Symbolic dynamics for a kinds of piecewise smooth maps

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Abstract

Symbolic dynamics is effective for the classification of orbital types and their complexity in one dimensional maps. In this paper, techniques of symbolic dynamics are used to analyze the chaotic dynamical properties of a two-parameter family of piecewise smooth unimodal maps with one break point. Boundary crisis and interior crisis are described via the kneading sequences, while for the period-3 window, a subshift of finite type is constructed. In addition, based on the symbolic model, the topological entropy of the map is computed, and the existence of chaotic sets of Smale horseshoe type is also proved.

Keywords: Piecewise smooth map, symbolic dynamics, crises, topological entropy, Smale horseshoe.

1. Introduction

Nonsmooth phenomena appear naturally in many systems, such as electrical circuits with switches [1], mechanical systems with impact [2], economic business cycles theory [3], and other natural and man-made systems [5, 6]. Piecewise smooth systems exhibit many dynamical properties that are different from those of smooth systems [7]. The main reason for this is the existence of the so-called switching manifolds, i.e., the existence of several borders in the phase space. The invariant sets of the systems may collide with the switching manifolds, resulting in the occurrence of border-collision bifurcations and the specific topological structure of attractors [8]. Avrutin and Schanz [9] studied border-collision period-doubling bifurcation scenario, which are composed of a sequence of pairs of border-collision and pitchfork bifurcations. Gardini et al. [10] proved that the border-collision bifurcation leads to an attracting 2-cycle in the piecewise smooth Matsuyama map and that the cycle loses stability through the subcritical flip bifurcations. Tramontana et al. [11] analytically calculated the border-collision bifurcation curves in a discontinuous growth model, and completely explained the abundant bifurcation structure in the system.

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Symbolic dynamics and topological horseshoe theory play an important role to prove the existence of chaotic invariant sets. see e.g. [12–17]. The subshift of finite type is an important branch of symbolic dynamical systems, which has been widely studied. It is generally divided into two directions. One focuses on the theoretical research of subshift of finite type [18–21], mainly including nonwandering sets, recurrence, chaos, topological transitivity, and mixing. The other is studying the dynamics of concrete nonlinear systems by subshift of finite type, such as Lozi maps [22], certain quadratic maps [23], and NMR-laser [24].

Sushko et al. [25] considered the following piecewise smooth map with a break point [25], and described the structure of the periodicity regions of the 2D bifurcation diagram:

$$x_n \to f\left(x_n\right) = \begin{cases} f_L\left(x_n\right) = r x_n, & 0 \le x_n < 1 - \frac{r}{a}, \\ f_R\left(x_n\right) = a x_n\left(1 - x_n\right), & 1 - \frac{r}{a} \le x_n \le 1. \end{cases} \tag{1}$$

The map (1) consists of a linear map and a Logistic map with a break point is $1 - \frac{r}{a}$, where r and a are real parameters. In this work, we will study the chaotic dynamics of (1) by using symbolic dynamics.

The remaining of this paper is organized as follows. Some definitions and theorems used in this paper are briefly recalled in Section 2. The crisis phenomena and the two-dimensional bifurcation diagram are discussed in Section 3. In Section 4, the subshift of finite type of the system is constructed in a periodic-3 window. In Section 5, the topological entropy of the system is computed based on the symbolic dynamics. It is proved that the topological entropy is positive so there exist chaotic sets of Smale horseshoe type in the system. At last, some conclusions are given in Section 6.

2. Preliminaries

In this Section, we briefly recall the basic definitions and results that will be used in what follows. See [26, 27] for details.

Definition 2.1. *Let $A = (a_{ij})$ be a $n \times n$ matrix. If $a_{ij} = 0$ or 1 for all i, j; $\sum_{i=1}^{n} a_{ij} \ge 1$ for all i and $\sum_{j=1}^{n} a_{ij} \ge 1$ for all j, then A is called a transition matrix.*

Definition 2.2. *Let*

$$\Sigma(n) = \left\{ s = \left(s_0 s_1 \cdots s_i \cdots\right) \mid s_i \in \{1, 2, \ldots, n\} \right\}. \tag{2}$$

We define a metric as

$$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{n^i}, x, y \in \Sigma(n). \tag{3}$$

$\Sigma(n)$ is called the one side symbolic space with n symbols.

Definition 2.3. *The shift map $\sigma : \Sigma(n) \to \Sigma(n)$ is defined as follows*

$$\sigma\left(s_0 s_1 s_2 \cdots s_i \cdots\right) = \left(s_1 s_2 \cdots s_i \cdots\right). \tag{4}$$

It is easy to verify that σ is a continuous surjection.

Definition 2.4. *Suppose that $A_{n\times n}$ is a transition matrix.*

$$\Sigma_A = \left\{ s = (s_0 s_1 \cdots s_j \cdots) \in \Sigma(n) \mid a_{s_i, s_{i+1}} = 1, \forall i \geq 0 \right\} \tag{5}$$

is a subset of $\sum(n)$. Then the subsystem (Σ_A, σ_A) of $(\Sigma(n), \sigma)$ is called the subshift of finite type induced by $A_{n\times n}$, where $\sigma_A = \sigma \mid \Sigma_A$.

Theorem 2.1. ([28, 29] Mañé) *Let N be a compact interval of the real line or the circle and let $f : N \to N$ be a (piecewise) C^2 map. Let U be a neighbourhood of the set $C(f)$ of critical points of f. Then*

(1) All periodic orbits of f contained in $N \backslash U$ of sufficiently large period are hyperbolic and repelling.

(2) If all periodic orbits of f which are contained in $N \backslash U$ are hyperbolic, then there exists $C > 0$ and $\lambda > 1$ such that

$$|Df^n(x)| \geq C\lambda^n. \tag{6}$$

Whenever $f^i(x) \in N \backslash (U \cup B_0(f))$ for all $0 \leq i \leq n-1$, where $B_0(f)$ is the union of the immediate basins of the periodic attractors of f.

3. Crises

The concept of crises in dynamical systems was introduced by Grebogi, Ott, and Yorke in Refs. [30, 31], which were used to describe the phenomenon of sudden qualitative changes of attractor structure caused by collisions between chaotic attractors and unstable orbits. In this Section, we use symbolic sequence to analyze the crisis of the map (1). The map has a unique critical point $c = 1 - r/a$, such that f is monotonically increasing in the interval $[0, c)$ and monotonically decreasing in the interval $(c, 1]$.

Definition 3.1. *For $x \in I$, its itinerary under f is an infinite symbolic sequence $S(x) = (s_0 s_1 \cdots)$, where*

$$s_j = \begin{cases} L, f^j(x) < c, \\ C, f^j(x) = c, \\ R, f^j(x) > c. \end{cases} \tag{7}$$

Definition 3.2. *The kneading sequence $K(f)$ of f is the itinerary of $f(c)$, i.e., $K(f) = S(f(c))$.*

Firstly, we calculate the critical parameter value for the boundary crisis. Since f satisfies piecewise monotonicity, the inverse functions of f has two branches:

$$\begin{cases} f_L^{-1}(y) = \frac{y}{r}, \\ f_R^{-1}(y) = \frac{1+\sqrt{1-4y/a}}{2}. \end{cases} \tag{8}$$

When $K(f) = RL^\infty$, we have

$$\begin{cases} \theta = f_L^{-1}(\theta), \\ f(c) = f_R^{-1} \circ f_L^{-1}(\theta), \end{cases} \tag{9}$$

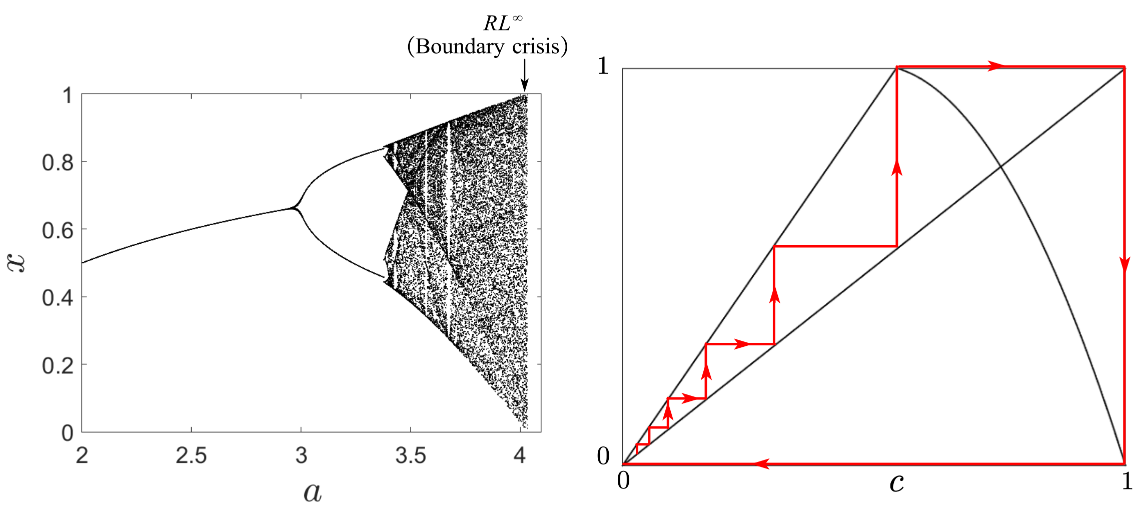
that is

$$\begin{cases} \theta = \frac{\theta}{r}, \\ a = \frac{r^2}{r-(1+\sqrt{1-4\theta/a})/2}. \end{cases} \tag{10}$$

To solve θ and a from (10), we consider the following iteration equation

$$\begin{cases} \theta_{n+1} = \frac{\theta_n}{r}, \\ a_{n+1} = \frac{r^2}{r-\left(1+\sqrt{1-4\theta_n/a_n}\right)/2}. \end{cases} \tag{11}$$

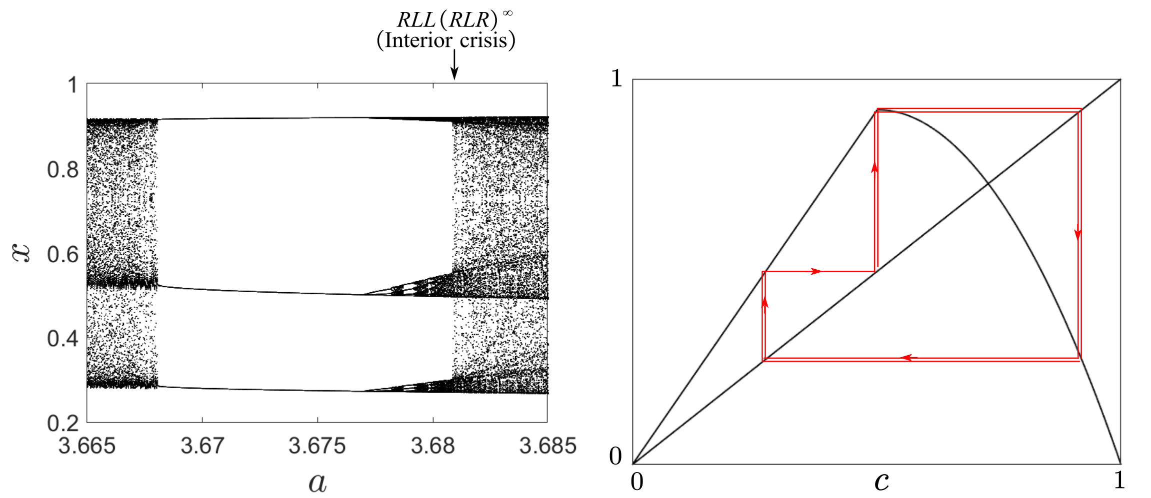
Fixing the parameter value $r = 1.835$ and choosing appropriate initial values a_0 and θ_0, the iteration sequence $a_0, a_1, a_2, \ldots$ quickly converges to $4.0326047\ldots$, which corresponds to the kneading sequence $K(f) = RL^\infty$. This implies that the chaotic orbit collides with the unstable fixed point $x = 0$ with symbol sequence L^∞, resulting in the sudden disappearance of the chaotic orbit, that is, a boundary crisis occurs. In fact, the kneading sequence RL^∞ can be regarded as a homoclinic orbit of the unstable fixed point $x = 0$, as shown in Figure 1(b).



(a) (b)

Figure 1: (a) The bifurcation diagram of map (1). (b) A homoclinic orbit of the unstable fixed point $x = 0$.

There is a period-3 window in Figure 1(a), and its partial enlargement is shown in Figure 2(a). When the kneading sequence $K(f) = RLL(RLR)^\infty$, the three chaotic bands collide with the unstable period-3 orbit with symbolic sequence $(RLR)^\infty$, which leads the three chaotic bands suddenly change and merge into a whole chaotic band. Thus, an interior crisis occurs, and its critical parameter values $3.6808107\ldots$ can be obtained through an iteration procedure

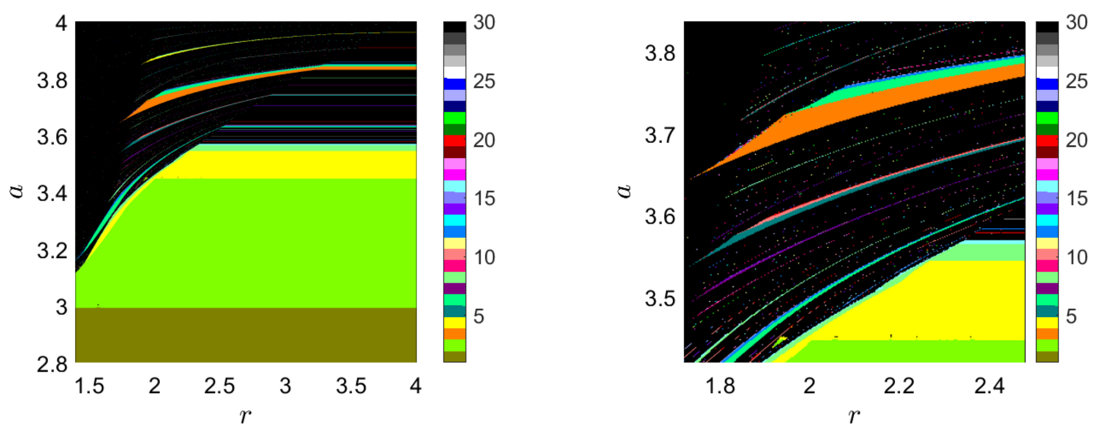


(a) (b)

Figure 2: (a) The partial enlargement of Figure 1(a). (b) The orbit corresponds to the symbolic sequence $RLL(RLR)^{\infty}$.

as above. However, when the parameter value exceeds the critical value of the interior crisis, most of the points of the chaotic orbit remain concentrated in the region where the original three chaotic bands were located.

The bifurcation diagram with respect to the parameters r and a are shown in Figure 3(a). The color bar is located on the right and different colors in the color bar represent different periods of the system (1). As the value of parameters r and a increase, complex dynamical behaviors occur.



(a) (b)

Figure 3: (a) The bifurcation diagram in (r, a)-plane, with an enlargement in (b).

4. Symbolic dynamics

In this Section, we use the finite type subshift symbolic dynamics to analyze the chaotic motion of the map $f(x)$. We choose the parameters $r = 1.835, a = 3.67$ to demonstrate our construction process. For such parameter values, the system has period three orbits, and hence exhibits chaos of Li-Yorke type [32]. We prove the chaos of Smale horseshoes type in this Section.

The map f has two periodic-3 orbits and they are given approximately by

$$\Gamma^s = (a_1, a_2, a_3) = (0.279845, 0.513531, 0.916828),$$

$$\Gamma^u = (b_1, b_2, b_3) = (0.291678, 0.535230, 0.912945).$$

Since $(f^3)'(a_i) \approx 0.557, (f^3)'(b_i) \approx 1.438 (i = 1, 2, 3)$, we obtain that Γ^s is stable and Γ^u is unstable.

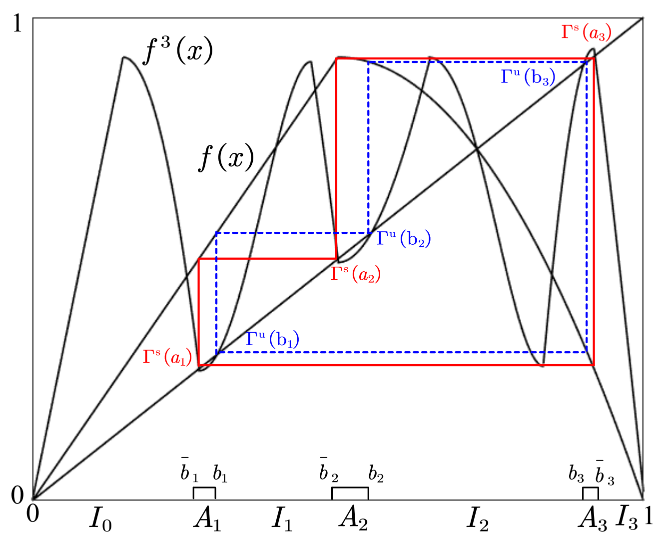


Figure 4: The interval coverage relation of map $f(x)$, where the solid red lines and dashed blue lines denote the graphical iteration, respectively.

The covering relation of interval of f is shown in Figure 4, where $b_2 = 0.535230$ and $\bar{b}_2 = 0.497518$ are the two preimages of b_3. $\bar{b}_1 = 0.271127$ and $\widetilde{b}_1 = 0.838258$ are the two preimages of $\bar{b}_2$. The preimage of $\bar{b}_1$ in the interval $[1/2, 1]$ is $\bar{b}_3 = 0.919671$.

Let

$$I_0 = \left[0, \bar{b}_1\right], \quad I_1 = \left[b_1, \bar{b}_2\right], \quad I_2 = \left[b_2, b_3\right], \quad I_3 = \left[\bar{b}_3, 1\right],$$
$$A_1 = \left(\bar{b}_1, b_1\right), \quad A_2 = \left(\bar{b}_2, b_2\right), \quad A_3 = \left(b_3, \bar{b}_3\right).$$

Note that f maps A_1 and A_3 monotonically onto A_2 and A_1, respectively, but f has a critical point that belongs to A_2 so f is not monotonic in this interval. Since, however, the maximum value of f is 0.9175, it follows that $f(A_2)$ is contained in A_3. Then we have

$$f(I_0) = I_0 \cup A_1 \cup I_1, \quad f(I_1) = I_2, \quad f(I_2) = I_1 \cup A_2 \cup I_2,$$

$$f\left(A_1\right)=A_2, \quad f\left(A_2\right)\subset A_3, \quad f\left(A_3\right)=A_1.$$

Taking I_1, I_2 as the vertices and the covering relationship as the edges, the directed graph is given in Figure 5.

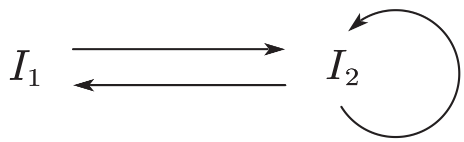


Figure 5: The directed graph.

The transition matrix corresponding to Figure 5 is

$$A=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Theorem 4.1. *Except* $0,\ a_i,\ b_i (i=1,2,3)$ *and the periodic points lie in* $I_1\cup I_2$*, the map* f *has no other periodic points.*

Proof. Because $A_1\cup A_2\cup A_3$ is the stable set of the periodic orbit Γ^s, there is no other periodic points in $A_1\cup A_2\cup A_3$. For $x\in I_1\cup I_2$, either its forward orbit $\gamma^+(x)\in I_1\cup I_2$ or its ω-limit set $\omega(x)=\Gamma^s$. For the second case, x is not a periodic point. If $x\in I_0\backslash\{0\}$, then there exists $n>0$ such that $f^n(x)\in A_1\cup I_1$. Thus, $f^l(x)\notin I_0\backslash\{0\}$ for $l\geq n$, and so x is not a periodic point. If $x\in I_3$, then $f(x)\in I_0$, so x is again not a periodic point. □

Theorem 4.2. $\Lambda=\{x\mid f^n(x)\in I_1\cup I_2, \forall n\geq 0\}$ *is a hyperbolic invariant set of* f*, i.e., there exists* $C>0$ *and* $\lambda>1$ *such that* $|Df^n(x)|\geq C\lambda^n$ *for all* $x\in\Lambda$*.*

Proof. Denoted by c the critical point of $f(x)$. Then $I=[f^2(c), f(c)]$ is an invariant interval of $f(x)$. We can see that Λ does not contain Γ^s and the critical point. Thus, Λ is a hyperbolic invariant set of f by Theorem 2.1. □

Lemma 4.1. *If* $s_0s_1\cdots s_{n-1}\cdots$ *is an allowable symbolic sequence of* f*, then* $\bigcap_{j=0}^{\infty} f^{-j}\left(I_{s_j}\right)$ *is a single-point set.*

Proof. Let $s_0s_1\cdots s_{n-1}$ be an allowable string of symbols with length n. By the interval coverage relation of map $f(x)$, we have

$$I_{s_0}\to I_{s_1}\to\cdots\to I_{s_{n-1}},$$

and so $I_{s_0}\cap f^{-1}\left(I_{s_1}\right)\cap\cdots\cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$ is nonempty.

We can show that $I_{s_0}\cap f^{-1}\left(I_{s_1}\right)\cap\cdots\cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$ is a single closed interval by the monotonicity of f on I_1 and I_2. To this end, assume that $I_{s_1}\cap f^{-1}\left(I_{s_2}\right)\cap\cdots\cap f^{-(n-2)}\left(I_{s_{n-1}}\right)$ be the single closed interval in I_{s_1}. If $s_1=1$, then

$$f^{-1}\left(I_{s_1}\cap f^{-1}\left(I_{s_2}\right)\cap\cdots\cap f^{-(n-2)}\left(I_{s_{n-1}}\right)\right)=f^{-1}\left(I_{s_1}\right)\cap f^{-2}\left(I_{s_2}\right)\cap\cdots\cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$$

has two closed intervals, one in I_0, and the other in I_2. If $s_1 = 2$, then there are also two closed intervals, one in I_1, and the other in I_2. Therefore, $I_{s_0} \cap f^{-1}(I_{s_1}) \cap \cdots \cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$ is a single closed interval in I_{s_0}.

For $x, y \in \bigcap_{j=0}^{\infty} f^{-j}\left(I_{s_j}\right)$ $(x < y)$, due to the monotonicity of f on the intervals I_1 and I_2, we have $f^q([x, y]) \subset I_{s_q}$ for any $q > 0$, so $[x, y] \subset \Lambda$. However, by Theorem 4.2 we have

$$|f^n(x) - f^n(y)| = |(f^n)'(\xi)||x - y| \geq C\lambda^n |x - y|, \quad (\xi \in [x, y]).$$

When n is sufficiently large, this contradicts due to the fact that the distance between $\{f^n(x)\}$ and $\{f^n(y)\}$ is bounded. Therefore, $\bigcap_{j=0}^{\infty} f^{-j}\left(I_{s_j}\right)$ is a single point set. □

Remark 4.1. *If $s_0 s_1 \cdots s_{n-1}$ is not an allowable string of Σ_A, then*

$$I_{s_0} \cap f^{-1}(I_{s_1}) \cap \cdots \cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$$

is an empty set. In fact, if $s_0 s_1 \cdots s_{n-1}$ is not an allowable string, then there exists $p(0 \leq p \leq n-2)$ such that $s_p = s_{p+1} = 1$. Assume that $I_{s_0} \cap f^{-1}(I_{s_1}) \cap \cdots \cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$ is not empty, then there exists x such that $f^p(x), f^{p+1}(x) \in I_1$. This leads to a contradiction.

Remark 4.2. *If $s_0 s_1 \cdots s_{n-1}$ and $s'_0 s'_1 \cdots s'_{n-1}$ are two different allowable strings in Σ_A. then $I_{s_0} \cap f^{-1}(I_{s_1}) \cap \cdots \cap f^{-(n-1)}\left(I_{s_{n-1}}\right)$ and $I_{s'_0} \cap f^{-1}\left(I_{s'_1}\right) \cap \cdots \cap f^{-(n-1)}\left(I_{s'_{n-1}}\right)$ are disjoint. Assume that this does not hold. Let $p(0 \leq p \leq n-1)$ be the smallest integer number such that $s_p \neq s'_p$. Then there exists x such that $f^p(x) \in I_{s_p}, I_{s'_p}$, this leads to a contraction.*

Theorem 4.3. *$f : \Lambda \to \Lambda$ topologically conjugates to $\sigma : \sum_A \to \sum_A$, where*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof. Let $h : \Lambda \to \Sigma_A,\ h(x) = (s_0 s_1 \cdots s_j \cdots),\ \left(f^j(x) \in I_{s_j}\right)$. We prove next that h is a homeomorphism from Λ to Σ_A and satisfies $h \circ f = \sigma_A \circ h$.

By Lemma 4.1, Remark 4.1 and Remark 4.2, h is a bijection. To prove that h is a continuous map, we choose $x \in \Lambda$ and assume that $h(x) = (s_0 s_1 \cdots s_j \cdots)$. For any $\varepsilon > 0$, take n sufficiently large such that $1/2^n < \varepsilon$. Since f is continuous, there exists $\delta > 0$ such that $f^i(O_\delta(x)) \subset I_{s_i}(0 \leq i \leq n)$, where $O_\delta(x) = \{y \in I \| y - x \| < \delta\}$. Thus, for any $y \in O_\delta(x) \cap \Lambda$ we have

$$d(h(x), h(y)) < \frac{1}{2^n} < \varepsilon.$$

This proves the continuity of h.

Λ is a compact set, h is a continuous bijection and Σ_A is a Hausdorff space, so h is a homeomorphism.

Finally, for $x \in \Lambda$, let $s = h(x)$ and $t = h(f(x))$, according to $f^j(x) \in I_{s_j}$ and $f^j(f(x)) = f^{j+1}(x) \in I_{s_{j+1}}$, we obtain $t_j = s_{j+1}$, i.e., $h \circ f = \sigma_A \circ h$. □

5. Topological entropy and chaotic invariant sets

The topological entropy is a non-negative topological invariant, which can be used to measure complexity of dynamical systems. In this Section we shall use the subshift of finite type constructed in the last Section to compute the topological entropy of the map f. First, we briefly recall the definition of topological entropy given by Bowen [33].

Let (X, d) be a metric space. A subset $F \subset X$ is called a $(n, \varepsilon)-$spanned set if for any $x \in X$ there exists $y \in F$ such that

$$d\left(g^i(x), g^i(y)\right) \leq \varepsilon, i = 0, 1, \cdots, n-1.$$

A subset $E \subset X$ is called a $(n, \varepsilon)-$separated set if for any $x, y \in E (x \neq y)$ there exists i with $0 \leq i < n$ such that

$$d\left(g^i(x), g^i(y)\right) > \varepsilon.$$

Denote by $r_n(\varepsilon, g)$ the minimal $(n, \varepsilon)-$spanned set of g and $s_n(\varepsilon, g)$ the maximal $(n, \varepsilon)-$separated set of g. Let

$$r(\varepsilon, g) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log r_n(\varepsilon, g),$$

$$s(\varepsilon, g) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log s_n(\varepsilon, g).$$

The topological entropy of g is defined as

$$h(g) = \lim_{\varepsilon\to 0} r(\varepsilon, g) = \lim_{\varepsilon\to 0} s(\varepsilon, g).$$

The following lemma gives a method to compute the topological entropy of $f \mid \Lambda$.

Lemma 5.1. **[26]** *Let σ be the shift map on symbolic space $\Sigma(n)$ and X a closed invariant subset of $\Sigma(n)$. Denote by ω_n the number of words of length n in X, i.e.,*

$$w_n = \#\left\{(s_0, \dots s_{n-1}) : s_j = x_j \text{ for } 0 \leq j < n \text{ for some } x \in X\right\}.$$

Then

$$h(\sigma \mid X) = \lim_{n\to\infty} \sup \frac{\log\left(w_n\right)}{n}.$$

Let $\sigma_A : \Sigma_A \to \Sigma_A$ be a subshift of finite type associated with the transition matrix A. Then

$$h\left(\sigma_A\right) = \log \rho(A),$$

where $\rho(A)$ is the spectral radius of A.

Lemma 5.2. **[34]** *Let J be a compact interval and $\varphi : J \to J$ a continuous map. If φ has topological entropy $h(\varphi) > 0$ then, for any λ with $0 < \lambda < h(\varphi)$ and any $N > 0$, there exist pairwise disjoint closed intervals $J_1, \dots, J_p$ and an integer $n > N$ such that $(1/n) \log p > \lambda$ and*

$$J_1 \cup \ldots \cup J_p \subseteq \textit{ int } \varphi^n\left(J_i\right) \quad (i = 1, \ldots, p).$$

Corollary 5.1. *The map f has chaotic invariant sets of Smale horseshoe type.*

Proof. Since the characteristic equation of the transition matrix A is

$$|\lambda E - A| = \lambda^2 - \lambda - 1 = 0.$$

The spectral radius of A is $\lambda_1 = \frac{1+\sqrt{5}}{2}$. By Lemma 5.1, the topological entropy of $f : \Lambda \to \Lambda$ is

$$h(f \mid \Lambda) = h(\sigma_A) = \log \rho(A) = \log \left(\frac{1+\sqrt{5}}{2} \right).$$

Since the topological entropy of f is positive, it follows from Lemma 5.2 that f has chaotic sets of Smale horseshoe type. □

6. Conclusions

In this paper, we discuss the chaotic dynamics of the unimodal piecewise smooth map by the techniques of symbolic dynamics. It is shown that the types of the crisis can be determined by the kneading sequences. In the period-3 window, it is proved that the topological entropy of the system is positive by constructing a subshift of finite type, this implies that there exist chaotic sets of horseshoe type in such parameter region. Though the proof is made for a concrete piecewise smooth map, many of the arguments can be applied to other unimodal maps.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflict of interest

The authors declare that they have no conflict of interest.

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