D, D_1, D_2, D_3 . Similarly, if T has diameter 3 we can place an end vertex of the central edge in each of the four dominating sets. In these cases we have $f_{\leq 2}(3, T) \leq 4 \leq n$.

If T has diameter 5, let $v_1 \dots v_6$ be a diametrical path. Place 4 dominators in v_4 and for each vertex x at distance 3 from v_4 , $x \in V_i$, place a D_i -dominator in x and a D -dominator in b , the second last vertex on the unique path $xabv_4$ from x to v_4 . In all cases we obtain $f_{\leq 2}(3, T) \leq n$.

Assume T has diameter 6. Let $P = v_1 v_2 \dots v_7$ be a diametrical path in T . If $\deg(v_i) \geq 3$ for $i \neq 4$ there is an edge e in T such that the two trees in $T - e$ have at least 4 vertices and by induction the result follows. So we may assume that $\deg(v_2) = \deg(v_3) = \deg(v_5) = \deg(v_6) = 2$. We easily see that a path P_7 on seven vertices has $f_{\leq 2}(3, P_7) = 7$, i.e. P_7 satisfies Theorem 4, so assume $\deg(v_4) \geq 3$.

Let l denote the length of a longest path emanating from v_4 in $T - E(P)$, $l \leq 3$. For $l = 1$ we place 4 dominators in each of v_3, v_5 . For $l = 2, 3$ we place 4 dominators in v_4 and each vertex x at distance 3 from v_4 is chosen to

class-dominate itself, while we on $xabv_4$, the unique path from x to v_4 choose b for D -domination. That gives $f_{\leq 2}(3, T) \leq n$. This proves Theorem 4. ■

The inequality of Theorem 4 is best possible as shown by the following examples.

$$f_{\leq 2}(3, P_7) = 7, f_{\leq 2}(3, P_8) = 8.$$

A path on 9 vertices with a pendent edge from its central vertex has $f_{\leq 2}(3, T) = 10 = n$.

However, it can be proven that $f_{\leq 2}(3, T_{11}) \leq 10$ for any tree on n vertices and $f_{\leq 2}(3, T_{12}) \leq 11$ for any tree on 12 vertices. For any tree T_{13} on 13 vertices we have $f_{\leq 2}(3, T_{13}) \leq 12$. So possibly there is a stronger inequality for trees with sufficiently many vertices. Some references to domination in partitioned graphs are given below.

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