Cantor's illusion

Richard L. Hudson 4-4-2024

abstract

This analysis shows Cantor's diagonal definition in his 1891 paper was not compatible with his horizontal enumeration of the infinite set M. The diagonal sequence was a counterfeit which he used to produce an apparent exclusion of a single sequence to prove the cardinality of M is greater than the cardinality of the set of integers N.

keywords: Cantor, diagonal, infinite

1. the argument

Translation from Cantor's 1891 paper [1]:

Namely, let *m* and *n* be two different characters, and consider a set [*Inbegriff*] *M* of elements

$$E = (x_1, x_2, \dots , x_v, \dots)$$

which depend on infinitely many coordinates $x_1, x_2, \dots , x_v, \dots$, and where each of the coordinates is either *m* or *w*. Let *M* be the totality [*Gesamtheit*] of all elements *E*.
To the elements of *M* belong e.g. the following three:

$$E^I = (m, m, m, m, \dots),$$
$$E^{II} = (w, w, w, w, \dots),$$
$$E^{III} = (m, w, m, w, \dots).$$

I maintain now that such a manifold [*Mannigfaltigkeit*] *M* does not have the power of the series 1, 2, 3, …, *v*, ….

This follows from the following proposition:
"If $E_1, E_2, \dots, E_v, \dots$ is any simply infinite [*einfach unendliche*] series of elements of the manifold *M*, then there always exists an element E_0 of *M*, which cannot be connected with any element E_v."
For proof, let there be

$$E_1 = (a_{1.1}, a_{1.2}, \dots , a_{1,v}, \dots)$$
$$E_2 = (a_{2.1}, a_{2.2}, \dots , a_{2,v}, \dots)$$
$$E_u = (a_{u.1}, a_{u.2}, \dots , a_{u,v}, \dots)$$
$$\dots\dots\dots$$

where the characters $a_{u,v}$ are either *m* or *w*. Then there is a series $b_1, b_2, \dots b_v, \dots$, defined so that b_v is also equal to *m* or *w* but is *different* from $a_{v,v}$.
Thus, if $a_{v,v} = m$, then $b_v = w$.
Then consider the element

$$E_0 = (b_1, b_2, b_3, \dots)$$

of *M*, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer u, otherwise for that u and for all values of v.

$$b_v = a_{u,v}$$

and so we would in particular have

$$b_u = a_{u,u}$$

which through the definition of b_v is impossible. From this proposition it follows immediately that the totality of all elements of *M* cannot be put into the sequence [*Reihenform*]: $E_1, E_2, \dots, E_v, \dots$ otherwise we would have the contradiction, that a thing [*Ding*] E0 would be both an element of M, but also not an element of M.
(end of translation)

2. Cantor's enumeration

The symbols {0, 1} will be substituted for {m, w} for visual clarity.
Cantor defines an infinite set M consisting of elements En. Each En is an infinite one dimensional horizontal sequence composed of two symbols 0 and 1. He does not specify a rule of formation for sequences, thus they are assumed to result from a random process such as a coin toss. There is one sequence per row, and all sequences are unique differing in one or more positions. He then assigns coordinates to the array of symbols using a two dimensional (u, v) grid.

u \ v	1	2	3	4	5	6	7	8	...
1	0	0	1	1	0	0	1	1	
2	1	1	1	1	1	1	1	1	
3	0	1	0	1	0	1	0	1	
4	1	0	1	1	1	0	1	1	
5	0	1	1	1	0	0	1	0	
6	0	1	0	1	0	1	0	1	
7	1	1	1	1	0	0	0	0	
8	1	0	1	0	0	1	1	1	
9	1	0	1	0	1	1	0	0	
10	1	0	0	1	0	1	0	0	
⋮	⋮							⋮	
D	0	1	0	1	0	1	0	1	
E_0	1	0	1	0	1	0	1	0	...

fig.1

2.1 orientation

Cantor then defines a diagonal sequence D (red) composed of symbols with coordinates (u, u). The negation of a sequence differs in all positions. Using D as a template, he interchanges all 0's and 1's to produce E_0 as the negation of D or (not D). He declares, E_0 as a horizontal sequence, cannot be in the enumeration since it will conflict with each coordinate (u, u).

2.2 issues

u \ v	1	2	3	4	5	6	7	8
1	0							
2		1						
3	1		0					
4		0		1				
5			1		0			
6				0		1		
7					1		0	
8						0		1
9							1	
10								0

fig.2

1. A copy of a geometric form inherits the properties of the original, thus E_0 should also be a diagonal sequence. Neither D nor E_0 are compatible with the horizontal enumeration.

2. There is an inconsistency in Cantor's sequence definition. The horizontal sequences were formed independently of each other, and entered randomly in the enumeration. D was formed using a specific rule of formation dependent on one element from each horizontal sequence and could only be a qualified sequence in a diagonal enumeration as in fig.2. If the enumeration consisted of diagonal sequences, there would be no interference of D and E_0 since they are parallel. In the original enumeration all horizontal sequences were parallel and did not interact. At this point Cantor is comparing two different enumerations, a diagonal form with a horizontal form. Both forms cannot coexist in the same enumeration without interference.

u \ v	1	2	3	4	5	6	7	8
1	0							
2		1						
3			0					
4				1				
5					0			
6	0	1	0	1	0	1	0	1
7							0	
8								1
9								
10								

fig.3

Fig.3 eliminates the clutter of a full enumeration to emphasize the relation of a diagonal and horizontal form. As shown the diagonal D could exist anywhere in the enumeration since duplicates cannot be detected with a single comparison such as coordinate (6, 6). If u6 was replaced with E_0 then a conflict would appear at coordinate (6, 6), which can't be 0 and 1 simultaneously. Since the sequences are formed from two symbols, there are two subsets M0 and M1, one containing sequence S, the other containing its negation (not S). If D is a member of M0 then by symmetry E_0 is a member of M1, making both members of M.

3. refutation

For this purpose the symbols {0, 1} are substituted for {m, w}, for visual clarity. A sequence or string is represented as s.

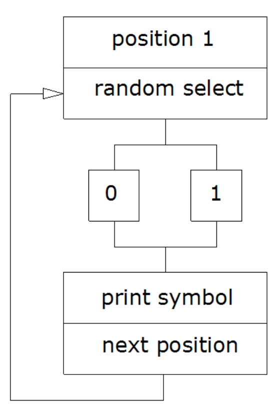

fig.4

Fig.4 is a basic flow chart for forming any s in the process of generating a binary tree graph T, a model that represents the Cantor set M in terms of sets and subsets.

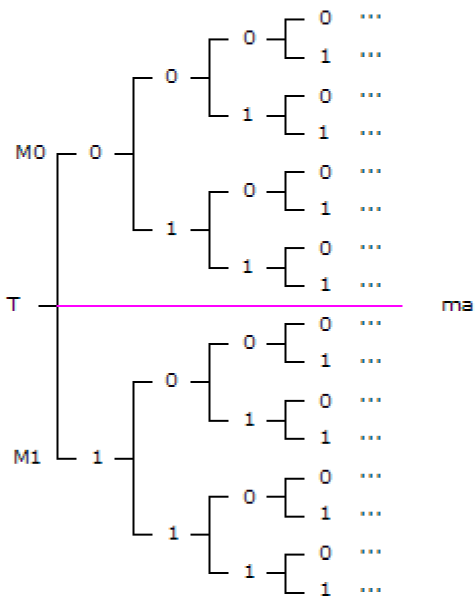

fig.5

Any s must begin with 0 or 1. The set M can be divided into two subsets M0 and M1. Each selection is independent of all others. and T contains copies of itself at every branch, thus the perpetual loop in fig.4. The following sample is an array of symbols using Cantor's coordinate system (v, u) for column and row. Each s has no last v and the list has no last row.

01111...
10000...
00111...
11000...
00011...
⋮

D=00101...
E_0=11010...

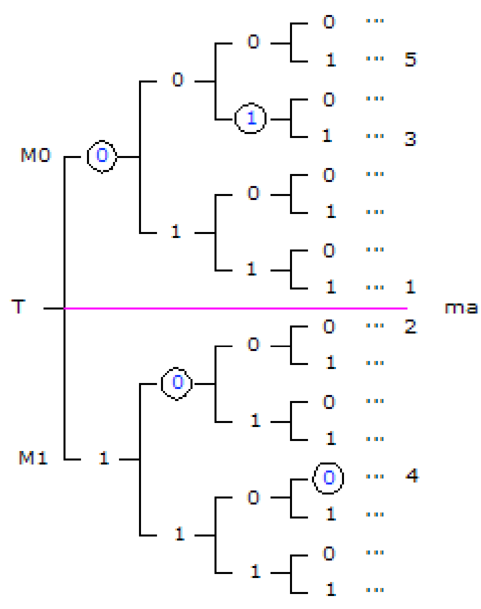

fig.6

Fig.6 tracks the path of D with row numbers from the sample on the right. As a sequence, it is not a contiguous path in the tree, but jumps between subsets M0 and M1 which is not possible. A path must continuously progress in v remaining in its initial subset for its entire existence. Each element of D is already assigned to a horizontal s.

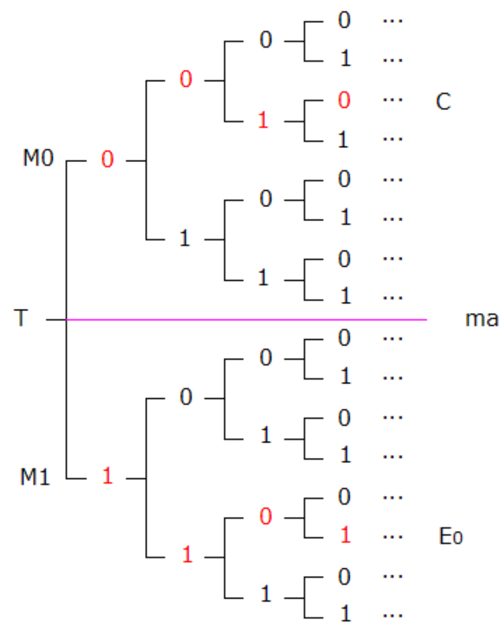

fig.7

D is the counterfeit for the existing path C, 3rd from the top in column 4.
Fig.7 has a mirror axis ma. Any s can be rotated 180º about ma to form its negation (not s). The beginning of C and E_0 are shown in red. In the tree graph the spacing of branches was decreasing for the purpose of confining the illustration to a single page.

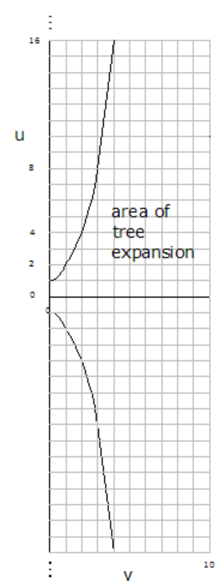

fig.8

A more realistic perspective is shown in fig.8 with an exponential growth rate of 2^v for both M0 and M1, with the (u,v) plane of each graph spaced apart in 3D space.

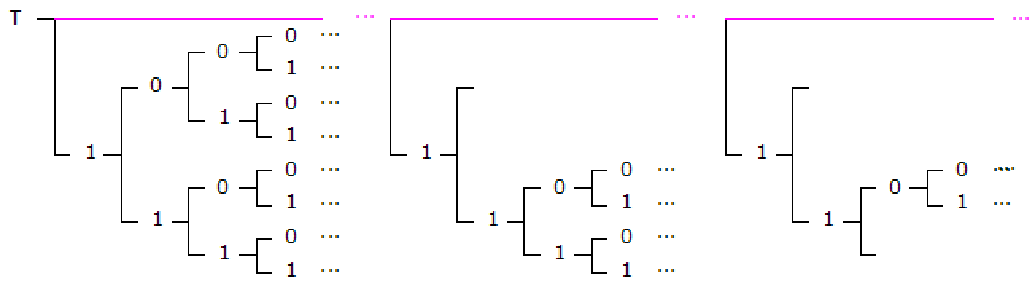

fig.9

C=00101..
E_0=11010..
C determines which subsets are excluded in forming E_0.
Position 1 can't be 0 which excludes subset M0.
Position 2 can't be 0 which excludes subset M10.
Position 3 can't be 1 which excludes subset M111.
Since there is no last selection, the final subset containing E_0 cannot be determined, but E_0 is definitely in subset M1, since it is determined by position 1.

conclusion

1. The diagonal D cannot be formed using the flow chart in fig.4.
2. The tree graph in fig.7 shows C and E_0 do not intersect, being members of different subsets. This contradicts Cantor's declaration of a missing E_0 in section 2.1.
3. The set N cannot be exhausted, which is the source for u and v.
4. Cantor's contradiction, that a thing cannot be in two different locations simultaneously, is a logical truth. The question then becomes which location is correct. Since there is access to the beginning of a sequence, the first symbol determines which subset.
5. Cantor's argument uses misdirection in the form of the diagonal D. This paper shows E_0 must be a member of T.

reference

[1] THE LOGIC MUSEUM Copyright © E.D.Buckner 2005