# MARTIN-LÖF ON THE VALIDITY OF INFERENCE

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Abstract. An inference is valid if it guarantees the transferability of knowledge from the premisses to the conclusion. If knowledge is here understood as demonstrative knowledge, and demonstration is explained as a chain of valid inferences, we are caught in an explanatory circle. In recent lectures, Per Martin-Löf has sought to avoid the circle by specifying the notion of knowledge appealed to in the explanation of the validity of inference as knowledge of a kind weaker than demonstrative knowledge. The resulting explanation is the main topic of this article.

## 1. Introduction

While preparing a lecture for the conference *Days of Judgement* in Leiden, September 2009, Per Martin-Löf became aware of a circularity in the explanations of the notions of inference and demonstration. An inference is valid if it guarantees the transferability of knowledge from the premisses to the conclusion: the conclusion can be known provided all of the premisses are known. In mathematics, being known amounts to being demonstrated. A demonstration, however, is a chain of valid inferences. The explanation of what it is for an inference to be valid therefore appeals implicitly to the notion of demonstration, which in turn explicitly appeals to the notion of valid inference.

A solution to this circularity problem was first offered in the lecture 'Is logic part of normative ethics?' (Martin-Löf, 2015) by the introduction of a dialogical perspective on inference: when justifying an inference it is enough to assume that someone else has asserted the premisses and under that assumption show that you may then assert the conclusion. In lectures from the ensuing years, this explanation of the validity of inference has been developed in various ways. Especially important has been an account of the content of a speech act as a task. Martin-Löf (2003) argued convincingly that one must distinguish the notion of proposition from the notion of the content of a speech act. The explanation of a content in this sense as a task makes it possible to see the dialogical explanation of the validity of inference as an instance of a more general non-dialogical explanation. The dialogical perspective does, however, lead to the formulation of dialogue rules for type theory. These rules in effect spell out the knowledge that is expressed by a judgement in type theory and that is transferrable from premisses to conclusion according to the novel account of inferential validity.

The aim of this article is to offer a detailed presentation of Martin-Löf's account of the validity of inference, including explanations of the various notions that the account makes use of, such as the new notion of content. The presentation is based on transcripts of lectures from recent years: the one already mentioned (Martin-Löf, 2015), 'Judgement and inference' (Martin-Löf, 2016), 'Assertion and request' (Martin-Löf, 2017a,b), 'Logic and ethics' (Martin-Löf, 2020), first given in 2019, and 'Epistemic assumptions: are they assumed to be backwards vindicated or forwards vindicable?' (Martin-Löf, 2019). All but one of these have until recently remained unavailable to the wider public, but they can now be found in an online repository. Since these lectures deal with many topics besides inference, and since they approach the explanation of validity from a variety of perspectives, it seemed to me to be of interest to attempt a synthesis of what they say about validity in particular.

The circle that motivates the search for a new account of the validity of inference is spelled out in more detail in Section 2. Besides Martin-Löf's lectures, I there rely on the writings of Sundholm (1997, 2012). Section 3 is primarily about terminology. A name—'assertoric knowledge'—is there given to the weaker form of knowledge that, according the new account, is transferrable from premisses to conclusion in a valid inference. The novel notion of content is introduced in Section 4. Relying on this notion of content, a correctness criterion for judgements is given in Section 5. The new account of validity of inference is then given in Section 6. Section 7 introduces another novelty of Martin-Löf's recent lectures, namely dialogue rules for type theory. The close affinity between the dialogue rules and the standard meaning explanations of type theory is illustrated in the final Section 8. This section does not rely directly on Martin-Löf's lectures, but is an original contribution by the author. The justification of an elimination rule offered there on the basis of the dialogue rules is in essence the same as the corresponding justification on the basis of the meaning explanations. We might say that the new accounts of content, correctness, and validity provide the meaning explanations with a new form, but do not change their substance.

No reference will be made to related works by other logicians or philosophers, even where the affinity is obvious (general speech act theory, accounts of inference, knowledge account of assertion, knowledge by testimony, dialogical logic). Some such references can be found in the original lectures and more will, I believe, be given in future work. The aim here has been to present Martin-Löf's ideas themselves. In the two final sections, some familiarity with Martin-Löf's type theory will be assumed. The new accounts of content, correctness, and validity, however, presuppose no such knowledge.

## 2. Inference

An inference in general may be explained pictorially by means of a figure such as the following:

$$\text{(Inf)} \qquad \frac{J_1 \ \ldots \ J_n}{J}$$

The judgements $J_1, \ldots, J_n$, above the line, are the premisses, and the judgement $J$, below the line, is the conclusion. In words we may explain inference either as the act of passing from the judgements $J_1, \ldots, J_n$ to the judgement $J$ or as the act of making the judgement $J$ with reference to the judgements $J_1, \ldots, J_n$. These two verbal explanations are related to each other as process to product: in the first we emphasize the process of passing from certain judgements to another judgement, and in the second we emphasize the product of that process, namely the judgement that is reached through that process. Nothing to be said here will hinge on which of the two explanations is adopted. All that matters is that an inference in general can be displayed in a figure such as (Inf), with a clear distinction between premisses and conclusion.

Judgement and inference are epistemic notions. A judgement, conceived of as an object rather than as an act, is a possible object of knowledge in the sense that it has the form of things known. Of course, not everything of the form of a judgement can be known, but when something of that form cannot be known, it is not owing to its form, but owing to its content. Because of this feature of judgements, it makes sense to speak of knowing a judgement, and therefore also to speak of knowing the premisses and conclusion of an inference.

The notion of judgement differs from the notion of proposition: a proposition does not have the form of things known, hence knowing a proposition is impossible on formal grounds. One might know a proposition $A$ in the sense of being acquainted with it, but knowing $A$ in this sense is quite different from knowing that $A$ is true, which is to know a judgement.

When we here speak of the validity of inference, we have in mind the validity of an actually made inference. That an inference figure is valid means that any actually made inference of the form set out in the figure is valid. To argue that an inference figure—such as a rule of inference—is valid we therefore assume given an arbitrary actual inference that instantiates the figure and argue that it is valid.

The precise characterization of what it is for an inference to be valid is our main concern in this article. We shall reach it by improving upon the following characterization:

(Valid-1) An inference is valid if the conclusion can be known under the assumption that the premisses are known.

Our starting point is thus the idea that an inference is valid when knowledge is transferable from the premisses to the conclusion: knowing the premisses is sufficient for knowing the conclusion. When justifying an inference, one must therefore show, under the assumption that the premisses are known, that the conclusion can be known.

Inference, being an epistemic notion, differs from consequence, which is better described as an alethic notion. The relation of consequence holds between propositions $A_1, \ldots, A_n$ as antecedents and a proposition $A$ as consequent if $A$ is true provided $A_1, \ldots, A_n$ are all true. Formal consequence, commonly known as logical consequence, holds between propositional functions $A_1(x_1, \ldots, x_m), \ldots, A_n(x_1, \ldots, x_m)$ as antecedents and a propositional function $A(x_1, \ldots, x_m)$ as succedent if consequence holds between the propositions $A_1(a_1, \ldots, a_m), \ldots, A_n(a_1, \ldots, a_m)$ as antecedents and the proposition $A(a_1, \ldots, a_m)$ as consequent whenever $a_1, \ldots, a_m$ are objects of appropriate type. (Among the types we count individual domains, function types, and a type of propositions.) In this explanation, proposition has taken the place of judgement and truth the place of knowledge.

Although the explanation (Valid-1) thus suffices for distinguishing inference from consequence, it is not precise enough to be entirely satisfactory. In mathematics, it is natural to count a judgement as known if it has been demonstrated. Under this account of knowledge, the explanation of the validity of inference becomes: the conclusion $J$ can be demonstrated under the assumption that the premisses $J_1, \ldots, J_n$ have been demonstrated. That a judgement has been demonstrated means, however, that it is the final conclusion in a chain of valid inferences. (We count an axiom as the conclusion of a valid inference with no premisses.) At least for mathematical discourse, the explanation of inferential validity is therefore circular: it appeals to the notion of demonstrated judgement, which in turn is explained in terms of validity.

Such an obvious circle is unacceptable for any pair of explanations, let alone for explanations of such basic notions as inference and demonstration. Believing that these notions can be explained, we are led to look either for other explanations altogether or for emendations of the already given explanations. Since it is not clear what alternative explanations should look like, the latter path is preferable. The circle arises when the notion of knowledge appealed to in the explanation of inferential validity is specified to mean demonstrative, or apodeictic, knowledge. The circle could be avoided by invoking instead knowledge of a kind that is weaker than demonstrative knowledge.

## 3. ‘Assertoric knowledge’

Martin-Löf in his lectures has sought to distinguish this weaker kind of knowledge from the stronger kind in various ways:

| | |
|---|---|
| knowledge in the strong sense | knowledge in the weak sense |
| knowledge in the qualified sense | knowledge in the unqualified sense |
| knowledge$_1$ | knowledge$_2$ |
| apodeictic knowledge | assertoric knowledge |

We shall prefer the last of these pairs of terms, which derives from two of the modalities in Kant’s table of judgements. (Martin-Löf does in fact not use the compound ‘assertoric knowledge’ in his lectures, but he does connect knowledge in the weak sense with the assertoric modality of Kant’s table of judgements.)

Apodeictic knowledge is knowledge supported by demonstration or other forms of scientific justification. Assertoric knowledge is knowledge of the kind required for correctly making a judgement. Precisely what such knowledge amounts to will be made clear in what follows. That assertoric knowledge is indeed weaker than apodeictic knowledge will then also be made clear.

Instead of ‘assertoric knowledge’ we could also say ‘judicative knowledge’. We shall namely take assertion and judgement to be subject to the same correctness criterion, hence for us there will be no semantical difference between them, just a difference of emphasis: ‘judgement’ suggests the inner mental act only, whereas ‘assertion’ is used primarily for the outer speech act. Although a judgement as a mental act need not as such be accompanied by any form of expression, we shall assume that it has linguistic form, so that it can, in principle, be expressed. The relation between judgement and assertion may then be explained by saying that an assertion is an externalized judgement, and a judgement is an internalized assertion. Because of this relation between judgement and assertion, we shall allow ourselves to use the two terms interchangeably. In particular, we shall allow ourselves to think of judgement as a speech act.

## 4. Content

A speech act quite generally is most simply analyzed as consisting of the two parts of mood and content. Whereas speech acts of different kinds may have the same content, the mood is unique to the kind of speech act in question and may indeed be said to determine that kind of speech act. The mood that determines assertion, or judgement, Martin-Löf calls the assertoric mood and indicates symbolically by means of Frege's assertion sign, $\vdash$. A judgement, or an assertion, will thus be assumed to have the following form, where $C$ is an arbitrary content:

$$\vdash C$$

The $C$ that appears here is not a proposition in the modern sense, but what Martin-Löf calls a content. The notions of content and proposition are closely related, but not the same: Martin-Löf explains their relation by saying that a proposition is an objectified content, namely a content formed so that it may occur as grammatical subject.

For a better understanding of what a content in the present sense is, it may be useful first to consider contents purely formally and ask what they look like in the syntax of type theory. In type theory it is usual, whenever $A$ is a proposition, to speak of

$$A \text{ true}$$

as a judgement, namely the judgement that the proposition $A$ is true. One might then say that the proposition $A$ is the content of this judgement. On the present understanding, however, it is not $A$, but the whole of '$A$ true' which is a content. The corresponding judgement arises only when this content is equipped with the assertoric mood, the result of which may be written as follows:

$$\vdash A \text{ true}$$

In general, all judgements are equipped with the assertoric mood. The two standard forms of judgement in type theory will therefore not be written as $a : \mathscr{C}$ and $a = b : \mathscr{C}$, which are forms content, but rather as follows:

$$\vdash a : \mathscr{C}$$

$$\vdash a = b : \mathscr{C}$$

With the introduction of this novel notion of content, what we are used to seeing as a judgement in type theory is thus rather a content. A judgement arises only when a content is supplied with assertoric mood. (Since all judgements have assertoric mood, the mood sign, $\vdash$, may still be left out from formal presentations of type theory.)

This excursion into the syntax of type theory gives us a sense of what a content looks like, what it is from a purely formal point of view. It remains to explain it also semantically.

Under the Brouwer–Heyting–Kolmogorov interpretation, a proposition is explained to be an intention (Heyting) or a task (Kolmogorov). A proposition is true if it is fulfillable as an intention, or solvable as a task. A proof, or truthmaker, of the proposition is something that fulfils it as an intention, or that solves it as a task. Since a proposition is an objectified content, it is natural to explain the notion of content in essentially the same way: a content is an intention, or a task. A task, in turn, Martin-Löf explains simply as something to do, or to be done.

## 5. Correctness

Assertoric knowledge is then explained in two steps which Martin-Löf describes as a phenomenological step and a constructivist step, respectively.

The content of an assertion is a task, or an intention. Under the related Brouwer–Heyting–Kolmogorov interpretation of propositions, the truth of a proposition is understood to be its solvability as a task, or its fulfillability as an intention. The phenomenological step appeals to the corresponding notion at the level of content: to know assertorically an assertion means to know its content to be solvable, doable, or fulfillable. The constructivist step identifies knowledge of such a potentiality as knowledge-how: the agent making the judgement knows how to perform the task, or how to fulfil the intention, that constitutes its content.

The two steps may be presented as two equations:

$$\begin{aligned} \text{to know assertorically } \vdash C &= \text{to know } C \text{ to be doable} \\ &= \text{to know how to do } C \end{aligned}$$

Instead of the pair doable/do, we could also use solvable/solve or fulfillable/fulfil.

The correctness criterion for acts of judgement, or assertion, lays down when it is correct to make a judgement, or an assertion. Martin-Löf adopts a knowledge account of the correctness of assertion, according to which an assertion, or judgement, $J$ is correct if and only if the agent making $J$ assertorically knows $J$. With the above account of assertoric knowledge, we are led to the following correctness criterion:

(Correct) A judgement, or an assertion, is correct if and only if the agent making it knows how to (is able to, can) perform the task that constitutes its content.

Variations on 'perform the task' are possible, for instance, 'do the task', 'solve the task' or 'fulfil the intention'.

We shall see below that, according to the criterion (Correct), it is correct to make a judgement that one has demonstrated. Correctness does, however, not require demonstration or scientific justification more generally. It is, for instance, possible for me to know—in the sense of assertoric knowledge—a mathematical theorem $J$ without my actually having worked through its proof. I may know the theorem $J$ because, for instance, I know of a textbook where it can be found. In order to fulfil the task that constitutes the content of $J$, it is enough for me to refer to this textbook. In general, knowledge that we base on trust rather than on demonstration or scientific justification more generally is only assertoric knowledge. Assertoric knowledge is therefore strictly weaker than apodeictic knowledge

The correctness criterion (Correct) has been reached from the knowledge account of assertion via a phenomenological account of the content of an assertion as a task and a constructivist account of knowledge of the solvability of a task as knowledge how the task is solved. The result is a clearly constructivist correctness criterion. An alternative to this criterion might say simply that an assertion is correct if the agent making it knows its content to be true. What the correctness criterion in this formulation amounts to more precisely will, however, not be clear until one has explained what is then meant by content, by truth of a content, and by knowledge of the truth of a content. What alternative accounts of these notions might look like will not be discussed here. We shall take Martin-Löf's constructive correctness criterion for granted.

## 6. Validity of inference

When the knowledge appealed to in (Valid-1) is specified to mean assertoric knowledge, we obtain the following three equivalent explanations of what it is for an inference to be valid. An inference is valid if

(Valid-2a) the conclusion can be assertorically known under the assumption that the premisses are assertorically known.

(Valid-2b) one can come to know how to do the task constituting the content of its conclusion under the assumption that, for each premiss, one knows how to do the task that constitutes its content.

(Valid-2c) correctness is transferrable from the premisses to the conclusion.

In (Valid-2a), the 'known' in (Valid-1) is replaced by 'assertorically known'. By spelling out the explanation of assertoric knowledge given in the previous section, we reach (Valid-2b). The more compact formulation (Valid-2c) relies on the correctness criterion (Correct). A slight variation on that formulation says that the conclusion can be seen to be correct under the assumption that all of the premisses are correct.

Since assertoric knowledge is weaker than apodeictic knowledge, it is clear that validity is explained here without reference, implicit or explicit, to apodeictic knowledge. The circle in the explanations of inference and demonstration has therefore been avoided.

In (Martin-Löf, 2015), the explanatory circle was avoided by means of a dialogical, or interactive, account of inferential validity:

(Valid-3) An inference of the form (Inf) is valid if I can take responsibility for $J$ provided others have taken responsibility for $J_1, \ldots, J_n$.

This account of validity is based on the conception of assertion as the taking on of an obligation: by making an assertion, one takes on the obligation, on the speaker's request, to do the task that makes up the content of the assertion. By the ought-implies-can principle—the principle that an obligation must be possible to fulfil—and the account (Correct), I can take responsibility for $J$ precisely when it is correct for me to assert $J$. Namely, if I can take responsibility for $J$, then, by the ought-implies-can principle, the obligation that I take on by asserting $J$ is one that I am able to fulfil, which is to say that I can perform the task that makes up the content of $J$, whence by (Correct) my assertion of $J$ is correct. In the other direction, if I know how to perform this task, I can certainly take on the obligation that it is to assert $J$.

It follows that (Valid-3) is a special case of (Valid-2). If others have taken responsibility for the judgements $J_1, \ldots, J_n$, then I shall know how to fulfil the tasks that make up their contents, since I can then simply refer to those who made these judgements. By (Valid-2), I can then come to know how to do the task that constitutes the content of the conclusion $J$, whence I can take responsibility for $J$.

Although the dialogical perspective that finds expression in (Valid-3) is thus not necessary for avoiding the circle in the explanations of the notions of inference and demonstration, we shall soon make use of it in shedding light on the correctness criterion (Correct) through dialogue rules.

In rule form, the explanation of the validity of inference may be written as follows:

$$\text{(V)} \qquad \frac{\mathrm{val}\left(\dfrac{J_1 \;\ldots\; J_n}{J}\right) \qquad \mathrm{cor}\, J_1 \;\ldots\; \mathrm{cor}\, J_n}{\mathrm{cor}\, J}$$

Here cor is a metalinguistic predicate applying to judgements and is to be read 'is correct', whereas val is a metalinguistic predicate applying to inference figures and is to be read 'is valid'. Instead of explaining valid inference as in (Valid-2), we could say that the rule (V) is meaning determining for the predicate val. It would then of course remain to explain the predicate cor, which we have done in (Correct) above.

We explained a judgement to be demonstrated to mean that it is the final conclusion in a chain of valid inferences. The metalinguistic predicate dem applying to judgements and to be read 'is demonstrated' can therefore, likewise, be said to be determined by the following rule:

$$\text{(D)} \qquad \frac{\mathrm{dem}\, J_1 \;\ldots\; \mathrm{dem}\, J_n \qquad \mathrm{val}\left(\dfrac{J_1 \;\ldots\; J_n}{J}\right)}{\mathrm{dem}\, J}$$

Since this rule is meaning determining for the predicate dem, and the predicate cor satisfies the structurally identical rule (V), any judgement of which dem holds is also a judgement of which cor holds:

$$\mathrm{dem}\, J \to \mathrm{cor}\, J$$

A more detailed argument for this entailment may be given by induction on the length of the demonstration of $J$. If $J$ is an axiom, that is, inferred from zero premisses, then $\mathrm{dem}\, J$ is inferred by the rule (D) from $\mathrm{val}(\frac{}{J})$, but then we may assert $\mathrm{cor}\, J$ as well, by (V). If $J$ is not an axiom, then $\mathrm{dem}\, J$ is inferred by the rule (D) from $\mathrm{dem}\, J_1, \ldots, \mathrm{dem}\, J_n$ and

$$\mathrm{val}\left(\frac{J_1 \;\ldots\; J_n}{J}\right)$$

By induction, we may assume $\mathrm{cor}\, J_1 \;\ldots\; \mathrm{cor}\, J_n$, but then $\mathrm{cor}\, J$ by (V).

## 7. Dialogue rules

Under the Brouwer–Heyting–Kolmogorov interpretation, propositions of a particular form may be explained in terms of one or more rules that lay down what it means canonically to fulfil it as an intention, or solve it as a task. For instance, propositions of the form $A \wedge B$ are explained in terms of the following rule:

$$\frac{a : A \qquad b : B}{\langle a, b\rangle : A \wedge B}$$

The rule explains the proposition $A \wedge B$ by displaying the form of a canonical fulfilment of it: it is a pair $\langle a, b\rangle$, where $a$ fulfils $A$ and $b$ fulfils $B$. What a fulfilment in general is of the intention $A \wedge B$ is then determined by the general stipulation that a fulfilment of a proposition $C$ is a method for obtaining a canonical fulfilment of $C$.

A novelty of type theory in recent years has been the development of rules that lay down what it means canonically to fulfil the intention, or do the task, that makes up the content of a judgement. The rules spell out the knowledge how, or ability, that an agent conveys to possess when making a judgement in the language of type theory. In other words, the rules spell out the correctness criterion (Correct) for judgements in type theory. Since this correctness criterion in effect characterizes the speech act of assertion, we may take the rules to determine the meaning of the assertoric mood for the judgements of type theory.

The novel rules employ, in addition to assertion, or judgement, the speech act of request. Two rules determine the speech act of request. Firstly, a request with content $C$ may always follow an assertion with the same content. Secondly, following the request, the agent making the assertion is obliged to answer the request. The answer may be another assertion, or it may be some other act, not necessarily a speech act. In any event, we may describe the interaction between assertion and request in the form of either of the following figures:

$$\frac{\vdash C \qquad C\,?}{C \text{ done}} \qquad\qquad \cfrac{\cfrac{\vdash C}{C\,?}}{C \text{ done}}$$

If a speech act, $\vdash C$, is made, then it may be followed by a request with the same content, which we write as '$C\,?$', where the question mark indicates the request mood. Following this request, the assertor is obliged to answer the request by performing the task $C$. The second way of writing the rule makes clear the order in which the acts are made. We shall, however, prefer a version of the first formulation.

An example that Martin-Löf has given several times is that of a child who asserts that it can swim. The mother's exclaiming 'Oh, can you?' is, in effect, a request with the same content, which the child is obliged to answer in some way. A canonical way of answering the request is by jumping into the water and swimming a few strokes.

This canonical way of answering the request is not another assertion, but a practical act. Many forms of content encountered in type theory are, by contrast, fulfilled by the making of one or more further judgements. A type-theoretical judgement in general therefore gives rise to a dialogue between a proponent, P, and an opponent, O, that Martin-Löf illustrates by the following diagram:

$$\begin{tikzcd} \mathrm{P} \arrow[rr, bend left, "\vdash C"] \arrow[rr, bend right, "\vdash C'"', "\text{must}"] & & \mathrm{O} \arrow[ll, "C\,?"', "\text{may}"] \end{tikzcd}$$

The rules that govern such dialogues are precisely the rules that lay down what it means canonically to fulfil the intention, or do the task, that makes up the content of a judgement of type theory. More precisely, if the content of a type-theoretical judgement $J$ is fulfilled by the making of another type-theoretical judgement, then a rule that describes the interaction between a proponent making $J$ and an opponent challenging $J$ lays down what it means canonically to fulfil the content of $J$.

Such a rule, which is called a dialogue rule, has the following general form:

$$\frac{\vdash C \qquad C\,?}{\vdash C'}$$

Instead of the general form of conclusion '$C$ done', indicating an act in general, we here have the special case $\vdash C'$, indicating a speech act of assertion. Actual dialogue rules show some variation on this general form. In particular, there may be more than one assertion occurring in the conclusion, and for some forms of content the request is accompanied by an assertion made by the opponent.

This is not the place to go through all the dialogue rules for the language of type theory, but a few examples will help to shed light on the new accounts of content, correctness, and validity. We shall use a simplified notation that leaves out the sign for the assertoric mood and replaces the request with a question mark to the right of the inference line:

$$\frac{C}{C'}\;?$$

Let us first consider three rules pertaining to disjunction:

$$\frac{d : A \vee B}{\begin{array}{c} \left\{\begin{array}{c} i(a) : A \vee B \\ d \Rightarrow i(a) : A \vee B \end{array}\right. \\ \hdashline \left\{\begin{array}{c} j(b) : A \vee B \\ d \Rightarrow j(b) : A \vee B \end{array}\right. \end{array}}\;? \qquad \begin{array}{c} \dfrac{i(a) : A \vee B}{a : A}\;? \\[2ex] \dfrac{j(b) : A \vee B}{b : B}\;? \end{array}$$

The dashed line indicates that the proponent has a choice between making a pair of judgements whose contents are displayed above the line and making a pair of judgements whose contents are displayed below the line. The double arrow, $\Rightarrow$, is to be understood as 'evaluates to'. The task $d : A \vee B$ is thus solved by two further judgements: either

$$\vdash i(a) : A \vee B \quad \text{and} \quad \vdash d \Rightarrow i(a) : A \vee B$$

or

$$\vdash j(b) : A \vee B \quad \text{and} \quad \vdash d \Rightarrow j(b) : A \vee B$$

The task $i(a) : A \vee B$ is, in turn, solved by the judgement $\vdash a : A$, whereas the task $d \Rightarrow i(a) : A \vee B$ is solved, not by another assertion, but by a calculation of $d$ with result $i(a)$. There is, quite generally, no dialogue rule with major premiss of the form $\vdash a \Rightarrow c : A$, since the task $a \Rightarrow c : A$ is not solved by an act of assertion, but by an act of calculation.[1]

The rules for implication are simpler in appearance than those for disjunction, but they bring in a new element, namely functions. For types $\alpha$ and $\beta$, we write $(\alpha)\beta$ for the type of functions from $\alpha$ to $\beta$.

$$\cfrac{d : A \supset B}{\begin{array}{c}\lambda(f) : A \supset B \\ d \Rightarrow \lambda(f) : A \supset B\end{array}}\ ? \qquad \frac{\lambda(f) : A \supset B}{f : (A)B}\ ?$$

When challenging a judgement of the form $\vdash f : (A)B$, the opponent has to assert a judgement of the form $\vdash a : A$ or one of the form $\vdash a = a' : A$. That is, the opponent has to provide either an argument or two equal arguments to $f$. The response by the proponent then has to be either $\vdash f(a) : B$ or $\vdash f(a) = f(a') : B$, respectively. All of this is captured by the following two rules:

$$\frac{f : (\alpha)\beta \qquad a : \alpha}{f(a) : \beta}\ ? \qquad \frac{f : (\alpha)\beta \qquad a = a' : \alpha}{f(a) = f(a') : \beta}\ ?$$

These rules are meaning determining for judgements of the form $\vdash f : (\alpha)\beta$, since they spell out what it means to solve a task of the form $f : (\alpha)\beta$. You have solved the task if you can (i) make the judgement $\vdash f(a) : \beta$ whenever you are provided with an object $a$ of type $\alpha$ and (ii) make the judgement $\vdash f(a) = f(a') : \beta$ whenever you are provided with identical objects $a$ and $a'$ of type $\alpha$. Knowing a judgement of the form $\vdash \lambda(f) : A \supset B$ therefore does not require knowledge of all the possible ways of constructing a proof of $A$: it is enough to know how, whenever a proof $a$, or equal proofs $a$ and $a'$, of $A$ are provided, to solve the tasks $f(a) : B$ or $f(a) = f(a') : B$, respectively.

## 8. Dialogue rules and meaning explanations

In order to get a better grasp of how the dialogue rules explain the meaning of judgements in type theory, it may be useful to see how they can be used to justify elimination rules. Let us consider the elimination rule for conjunction, and let us see that it is indeed valid in the sense of (Valid-2).

($\wedge$-elimination) $$\frac{d : A \wedge B}{\mathbf{fst}(d) : A}$$

To show that this inference figure is valid I must explain to you how to do the task $\mathbf{fst}(d) : A$ under the assumption that you know how to do the task $d : A \wedge B$, where $A$ and $B$ are arbitrarily given propositions. Neither of these tasks has been defined by the dialogue rules displayed in the previous section. The two rules defining tasks of the form $d : A \vee B$ and $d : A \supset B$ are, however, instances of a more general rule that also defines the tasks $\mathbf{fst}(d) : A$ and $d : A \wedge B$. This more general rule has the

[1]It is possible to formulate dialogue rules also for assertions of the form $\vdash a \Rightarrow c : A$, but it would complicate the presentation here to enter into those rules. Instead we take the more intuitive approach that the task $a \Rightarrow c : A$ is solved by an act of calculation.

following form:

(Element)
$$\cfrac{\cfrac{d : C}{c : C}\,?}{}\qquad \begin{array}{c} \dfrac{d : C}{c : C}\ ? \\ d \Rightarrow c : C \end{array}$$

In this rule, $C$ is any set formed by a formation rule. Corresponding to this formation rule there are one or more introduction rules, and they determine the form of canonical elements of $C$. In the rule (Element), $c$ is such a canonical element, and $d \Rightarrow c : A$ is the task of calculating $d$ to $c$.

Justifying ($\wedge$-elimintation) therefore requires that I explain how you provide a canonical element $c$ of $A$ and calculate $\mathbf{fst}(d)$ to $c$, under the assumption that you can provide a canonical element of $A \wedge B$ and know how to calculate $d$ to that canonical element. A canonical element of $A \wedge B$ has the form $\langle a, b \rangle$, where $a$ is an element of $A$, and $b$ is an element of $B$. I may therefore assume that you know how to solve the following two tasks:

$$\langle a_0, b_0 \rangle : A \wedge B$$
$$d \Rightarrow \langle a_0, b_0 \rangle : A \wedge B$$

Since you know how to do the task $\langle a_0, b_0 \rangle : A \wedge B$, you can provide a canonical element $c_0$ of $A$ and solve the task

$$a_0 \Rightarrow c_0 : A$$

This follows from the definition of tasks of the form $\langle a, b \rangle : A \wedge B$, which is similar to the dialogue rules given for the forms of content $i(a) : A \vee B$, $j(b) : A \vee B$ and $\lambda(f) : A \supset B$.

The definition of the function **fst** is given by the equality rule for conjunction,

$$\frac{a : A \qquad b : B}{\mathbf{fst}(\langle a, b \rangle) = a : A}$$

Calculation, as understood here, is definitional reduction: the continuous replacement of definiendum by the corresponding definiens. Now that I have explained to you the definition of **fst**, you therefore know how to solve the task

$$\mathbf{fst}(\langle a_0, b_0 \rangle) \Rightarrow c_0 : A$$

You solve the task

$$\mathbf{fst}(d) \Rightarrow c_0 : A$$

by first calculating $d$ to $\langle a_0, b_0 \rangle$ inside the frame $\mathbf{fst}(\ \ )$ and then continue with the calculation of $\mathbf{fst}(\langle a_0, b_0 \rangle)$ to $c_0$,

$$\mathbf{fst}(d) \Rightarrow \mathbf{fst}(\langle a_0, b_0 \rangle) \Rightarrow c_0 : A$$

You solve the task $\mathbf{fst}(d) : A$ by providing the canonical element $c_0$ of $A$ together with this calculation.

This shows that the rule ($\wedge$-elimination) is valid in the sense of (Valid-2). Those familiar with the meaning explanations for the language of type theory that were first presented in detail in (Martin-Löf, 1984) will see that there is no essential difference between the justification of ($\wedge$-elimination) just given and that given on the basis of the meaning explanations. In the latter case also, one has to explain how to calculate $\mathbf{fst}(d)$ to a canonical element of $A$ under the assumption that one knows how to calculate $d$ to a canonical element of $A \wedge B$. The same observation can be made for the justification of all other elimination rules. The new definition of validity (Valid-2) and the concomitant explanation of the meaning of the forms of judgement of type theory in terms of dialogue rules do therefore not require a revision of the ways of justifying elimination rules first spelled out in (Martin-Löf, 1984).

The dialogue rules can indeed be seen just to be the meaning explanations in a different presentation. The rule (Element), for instance, is just a different presentation of the explanation of being an element of a set $A$ as being 'a method (or program) which, when executed, yields a canonical element of $A$ as result' (Martin-Löf, 1984, p. 9). A more detailed exploration of the affinity between the dialogue rules and the meaning explanations will have to wait for another occasion.

**Acknowledgements.** I am grateful to Per Martin-Löf, Göran Sundholm, and Shahid Rahman for discussions on topics of relevance to this article, and to the same as well as to Bruno Bentzen, Will Stafford and an anonymous referee for comments on an earlier draft. While researching and writing the article, I was supported by a *Lumina quaeruntur* fellowship (LQ300092101) from the Czech Academy of Sciences.

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