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Beyond semantic pollution: Towards a practice-based philosophical analysis of labelled calculi

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Abstract

This paper challenges the negative attitudes towards labelled proof systems, usually referred to as semantic pollution, by arguing that such critiques overlook the full potential of labelled calculi. The overarching objective is to develop a practice-based philosophical analysis of labelled calculi to provide insightful considerations regarding their proof-theoretic and philosophical value. To achieve this, successful applications of labelled calculi and related results will be showcased, and comparisons with other relevant works will be discussed. The paper ends by advocating for a more practice-based approach towards the philosophical understanding of proof systems and their role in structural proof theory.

Keywords: semantic pollution, labelled calculi, practice-based philosophy of logic, proof theory

1 Introduction

Semantic-based or, more commonly, *labelled* calculi are usually obtained by incorporating semantic expressions into the syntax of the corresponding non-labelled calculi – such as sequent, natural deduction and tableau systems. There are at least two specific types of semantic elements that have been used to develop such frameworks: algebraic and, mainly, relational. Labelled calculi have been widely used and applied, especially in structural proof theory, due to their flexible and intuitive characterization. However, because they incorporate semantic information into the formalism, these calculi have been criticized for their perceived *impurity*, which is usually referred to as *semantic pollution* – "an epithet attributed [...] to Rajeev Goré in conversation" (Read, 2015, 650). Some major opinions are:

"Despite their impact, labelled proof systems have been criticized as impure, in contrast to the more traditional proof systems."[1] (Negri, 2007, 109)

"[...] some objections have recently been raised about the internalisation of the semantics into the syntax of sequent calculi. The main concerns regard the purity of methods, in the sense that the internalisation of Kripke semantics contaminates the syntactic purity of sequent calculi." (Boretti, 2009, 33)

"The potentially controversial aspects – which are certainly not to the present author's taste – of the use of 'labelled' sequents appear in a pronounced form in Negri (2005), where we find not only labels representing the points in Kripke models but also (what look very much like) explicit accessibility relation statements. This is a clear case of what Poggiolesi (2010), summarizing desiderata for proof systems from various sources, calls a failure of 'semantic purity'. [...] Raj Goré has described such failures (in conversation) as 'semantic pollution', though without necessarily endorsing the desideratum in question." (Humberstone, 2011, Remark 1.21.8, pp. 111–112)

"[A calculus should] not make any use of semantic parameters beyond the language of formulas. The labelled method, on the contrary, is a semantic method since it imports in its language the whole structure of Kripke semantics in an explicit and significant way." (Poggiolesi & Restall, 2012, 49)

To provide an initial understanding of labelled calculi (without endorsing any specific approach, as multiple will be explored in this article), we will introduce a general categorization of different labelling techniques based on the *strength* of the semantic commitment expressed by labels as follows:[2]

1. *Weak* labelling: there are deductive systems using multiple types of labels, but only in a limited way, that is, they play a minor role in the inferential process. Although the motivation for using labels is primarily semantic, their interpretation is not. As such, the apparatus of labels in these systems is not strong enough to construct falsifying models, see e.g., Rescher and Urquhart (1971), Mints (1997).
2. *Medium* labelling: there are labelled deductive systems that use prefixes to label formulas. This labelling technique is relatively simple and natural, as it does not require any special rules for the labels. Whereas this method does not directly encode a semantics into the syntax, it can still be used to construct falsifying models relatively easily,[3] see e.g., Massacci (1998, 2005), Kashima (2003).
3. *Strong* labelling: there are labelled proof systems in which labellings preserve the semantics of the logic as much as possible. In these deductive systems, in addition to the rules for logical constants, there are rules that govern the behaviour of

[1]It may appear surprising to associate such a critical statement with S. Negri, a major advocate of labelled calculi. However, in the mentioned paragraph, which dates back to 2007, she was simply surveying critical attitudes towards labelled systems (without discussing them any further). We will examine Negri's work more closely on pp. 16-18.

[2]The description can be found in Indrzejczak (2021, 204-206), where one can also find other references besides the one mentioned throughout the paper. A survey of several labelling techniques can be found in Poggiolesi (2010, Ch. 4) as well.

[3]So, "we obtain quite satisfactory results with the help of a relatively modest apparatus" (Indrzejczak, 2021, 205).

labels, and often some rules that relate both levels. In these systems, countermodels can be easily extracted from labels, see e.g. Basin, Matthews, and Viganò (1997, 1998), Viganò (2000), Negri (2007, 2009), Negri and von Plato (2011).

The criticism of semantic pollution has been mainly directed towards strongly labelled proof systems, which fully internalize a semantics, mainly model-theoretic, within the syntax of the chosen framework. However, before delving into our analysis, let me briefly explain the methodology employed and the expected outcomes.

Methodology and aim.

While investigating inferentialism, modal logics and labelled rules, Stephen Read raised an intriguing question without exploring it further:[4]

> "One may ask how [inference rules] relate to our inferential practice. That is part of a wider question, how any part of the theoretical systematization of logic relates to practice." (Read, 2008, 15)

In this article, I will offer a philosophical analysis of the different attitudes towards the use of labels in proof systems filtered through considerations arising from what could be called "proof-theoretic practice". More precisely, by examining how proof theorists have responded to the creation and implementation of labelled calculi, I will suggest a systematization of four types of responses that summarize how the *accusation* of semantic pollution has been addressed. My investigation adopts a *bottom-up* methodology, which entails primarily relying on actual case studies from the works of working proof theorists to assess my conclusions.[5] Despite practice-based philosophical analyses on logic not being widely recognized or conducted, there have been some case studies that support a practice-first perspective, such as those examining formal languages and formal semantics (see e.g., Dutilh Novaes (2012b) and Maddirala (2014)).[6] However, it is important to note that established discussions on semantic pollution, to which I will soon refer, took a different, mainly *top-down* approach that disregards (almost completely) the actual practice – which is the core focus of my investigation. Along similar lines, C. Dutilh Novaes has argued that:

> "[...] the mismatch between philosophical theorizing and actual (recent) practices when it comes to logic is a reason for us to take seriously the idea of expanding the research agenda within philosophy of logic." (Dutilh Novaes, 2012b, 76)

[4]Proof-theoretic inferentialism is a philosophical perspective that can be traced back to the work of Gentzen (1969), and, roughly speaking, its main objective is to clarify the meaning of logical operators by identifying the inference rules that govern their use in derivations. In Section 2.4, I will provide a more detailed analysis of inferentialism and its relationship to labelled calculi.

[5]I am relying on B. Martin's terminology and idea according to which a philosophy of logical practice "takes a bottom-up approach, beginning with case studies of instances of practice within the field. [...] In comparison, philosophy of logic traditionally has used a top-down approach, beginning with certain philosophical presumptions about the subject" (Martin, 2022, 274), (Martin, 2023, 160).

[6]More than a decade ago, Dutilh Novaes observed that "no such practice-based turn has yet taken place within the philosophy of logic" (Dutilh Novaes, 2012b, 72). More recently, Martin has ascertained that the situation hasn't yet changed: "unlike the philosophy of science and the philosophy of mathematics, the philosophy of logic has yet to recognise the importance of building its understanding of the field upon the actual practice of its researchers" (Martin, 2023, 169). More information on the *practical turn* in the philosophy of mathematics and science can be found, e.g., in Mancosu (2008), Giardino, Moktefi, Mois, and van Bendegem (2012) and Soler, Zwart, Lynch, and Israel-Jost (2014).

In the context of a practice-based analysis, our first step is to determine our research questions and goals. Tentatively, we can outline the main tasks and objectives as follows: to consider how logic was and is actually practiced; to take into account historical and recent developments in the field; to clarify underlying assumptions, background ideas, motivations and justifications; to compare alternative theories and methodologies; to draw philosophically significant conclusions from formal results; to understand logical practices and activities, the tools and methods involved; to relate the development of logic to to its extra-logical applications (e.g., mathematics, AI, computer science, philosophy).

The aim of a philosophy of logical practice is, generally said, to clarify and shed light on what we do when we engage in logical activities and practices, and how we do it. In this article, we will explore this approach by presenting a case study on labelled calculi, their methodology, and on how practitioners have responded to concerns about their alleged impurity. It's important to note that a practice-based philosophy of logic doesn't seek to replace traditional forms of investigation; rather, it offers an alternative approach that complements traditional methods (Dutilh Novaes, 2012b, 76).[7] We believe that a practice-based approach has the potential to provide new insights into traditional issues, including the analysis of semantic pollution of proof systems.

2 Analysing semantic pollution: 4 perspectives

In this section, my aim is to provide a philosophical analysis of semantic pollution in labelled systems by filtering my considerations through the role they have played and continue to play in structural proof theory. More specifically, we will examine how the accusation of semantic pollution has influenced the development of the field and how this charge has interfered with the implementation of labelled proof systems. As previously mentioned, accusations of semantic pollution have primarily been directed at strongly labelled systems, but it is important to recognize that such critiques are not solely limited to them. We have generally characterized labelling techniques by saying that they embody *semantic elements* within the syntax of the corresponding non-labelled framework without further characterizing the notion of semantic element. This intentional lack of specificity stems from the fact that the various case studies we will examine later have taken different approaches to determining what constitutes a semantic element that can be considered as polluting a calculus. The resulting analysis can be summarized as follows:

[7] Also Martin wrote: "It is not my aim to argue that philosophy of logic as it is currently practised should be wholly replaced [...] the philosophy of logical practice [should] sit alongside traditional philosophy of logic" (Martin, 2023, 152). However, we should be aware that there are limits to this tolerant attitude, and it is important to constantly question the appropriateness of the chosen method in addressing the specific question at hand, as explored in the context of the philosophy of mathematical practice by Rittberg (2019).

Negative Perspectives

Convinced-no answers	Strict rejection of labelled systems; no connection to any semantics is tolerated.
Relaxed-no answers	Rejection of calculi internalizing semantic elements explicitly within the syntax.

Positive Perspectives

Opportunistic-yes answers	Labelled systems are acceptable frameworks, for methodological/practical reasons.
Enthusiastic-yes answers	Labelled calculi are valuable tools for structural proof theory, as they enable precise and systematic studies of logics, and have worthwhile applications in philosophical contexts.

To begin, my research aims to integrate ongoing discussions in the field. The negative perspectives outlined above are primarily drawn from the reflections presented in Poggiolesi (2010), whereas my contribution lies in the overall systematization along with the presentation of what I refer to as positive perspectives. Before proceeding with the examination, it is important to note that the negative perspectives mainly use a top-down approach, which evaluates the value of labelled deductive systems based on pre-theoretical and philosophical ideas about what elements should pertain to proof systems. In contrast, the positive perspectives are derived from a bottom-up methodology, that starts from concrete examples of the use, applications, and results of labelled calculi by working proof theorists and philosophers.

2.1 First perspective: convinced-no answers

Convinced-no answers
Strict rejection of labelled systems.

In Poggiolesi (2010), as well as in following works, such as Martinot (202x), certain statements made by A. Avron during the 1990s, particularly in Avron (1996), have been identified (although not agreed on) as encapsulating what we might call a *full-blooded* (or *strong*) *syntactic purity principle* for sequent calculi.[8] This property demands complete independence of a calculus from any particular semantics and Avron argues in its favour as follows:

> "Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics. One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof system." (Avron, 1996, 2)

According to this line of reasoning, the form and structure of a proof system should not reveal any semantic information. More precisely, sequent calculi should not incorporate semantic expressions into the syntax of sequents, and the intended semantics of

[8] To be clear, Poggiolesi presents an analysis of Avron's idea, but, as we will discuss soon, she views it as problematic and does not endorse it (Poggiolesi, 2010, 29-31).

a given proof system should not be discernible merely from the shape of its structures. This property has been criticized for having overly strong implications for the practice of structural proof theory. In Poggiolesi (2010), Gentzen's sequent calculus for classical logic **LK** is considered, and Avron's ideal is criticized. Poggiolesi argues that, by endorsing a full-blooded view of syntactic purity, even **LK** *is* semantically polluted, as it can be interpreted simply as a notational variant of semantic tableaux or signed trees. To illustrate this, let us consider the derivability of $\neg\neg\varphi \to \varphi$ in three different frameworks – a model-theoretic semantics argument, a proof employing signed tableaux, and an **LK**-derivation:

(a) $\nvDash \neg\neg\varphi \to \varphi$;
(b) $\vDash \neg\neg\varphi$ and (c) $\nvDash \varphi$;
(d) $\nvDash \neg\varphi$;
(e) $\vDash \varphi$.
Contradiction from (c), (e).

$\rightsquigarrow$

$$\begin{array}{lcl}
\text{(a)} & \mathfrak{f} : \neg\neg\varphi \to \varphi & \\
\text{(b)} & \mathfrak{t} : \neg\neg\varphi & \text{(a)}, \mathfrak{f}\to \\
\text{(c)} & \mathfrak{f} : \varphi & \\
\text{(d)} & \mathfrak{f} : \neg\varphi & \text{(b)}, \mathfrak{t}\neg \\
\text{(e)} & \mathfrak{t} : \varphi & \text{(d)}, \mathfrak{f}\neg \\
\text{(f)} & \text{✗} & \text{from (c), (e)}
\end{array}$$

$\rightsquigarrow$

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\, R\neg}{\neg\neg\varphi \Rightarrow \varphi}\, L\neg}{\Rightarrow \neg\neg\varphi \to \varphi}\, R\to$$

The criticism raised by Poggiolesi highlights that (i) sequent rules can be interpreted in terms of the semantic definitions of each logical connective, (ii) the symbols "$\Rightarrow$" and "," resemble the metalinguistic expressions "*if ... then*", "*and*", "*or*", and (iii) the placement of formulas on the left or right side of the sequent arrow reminds us of truth values in a semantic setting.[9] Therefore, the only possible conclusion is that:

> "[...] either we reject the strong syntactic purity property as a too strong property, or we are forced to admit that even the original Gentzen system [**LK**] would not be a good sequent calculus." (Poggiolesi, 2010, 30)

Finally, it should be noted that, as a consequence of Poggiolesi's critique, adherence to a full-blooded principle of syntactic purity would entail deeming all sequent-based frameworks developed after Gentzen's **LK** as semantically impure. Hence, advocates of convinced-no answers, who firmly oppose labelled systems and argue for their exclusion from the set of "good" calculi, may inadvertently propose the exclusion of frameworks that they themselves would not consider semantically polluted. Moreover, by taking this consideration to its extreme consequences, those who regard semantic pollution as a detrimental characteristic of proof systems would essentially have to reject an important part of the past 70 years of advancements in structural proof theory as being inadequately developed – a perspective at least extremely uncommon to support.

2.2 Second perspective: relaxed-no answers

Relaxed-no answers
Rejection of calculi internalizing semantic elements *explicitly* within the syntax.

If one finds the principle of full-blooded syntactic purity too stringent, but still wishes to avoid proof systems that incorporate semantic elements within the syntax, they may consider adopting a more nuanced and delimited criterion. This approach endorses

[9] On a similar note, Humberstone critically argues that "one might even say [that] the left/right division of (sets of) formulas in a sequent, show[s] the desideratum of semantic purity [...] to be potentially problematic, or at least, in danger of turning into a matter of degree rather than of kind" (Humberstone, 2011, 112).

what is usually referred to as the *weak syntactic purity principle*. It roughly stipulates that a sequent calculus should not utilize any semantic elements in an *explicit manner*.[10] Specifically, this principle does not preclude a connection to a specific semantics, it only prohibits the explicit use of objects extrapolated from a semantic structure, usually relational or algebraic. One calculus that adheres to such an ideal principle, is Gentzen's **LK**, which is thus considered to be free of semantic pollution. However, this principle has spurred the development of alternative frameworks, such as Poggiolesi and Restall's proposal to reflect the structure of Kripke semantics at the proof-theoretic level, but without resorting to the presence of semantic parameters explicitly (see Poggiolesi and Restall (2012), but also Poggiolesi (2009a, 2009b, 2010)). Accordingly, their preferred framework is the tree-hypersequent calculus, which extends Gentzen's original framework in a "purely syntactical way" (Poggiolesi & Restall, 2012, 40).[11] Roughly, Poggiolesi and Restall's relaxed-no-answer approach recommends using tree-hypersequents instead of labelled deductive systems, as they offer a simpler vocabulary than labelled calculi. That is, they avoid introducing unnecessary tools to proof-theoretic work, such as possible worlds.

More precisely, the tree-hypersequent calculus incorporates specific elements that generalize the structure of standard sequents. Firstly, a tree-hypersequent can be seen as a tree where each node is associated with an ordinary sequent, and the edges between these nodes represent, at the syntactic level, the accessibility relation between possible worlds in Kripke semantics. However, as said, no semantic expressions, such as labels, are explicitly introduced. The idea is to reproduce *implicitly* tree-frames of Kripke semantics by using the tree-structure of tree-hypersequents to capture the *behaviour* of possible worlds and the accessibility relations between them. This is done by extending the language with two further *structural connectives*:

- the symbol "/" (i.e., $X \Rightarrow Y/V \Rightarrow Z$ denotes that the world-sequent $X \Rightarrow Y$ is linked by the accessibility relation R to the world-sequent $V \Rightarrow Z$);
- the symbol ";" (i.e., $X \Rightarrow Y/V_1 \Rightarrow Z_1; V_2 \Rightarrow Z_2$ denotes that the world-sequent $X \Rightarrow Y$ is linked by the accessibility relation R to both world-sequents, $V_1 \Rightarrow Z_1$ and $V_2 \Rightarrow Z_2$).

For example, rules for $\Box$ are given as follows in, among others, Poggiolesi (2010) and Poggiolesi and Restall (2012):

$$\frac{G[X \Rightarrow Y/\varphi, V \Rightarrow Z]}{G[\Box\varphi, X \Rightarrow Y/V \Rightarrow Z]}\; L\Box \qquad\qquad \frac{G[X \Rightarrow Y/ \Rightarrow \varphi; \underline{V}]}{G[X \Rightarrow Y, \Box\varphi/\underline{V}]}\; R\Box$$

The $L\Box$ and $R\Box$ rules in the tree-hypersequent calculus are used to move formulas between world-sequents. The $L\Box$ rule moves the formula from one component to another (root-first) and is seen as mirroring the semantic process of *saturation* of already existing (or, sometimes, opened) worlds. The $R\Box$ rule, on the other hand,

[10] "The weak notion [of syntactic purity] claims that a sequent calculus should not make any use of explicit semantic elements" (Poggiolesi, 2010, 29).

[11] Tree-hypersequent systems are equivalent, despite variations in construction methods and notation, to frameworks commonly known as *nested sequents* (first introduced by Bull (1992) and Kashima (1994)), and *deep sequents* (first discussed by Brünnler (2006, 2009)). The term "tree-hypersequent", along with its specific formalism and suggested interpretation, was introduced in Poggiolesi (2009a). For a comprehensive source, refer to, e.g, Lellmann and Poggiolesi (202x).

adds a new component to the premise (root-first) and is seen as encoding the semantic procedure of *opening* a new world from a given one.
As discussed by Boretti (2009), the rules displayed above and the suggested interpretation highlight that:

> "[...] tree-hypersequent calculi [...] and deep sequent systems [...], although absolutely legitimated on their formal base, do not seem to be unaffected by [semantic pollution concerns]. Whereas the semantics notions are explicitly internalised into the labelled calculi in the form of the syntactical counterparts of forcing [...] and accessibility relation [...], tree-hypersequents and deep sequent systems *hide* their relational semantics under a more complex syntax." (Boretti, 2009, 34-35, emphasis mine)

This position is also upheld in Read (2015), where it is explicitly stated that the explicit-implicit distinction, as endorsed by Poggiolesi and Restall, is not the correct way to consider the difference in the use of Kripke semantics between labelled and tree-hypersequent systems. Read notes that in tree-hypersequent calculi, the semantic content is still explicitly present, but is indicated by the symbols "/" and ";" instead of R. Similarly to Boretti, he argues that the tree-hypersequent rules for necessity only encode the semantic structure of modal formulas in an opaque and disguised manner, thus simply *obscuring* the semantic apparatus that is more evident in the notation of labelled sequents.[12] This suggests that although Kripke semantics is not exhibited in the same way as in labelled calculi, it is still displayed. To illustrate this, we consider the following labelled rules for $\Box$:

$$\dfrac{y:\varphi, xRy, \Gamma \Rightarrow \Delta}{x:\Box\varphi, xRy, \Gamma \Rightarrow \Delta}\ L\Box' \qquad\qquad {\scriptstyle (y \text{ fresh})}\ \dfrac{xRy, \Gamma \Rightarrow \Delta, y:\varphi}{\Gamma \Rightarrow \Delta, x:\Box\varphi}\ R\Box'$$

By examining two simple derivations of $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, we can compare the work done by "/" and R, respectively:

$$\begin{array}{cl}
\Rightarrow /\varphi \Rightarrow \varphi \qquad \Rightarrow /\psi \Rightarrow \psi & \\ \hline
\Rightarrow /\varphi, \varphi \to \psi \Rightarrow \psi & L\to \\ \hline
\Box\varphi \Rightarrow /\varphi \to \psi \Rightarrow \psi & L\Box \\ \hline
\Box(\varphi \to \psi), \Box\varphi \Rightarrow / \Rightarrow \psi & L\Box \\ \hline
\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi & R\Box \\ \hline
\Box(\varphi \to \psi) \Rightarrow \Box\varphi \to \Box\psi & R\to \\ \hline
\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & R\to
\end{array}
\qquad
\begin{array}{lcl}
& xRy, y:\varphi \Rightarrow y:\varphi \qquad xRy, y:\psi \Rightarrow y:\psi & \\ \hline
& xRy, y:\varphi \to \psi, y:\varphi \Rightarrow y:\psi & L\to' \\ \hline
& xRy, y:\varphi \to \psi, x:\Box\varphi \Rightarrow y:\psi & L\Box' \\ \hline
& xRy, x:\Box(\varphi \to \psi), x:\Box\varphi \Rightarrow y:\psi & L\Box' \\ \hline
{\scriptstyle (y \text{ fresh})} & x:\Box(\varphi \to \psi), x:\Box\varphi \Rightarrow x:\Box\psi & R\Box' \\ \hline
& x:\Box(\varphi \to \psi) \Rightarrow x:\Box\varphi \to \Box\psi & R\to' \\ \hline
& \Rightarrow x:\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & R\to'
\end{array}$$

In the labelled derivation, the application of $R\Box'$ (root-first) introduces a fresh (i.e., new) state variable y, which is related to the given state variable x by R. Similarly, in the tree-hypersequent derivation, each root-first application of $R\Box$ adds a new world-sequent that is related to the already given one by "/". On the other hand, the root-first applications of $L\Box'$ in the labelled derivation do not necessarily require the introduction of fresh state variables, meaning that they could have been used before. Likewise, in the tree-hypersequent derivation, the applications of $L\Box$ allow for the reuse of an

[12] "Kripke semantics is not merely implicit in the very notation of tree-hypersequents, rather, it is explicit but opaquely disguised in the notation of '/' and ';'." (Read, 2015, 659). See also Boretti (2009, 36).

already given world-sequent.
Given these insights, it might be appropriate to reframe the explicit-implicit dichotomy advocated by Poggiolesi and Restall in terms of differentiating between the *linguistic* and *structural* embodiment of the semantic apparatus. Indeed, the newly introduced symbols "/" and ";" explicitly display the semantics at the level of the *structure* of tree-hypersequents, rather than incorporating the semantics as an expansion of the *vocabulary* of sequents, as it is the case in labelled calculi. Nevertheless, what matters for the present investigation is that, even though the apparatus of Kripke semantics is presented differently in tree-hypersequent systems than in labelled calculi, it is still (i) explicitly displayed (although obscured in an unconventional notation) and (ii) plays a fundamental role.
In conclusion, I would like to bring to the fore two critical points. Firstly, Boretti's and Read's considerations illuminate the flaw in claiming that tree-hypersequent calculi can be entirely free from semantic pollution. The structural features of the calculus inherently embody the pollution, and the functional equivalence of symbols "/" and R, despite their notational differences, serves as evidence of the presence of the semantic apparatus in tree-hypersequent calculi just as in labelled systems. Secondly, Poggiolesi's and Restall's analysis takes a top-down approach that emphasizes their preference for the purity and simplicity of vocabularies of sequent calculi.[13] Although this approach aligns well with their goals, according to my investigation it provides only a partial perspective, which we believe needs to be supplemented with considerations grounded in practice, and this is the theme of the next perspective.
Let me clarify that my intention is not to propose the adoption of labelled deductive systems by *all* proof theorists, as there are also valid circumstances and reasons for preferring non-labelled frameworks. Indeed, I firmly agree that:

> "in dealing with Gentzen systems, no particular variant is to be preferred over all the others; one should choose a variant suited for the purpose at hand." (Troelstra & Schwichtenberg, 2000, 51)

In light of this consideration, indeed, the objective of this part of the paper was to examine the perspectives of proponents of non-labelled calculi who express concerns about the additional vocabulary introduced by labelled systems. Our analysis is specifically focused on practitioners who prioritize non-labelled frameworks over labelled ones based on principles of syntactic purity, as opposed to those who simply choose to use non-labelled calculi for practical reasons.

2.3 Third perspective: opportunistic-yes answers

Opportunistic-yes answers
Labelled deductive systems are acceptable frameworks, but for merely methodological/practical reasons.

[13] "The labelled vocabulary also allows us to introduce sequents significantly beyond what we need in modal deduction. So, let us continue our search for a simpler vocabulary for the structure of modal derivations" (Poggiolesi & Restall, 2012, 49). Alternatively: "can we really assert that these calculi [i.e., labelled ones] are computational instruments for modal logic, a logic where labels or relational atoms do not usually appear?" (Poggiolesi, 2010, 99).

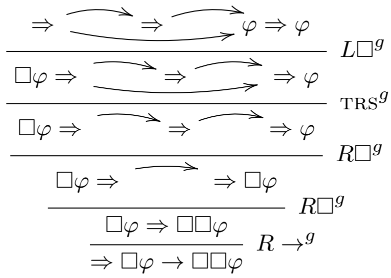
In the domain of structural proof theory, certain types of formalizations and structures are commonly utilized for their practical utility in achieving specific goals. Labelled proof systems serve as a starting point for investigating a wide range of logics that can be characterized by relational and algebraic semantics. Therefore, practitioners have methodological reasons for employing labelled systems. Our perspective suggests that labelled calculi are not rejected based on strict syntactic purity criteria but rather *opportunistically tolerated* due to their potential to facilitate the development of new translations of labelled sequents into other frameworks or the creation of entirely new frameworks to which they can be translated. The term "opportunistic", used metaphorically here, suggests the idea of seizing opportunities as they present themselves. From this perspective, the presence of semantic pollution in labelled systems is not necessarily viewed as a drawback but rather as an initial step towards comprehending the proof theory of specific logics. Essentially, labelled systems can be said to be tolerated due to their potential to enable progress in the field by establishing connections between different frameworks.
To gain a more comprehensive understanding of the concept of "opportunistic-yes" answers and the notion of semantic pollution through an analysis of how proof theorists have employed labelled calculi, let us examine three specific examples from practice. As previously mentioned, proceeding with concrete instances is a natural course of action for our inquiry.

Bridging labelled and graph-based systems

Restall proposes that, when working with labelled calculi, we consider "the behaviour of R and the labels as a part of the structural furniture of a sequent rather than its content" (Restall, 2006, 11). This approach, among others discussed therein, relies on a translation from labelled to graph-based sequent calculi, where a labelled sequent of the following shape $\mathcal{R}, X \Rightarrow Y$[14] can be represented by a directed graph. In this graph, each node represents a label occurring in the sequent, and an arc from x to y exists when xRy is in $\mathcal{R}$. We can represent xRy using $\Rightarrow \overset{\frown}{\longrightarrow} \Rightarrow$, where, as usual, the antecedent is on the left and the consequent is on the right. The label on a formula in X or Y indicates where the formula can occur on the graph, if $x : \varphi \in X$ (resp. $y : \varphi \in Y$), then the corresponding unlabelled formula will be placed in the antecedent (resp. consequent) position at the node of the graph identified by x (resp. y). In the resulting structure, each node in the graph contains traditional Gentzen sequents, where formulas are positioned on the left and/or right of the $\Rightarrow$ symbol. To illustrate this, we provide a simple example of a labelled derivation of $\Box\varphi \rightarrow \Box\Box\varphi$ and its graph-based translation. (Note that validating $\Box\varphi \rightarrow \Box\Box\varphi$ requires relational frames to have a transitive accessibility relation.)

[14] We recall that $\mathcal{R}$ and X, Y are collections (usually, sequences, multisets or sets) of relational atoms and labelled formulas, respectively.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRz, yRz, xRy, z : \varphi \Rightarrow z : \varphi}{xRz, yRz, xRy, x : \Box\varphi \Rightarrow z : \varphi}\,L\Box'}{yRz, xRy, x : \Box\varphi \Rightarrow z : \varphi}\,\text{TRS}}{xRy, x : \Box\varphi \Rightarrow y : \Box\varphi}\,R\Box'\ (z \text{ fresh})}{x : \Box\varphi \Rightarrow x : \Box\Box\varphi}\,R\Box'\ (y \text{ fresh})}{\Rightarrow x : \Box\varphi \rightarrow \Box\Box\varphi}\,R\rightarrow'
\quad \rightsquigarrow
$$



If a sequent has no arcs at all, then there is no difference between a proof of a sequent in this framework and a proof in a standard Gentzen sequent calculus, such as **LK**. In the context of our discussion, this paper is a good example as it emphasizes the importance of working with labelled calculi to create frameworks in which the semantics is reflected in the structure of sequents rather than in their vocabulary:

> "This is [...] a new way of representing the labelled sequent structure, pushing the relational statements and labels into the syntax of the proof theory, leaving the formulas to remain as the content." (Restall, 2006, 11)

This approach aligns with the opportunistic-yes respondent, whose practices and goals recognize and highlight the practical value of using labelled systems to obtain the desired alternative systems.

Bridging labelled and tree-hypersequent systems, and the "structural refinement" method

In their paper, Goré and Ramanayake (2012) introduce a subset of labelled sequents known as "labelled tree-sequents" (LTS), which have a tree-like structure without cycles.[15] They prove that LTS and tree-hypersequents (THS) are equivalent and show that this relationship extends to nested deep sequents. Specifically, they present two translation functions between LTS and THS that preserve provability, namely $\mathbb{LT}$: LTS $\mapsto$ THS and $\mathbb{TL}$: THS $\mapsto$ LTS. Examples of applications of $\mathbb{LT}$ and $\mathbb{TL}$ are given as follows, respectively:

[15] For further information on the relations between tree-hypersequent and labelled systems (among several other frameworks), one may also consult the work by Lellmann and Poggiolesi (202x) (esp. Section 5.5).

$$\dfrac{\mathcal{R}, \overbrace{xRy, y : \Box\varphi}^{principal}, \Gamma \Rightarrow \Delta, \quad \overbrace{y : \varphi}^{principal}}{\mathcal{R}, \Gamma \Rightarrow \Delta, \; \underbrace{x : \Box\varphi}_{principal}} \; R\Box' \qquad\qquad \dfrac{G[\overbrace{X \Rightarrow Y}^{x} / \overbrace{\Box\varphi \Rightarrow \varphi}^{y}]}{G[\underbrace{X \Rightarrow Y, \Box\varphi}_{x} / \underbrace{\varnothing}_{y}]} \; R\Box$$

$$\rightsquigarrow \qquad\qquad\qquad\qquad \rightsquigarrow$$

1. $\mathbb{LT}(\mathcal{R}, \Gamma \Rightarrow \Delta) = G[\overbrace{X \Rightarrow Y}^{x} / (\underline{X}; \overbrace{\varnothing}^{y})]$
2. $\mathbb{LT}(xRy, y : \Box\varphi \Rightarrow y : \varphi) = \overbrace{\Rightarrow}^{x} / \overbrace{\Box\varphi \Rightarrow \varphi}^{y}$
3. $\mathbb{LT}(\Rightarrow x : \Box\varphi) = \overbrace{\Rightarrow \Box\varphi}^{x}$

$$\rightsquigarrow$$

$$\dfrac{G[X \Rightarrow Y / \Box\varphi \Rightarrow \varphi]}{G[X \Rightarrow Y, \Box\varphi]} \; R\Box$$

1. $\mathbb{TL}(G[\overbrace{X \Rightarrow Y}^{x} / (\underline{X}; \overbrace{\varnothing}^{y})]) = \mathcal{R}, \Gamma \Rightarrow \Delta$
2. $\mathbb{TL}(\overbrace{\Rightarrow}^{x} / \overbrace{\Box\varphi \Rightarrow \varphi}^{y}) = xRy, y : \Box\varphi \Rightarrow y : \varphi$
3. $\mathbb{TL}(\overbrace{\Rightarrow \Box\varphi}^{x} / \overbrace{\varnothing}^{y}) = \Rightarrow x : \Box\varphi$

$$\rightsquigarrow$$

$$\dfrac{\mathcal{R}, xRy, y : \Box\varphi, \Gamma \Rightarrow \Delta, y : \varphi}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box\varphi} \; R\Box'$$

The objective of Goré and Ramanayake is to demonstrate that the translations they propose allow to transfer syntactic results (such as cut-admissibility) between these two frameworks. They illustrate this through a case study, where they establish the equivalence between the THS-based calculus **CSGL** (see Poggiolesi (2009b, 2010)) and the LTS-based calculus **G3GL** (see Negri (2005)) for provability logic **GL**[16] (see Goré and Ramanayake (2012, 291–297)). Finally, the authors argue that these results successfully show:

> "[...] that THS and LTS are notational variants, allowing us to transfer proof-theoretic results including syntactic cut-admissibility between these formalisms, thus alleviating the need for independent proofs in each system." (Goré & Ramanayake, 2012, 297)

In a similar vein, Lyon (2021c) has extensively studied and applied what he refers to as the *structural refinement method* (see also Lyon and van Berkel (2019) and Lyon (2021a, 2021b)).[17] The strategy involves a procedure in which the inherent external features of labelled systems – specifically, the explicit representation of semantic structures – are internalized. This internalization is achieved through the introduction of alternative yet equivalent rules, i.e., so-called *propagation rules*. Intuitively, these rules, when applied root-first, facilitate the *propagation* of a formula from a label x to another one y when a certain path of relational atoms between x and y exists. Examples of propagation rules can be given as follows (see (Lyon, 2021a, 418) and (Lyon, 2021c, 92)):

$$\dfrac{\mathcal{R}, \Gamma \Rightarrow y : \varphi}{\mathcal{R}, \Gamma \Rightarrow x : \Diamond\varphi} \; P\Diamond \qquad\qquad \dfrac{\mathcal{R}, \Gamma, \; x : \Box\varphi, \; y : \varphi \Rightarrow z : \psi}{\mathcal{R}, \Gamma, \; x : \Box\varphi \Rightarrow z : \psi} \; P\Box$$

[16]The logic **GL** is obtained by adding the axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ to the normal modal logic **K**. Correspondingly, frames for **GL** are obtained by restricting those for **K** with additional semantic constraints. Specifically, the accessibility relation is required to be irreflexive, transitive and Noetherian.

[17]The approach applied by Lyon to labelled calculi is an adaptation of the method employed by Tiu, Ianovski, and Goré (2012) (see also Ciabattoni, Lyon, Ramanayake, and Tiu (2021)). For more details on display calculi one can refer to, e.g., Wansing (2000, 13-17) and Bimbó (2014, 199-216).

where we use the labelled formula $x : \Diamond\varphi$, or $x : \Box\varphi$, to introduce (root-first) the labelled formula $y : \varphi$, with a path between x and y of some length n. Rules like $P\Diamond$ and $P\Box$ serve to constrain and simplify the structure of sequents, resulting in shorter and more concise proofs. Lyon has successfully proved that such *refined labelled calculi* and deep nested systems[18] are notational variants by defining translations that enable one to straightforwardly switch between labelled and nested notations. Furthermore, these translations have found practical applications, that is they have been used to develop algorithms that facilitate the translation of proofs between refined labelled and nested calculi. For instance, the structural refinement method was successfully applied to bridge nested and labelled sequent systems for intuitionistic modal logics, grammar logics, first-order intuitionistic logic, and deontic STIT logics, allowing for the algorithmic translation of labelled proofs into nested proofs.
As highlighted by Lyon:

> "[...] it is interesting to wonder if switching from labelled to nested notation possesses any utility in its own right, that is, does the option of converting notation offer any advantage? Two reasons come to mind which justify a positive answer. First, switching from labelled to nested notation ensures that the language of our calculi enjoys a degree of parsimony [...] Second, switching to a more restrictive notation has practical value as it provides a priori knowledge about the structure of proofs; e.g. one can be certain that all sequents within a given proof encode trees prior to observing any proof within the associated system. [...] Furthermore, translating in the opposite direction – from nested to labelled notation – appears to be worthwhile as well, [...] [since in some cases] the labelled notation [is] easier to work with and [allows] for simpler definitions." (Lyon, 2021c, 112-113)

The approaches discussed so far clearly align with the opportunistic-yes perspective, as they emphasize the importance of exploring and utilizing labelled proof systems to facilitate the development and study of their connections with alternative proof systems, such as tree–hypersequents and nested deep sequents. Thus, the practical scenarios we have showcased provide clear examples of how labelled calculi are actually valued and used by practitioners to address broader issues in the advancement of structural proof theory.

Bridging labelled and hypersequent systems

In their paper, Ciabattoni, Maffezioli, and Spendier (2013) compare hypersequent and labelled calculi by examining proof systems for various intermediate logics, including $\mathbf{Bd}_2$ (Bounded depth at most 2).[19] A hypersequent is a structure of the form: $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, where each $\Gamma_i \Rightarrow \Delta_i$ (for $i = 1, \ldots, n$) is a sequent. Here, Γ_i, Δ_i are multisets of formulas, and the symbol "|" mostly represents a meta-level disjunction. In their paper, based on the algorithm for constructing hypersequent rules from axioms classified according to a hierarchy based on the invertibility of logical and quantifier rules (see Ciabattoni et al. (2009)), the authors propose a classification of frame conditions into a hierarchy that intuitively reflects the difficulty of handling

[18] As remarked in footnote 11, tree-hypersequents and nested deep sequents, despite the different formalisms employed, are essentially the same.

[19] Intermediate logics refer to systems that fall between intuitionistic and classical logic and include a continuum of different systems, such as Gödel-Dummett logic, logics of bounded width/depth, Gödel n-valued logics, Jankov's logic, and others. For further details on this, one can refer to Dyckhoff and Negri (2012) in addition to Ciabattoni, Straßburger, and Terui (2009) and Ciabattoni et al. (2013).

the corresponding first-order formulas in a proof-theoretic context. The authors begin with a Hilbert system for $\mathbf{Bd}_2$, namely $\mathbf{IL} \oplus \xi \vee (\xi \to (\varphi \vee (\varphi \to \psi)))$, and construct a hypersequent calculus for it by transforming its characteristic axiom into the corresponding hypersequent rule:

$$\xi \vee (\xi \to (\varphi \vee (\varphi \to \psi))) \quad \rightsquigarrow \quad \frac{G \mid \Gamma', \Gamma \Rightarrow \Delta' \qquad G \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}$$

Furthermore, the authors examine relational frames for $\mathbf{Bd}_2$, specifically $\mathcal{F}_{\mathbf{IL}} \oplus \forall x, y, z((x \leq y \wedge y \leq z) \to (y \leq x \vee z \leq y))$, and transform the aforementioned frame condition, which is expressed as a first-order formula, into a labelled rule. Following the methodology developed by Dyckhoff and Negri (2012), the transformation is given as follows:

$$\forall x, y, z((x \leq y \wedge y \leq z) \to (y \leq x \vee z \leq y))$$

$$\wr$$

$$\frac{y \leq x, x \leq y, y \leq z, \Gamma \Rightarrow \Delta \qquad z \leq y, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}$$

The authors' proposed classifications of axioms and frame conditions offer significant guidance for future research in logic by providing insights into the development of more efficient proof procedures for intermediate logics. These classifications are particularly valuable as they enable comparisons across various frameworks, especially between those employing vocabularies enriched with semantic parameters, like labelled sequents, and further generalizations of the sequent calculus typically viewed as free from semantic pollution concerns, such as hypersequents. As the authors note:

> "the methods of turning Hilbert axioms into hypersequent rules and frame conditions into labelled rules [...] are closely related." (Ciabattoni et al., 2013, 81)

In conclusion, the authors' findings not only highlight the potential connections between various frameworks but also provide insights into advancing the study of the links between proof theory and the algorithmic implementation of proof systems. Such an approach clearly aligns with the opportunistic-yes stance.

Our overarching objective is to contribute to a philosophical analysis of semantic pollution grounded in practice and providing insights into how we engage in logical activities and practices. The analysis carried out so far suggests that practitioners in structural proof theory utilize labelled calculi for their practical value in achieving specific goals. To exemplify our perspective, we have provided three specific case studies. By examining how proof theorists have employed labelled calculi in these instances, we hope to have delivered a more comprehensive understanding of the concept of "opportunistic-yes" stance and its relation to the charge of semantic pollution. The concept of opportunistic-yes answers represents a departure from the traditional approach of rejecting calculi that explicitly (or, more precisely, linguistically) internalize semantic elements within the syntax. Although seeking "less semantically polluted calculi" might still hold validity, the opportunistic-yes approach perceives alleged pollution as a means to advance research rather than a barrier to avoid at all costs. Our practice-oriented approach underscores the idea that practitioners do not dismiss calculi based on weak or strong principles of syntactic purity but, rather, embrace the potential of labelled systems to establish links between different frameworks and facilitate progress in the field.

2.4 Fourth perspective: enthusiastic-yes answers

> ENTHUSIASTIC-YES ANSWERS
> Labelled calculi are valuable tools for structural proof theory, as they enable precise and systematic studies of logics, and have worthwhile applications in philosophical contexts.

In the debate regarding the role of semantic elements in proof systems, unsurprisingly, there are proponents who support labelled calculi as suitable proof systems. This position has gained widespread recognition and has been extensively studied by practitioners over a considerable period of time. Additionally, as previously noted in the Introduction, the philosophical perspective of proof-theoretic inferentialism suggests that deductive calculi can provide a definition of the meaning of logical connectives. Some scholars have extended this view to include labelled systems. It is important to mention, however, that whereas labelled calculi are widely recognized as suitable proof systems among practitioners, there is currently little agreement regarding their utility in supporting inferential views of the meaning of logical operators. In fact, only a limited number of scholars have endorsed the notion that labelled calculi can be utilized to defend inferentialist perspectives.

The term "enthusiastic", used metaphorically, implies that concerns regarding semantic pollution are minimal or non-existent, and the proof-theoretic (and, in some cases, philosophical) value of labelled calculi is acknowledged. This practical recognition of the proof-theoretic validity of labelled calculi is particularly noteworthy because it suggests that labelled frameworks can be used as stand-alone proof systems, without necessarily relying on other frameworks for validation. This recognition has been further strengthened by the endorsement of certain scholars who embrace philosophical perspectives that regard labelled calculi as valuable tools. This perspective is intriguing as it suggests that the apparent violation of syntactic purity criteria does not necessarily make the labelled framework void or unfit for practical applications. Rather, it retains the proof-theoretic significance of labelled calculi *per se*, which continue to be the subject of intense interest, applications, and discussions among practitioners. However, it's important to note that there is no unique methodology for constructing labelled proof systems. When it comes to labelling strategies and label choices, various alternatives are available, including possible worlds, natural numbers, and (sets of) truth values. Besides the references mentioned at the beginning of Section 1, additional remarkable examples of labelling methods can be found in various other sources. For instance, in Anderson and Belnap (1975), rules for relevant implication include formulas labelled with natural numbers to formally express the relevance condition. Intuitively, their labelling technique is used to ensure that for a formula ψ to follow

from a formula φ, the assumption φ must be *actually used* in proving the conclusion ψ. Additionally, Fitting (1983) employs truth values as labels to characterize intuitionistic and modal systems. The incorporation of information extracted from possible worlds semantics into labelled calculi has early instances in the work of Giambrone and Urquhart (1987). Other significant examples discussing procedures for translating complex relational semantics into labelled proof systems can be found in the works of Nerode (1991) and Orlowska (1991, 1992). These articles address the proof theory of various non-classical logics, including modal logics, epistemic logics for representing partial knowledge in groups of agents, and relevant logics. Furthermore, Simpson (1994) significantly contributes to the development of a methodology for transforming so-called geometric implications into well-constructed structural rules, particularly for intuitionistic modal logics. Similar methodologies have found extensive applications in Hein (2005) as well as in Negri (2005) and subsequent papers. Moreover, Gabbay (1996), Viganò (2000), and Basin, D'Agostino, Gabbay, Matthews, and Viganò (2000) serve as comprehensive sources of methodologies and applications, covering a wide array of non-classical logics characterized through various labelling methods.
Describing each framework, its benefits, and limitations would be a lengthy task, and therefore, in line with our practice-based methodology, we present two significant case studies that we believe offer insightful perspectives on the concept of "enthusiastic-yes" responses towards the utilization and formal applicability of labelled calculi, and their relationship to the critique of semantic pollution. The first case study examines and discusses a group of labelled calculi, extensively employed, which provides a systematic methodology to investigate the proof theory of a broad range of non-classical logics. The second case study aims at illustrating the interconnection between proof-theoretic inferentialism and labelled calculi.

Structural proof analysis

As already mentioned, there are several approaches in proof theory that use labelled systems. As a case study, we choose a widely used approach that can be traced back to Negri (2005) and Negri (2007) for modal and several other non-classical logics. The approach proposed therein involves starting with sequent calculi without primitive structural rules[20] and then adding the machinery of Kripke's relational semantics in a specific way. Firstly, the semantic conditions of each connective, including the reference to possible worlds, are transformed into well-formed right and left rules. Secondly, the frame conditions, if any, are examined and transformed into corresponding schematic

[20]The approach of working with sequent calculi lacking primitive structural rules (e.g., weakening and contraction) is commonly employed in proof-theoretic research. This methodology involves constructing calculi with logical rules in such a way that they *absorb* the effects of the structural rules, thus eliminating the need for primitive structural rules. The tradition of incorporating structural rules into logical rules can be traced back to Kleene, who is also acknowledged for coining the term "**G3**" to label these specific calculi, as well as for introducing the labels "**G1**" and "**G2**" for related sequent systems (see Kleene (1952, 440-516), and also Bimbó (2014, 60-62)). In this context, "**G**" stands for Gentzen, and the numbers denote the approach used for constructing the calculus. For example, the labels "**G3i**" and "**G3c**" are conventionally used in practice to denote the structural rules-free sequent calculi ("**G3**") for intuitionistic ("**i**") and classical ("**c**") logic, respectively. Researchers, including Dragalin (1988), Dyckhoff (1992, 1997) and Dyckhoff and Negri (2000, 2001), refined and extended Kleene's methodology in several ways. For more detailed information on these calculi, one can also refer to Troelstra and Schwichtenberg (2000), Negri and von Plato (2001), and Indrzejczak (2021).

sequent-style rules. To illustrate this, we will provide an example of the transformation of the clause for implication in intuitionistic logic into its corresponding labelled rule:

$$x \vDash \varphi \to \psi \text{ iff for all } y \text{ such that } x \leq y \text{: if } y \vDash \varphi, \text{ then } y \vDash \psi$$

$$\rightsquigarrow$$

$$\scriptstyle (y \text{ fresh})\ \displaystyle\frac{y : \varphi, x \leq y, \Gamma \Rightarrow \Delta, y : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \to \psi}\ {\scriptstyle R \to^I}$$

$$\frac{x \leq y,\ x : \varphi \to \psi, \Gamma \Rightarrow \Delta,\ y : \varphi \qquad y : \psi,\ x \leq y,\ x : \varphi \to \psi, \Gamma \Rightarrow \Delta}{x : \varphi \to \psi, x \leq y, \Gamma \Rightarrow \Delta}\ {\scriptstyle L \to^I}$$

The right rule, when applied root-first, requires the usage of a new, fresh label, whereas the left rule incorporates copies of the conclusion formula $x : \varphi \to \psi$ in the premises, absorbing the effects of the contraction rules.

Finally, in the context of Kripke's relational semantics for intuitionistic logic, frames are defined by a reflexive and transitive binary accessibility relation denoted by the symbol "$\leq$" between two states. By observing that all frame conditions are expressed as either universal axioms or geometric implications, we transform them into well-constructed sequent-style rules. First, we convert universal axioms into conjunctive normal form, i.e., $P_1 \wedge \cdots \wedge P_i \to Q_1 \vee \cdots \vee Q_m$, and then into suitably formulated rules. For example, in the case of intuitionistic logic, we apply the following transformations:

Reflexivity: $x \leq x \quad \rightsquigarrow \quad \dfrac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle \text{REF}}$

Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z \quad \rightsquigarrow \quad \dfrac{x \leq z,\ x \leq y,\ y \leq z, \Gamma \Rightarrow \Delta}{x \leq y,\ y \leq z, \Gamma \Rightarrow \Delta}\ {\scriptstyle \text{TRS}}$

Geometric implications, on the other hand, are formulas of the form $\forall \overline{z}(\alpha \to \beta)$, where α and β are geometric formulas, i.e., they contain neither $\forall$ nor $\to$. They are converted to conjunctions of formulas according to the following scheme: $\forall \overline{x}(P_1 \wedge \cdots \wedge P_i \to \exists y_1 Q_1 \vee \cdots \vee \exists y_m Q_m)$, and, subsequently, into the corresponding left rule (see, e.g., Negri (2003)). For example, the following transformation for the frame property usually referred to as "directedness", expressible as a first-order geometric formula, can be considered (see, e.g., Negri (2007), Dyckhoff and Negri (2012)):[21]

$$\forall x, y, z((x \leq y \wedge x \leq z) \to \exists u(y \leq u \wedge z \leq u))$$

$$\rightsquigarrow$$

$$(u \text{ fresh})\ \frac{y \leq u,\ z \leq u,\ x \leq y,\ x \leq z, \Gamma \Rightarrow \Delta}{x \leq y,\ x \leq z, \Gamma \Rightarrow \Delta}\ {\scriptstyle \text{DIR}}$$

Similarly, rules for modal logics are derived by applying the same methodology used for $\to$ to the semantic clause for $\Box$. As mentioned earlier, labelled calculi can be regarded as suitable tools for the development of proof analysis, which aims at investigating the

[21]The logic obtained by adding directedness to intuitionistic logic frames results in the so-called logic of *weak excluded middle* (sometimes referred to as Jankov logic or De Morgan logic as well), which is axiomatized by adding both axioms $\neg\varphi \vee \neg\neg\varphi$ and $\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$ to those of intuitionistic logic.

structure of proofs through computational tools and formal constructions.[22] In the context of the calculi developed by Negri (2005) and subsequent works, the essential results that can be obtained can be summarized as follows:

- Height-preserving admissibility of substitution of labels;
- Derivability of generalized initial sequents;
- Height-preserving admissibility of the weakening rules;
- Height-preserving invertibility of logical and relational rules;
- Height-preserving admissibility of the contraction rules;
- Cut-admissibility;
- Soundness;
- Completeness with respect to a semantics;
- Completeness with respect to an axiomatic system.
- Direct extraction of countermodels from failed root-first derivations.

At the heart of this example, we see that labelled calculi constructed according to the strategy just illustrated are technically just as suitable as any other standard system within the field of structural proof theory (along with the added benefit of being adaptable to a wider range of logical systems):

> "Our aim is to provide a general approach to the proof theory of non-classical logics through labelled sequent calculi that obey all the principles of good design usually required of traditional sequent systems. In particular, the calculi we shall present have all the structural rules – weakening, contraction, and cut-admissible; they support, whenever possible, proof search, and have a simple and uniform syntax that allows easy proofs of metatheoretic results." (Negri, 2007, 109)

One can find many applications of Negri's methodology beyond normal (alethic) modal and intermediate logics, including temporal logics (e.g. Boretti (2009); Boretti and Negri (2009)) and quantified modal logics (e.g., Orlandelli (2021)), intuitionistic modal logic (e.g., Marin, Morales, and Straßburger (2021), Girlando, Kuznets, Marin, Morales, and Straßburger (2023)), as well as intuitionistic multi-modal and epistemic logics (e.g., Garg, Genovese, and Negri (2012); Maffezioli, Naibo, and Negri (2013)), non-normal modal logics (e.g. Tesi (2021)), conditional and counterfactual logics (e.g., Negri and Sbardolini (2016), Poggiolesi (2016), Girlando (2019), Girlando, Negri, and Sbardolini (2019)), relevant logics (e.g., Kurokawa and Negri (2020), De Martin Polo (2023, 202x)), and many others.

Lastly, we wish to remark – echoing Negri's emphasis – that labelled calculi, due to their generality, prove valuable not only as deductive systems but also as instruments for obtaining meta-theoretic results, such as embedding proofs between logics. These results, previously achieved through complex and non-constructive proofs, become almost straightforward once the logics in question are equipped with the appropriate labelled calculus. For instance, Dyckhoff and Negri (2012) and Dyckhoff and Negri (2016) proved the embedding of intermediate logics into their modal companions, and the embedding of intuitionistic logic into provability logic, respectively.

[22] In their book Negri and von Plato (2011, xi), the authors trace the origins of the *structural proof analysis research program* to their earlier work, i.e., Negri and von Plato (1998). In that work, they presented a systematic method for incorporating axioms into sequent calculi while preserving the eliminability of cut. This methodology, based on the idea of representing axioms as rules of inference, has also served as the theoretical underpinning for the conversion of frame conditions into rules as exemplified above.

More recently, Tesi and Negri (2023), by using labelled calculi that internalize neighbourhood semantics, have extended the modal embedding to infinitary intuitionistic logic.

Proof-theoretic inferentialism

In his articles, Read (2008, 2015) advocates the use of labelled systems as an appropriate framework to support an inferentialist perspective on the meaning of logical constants. This philosophical position claims that the meaning of logical constants is not given by identifying objects as their meaning, but by stating the rules for their use in inferences. These rules usually state the grounds for asserting propositions (i.e., the conditions under which such assertions can be inferred) and the consequences of the asserted propositions (i.e., what can be inferred from asserting them).[23] Read argues for labelled natural deduction calculi, rather than sequent systems, where for each connective there is the pair introduction/elimination rules, instead of the pair left/right rules. Roughly, the meaning of logical operators can be captured through their introduction rules. Instead, as Gentzen observed, the elimination rules represent the consequences of the meaning conveyed by the introduction rules. Specifically, the focus of Read's research is on defending an inferentialist view of modalities, and the proposed rules take the following form:

$$\dfrac{\begin{matrix}(xRy)\\ \vdots \\ y:\varphi\end{matrix}}{x:\Box\varphi}\,{\scriptstyle I\Box} \qquad \dfrac{x:\Box\varphi \quad \begin{matrix}(xRy \Rrightarrow y:\varphi)\\ \vdots \\ i:\xi\end{matrix}}{i:\xi}\,{\scriptstyle E\Box} \qquad \dfrac{y:\varphi \quad xRy}{x:\Diamond\varphi}\,{\scriptstyle I\Diamond} \qquad \dfrac{x:\Diamond\varphi \quad \begin{matrix}(xRy, y:\varphi)\\ \vdots \\ i:\xi\end{matrix}}{i:\xi}\,{\scriptstyle E\Diamond}$$

where, in $I\Box$, $x \neq y$ and y must not appear in any other assumption on which $y : \varphi$ depends. Additionally, in $E\Box$ and $E\Diamond$, we require also that $y \neq i$, and that y must not appear in any other assumption on which $i : \xi$ depends. The notation $(xRy \Rrightarrow y : \varphi)$ denotes that we are assuming that φ is true at all worlds y accessible from x.[24] Turning our attention to Read's defence of inferentialism, we can characterize his argumentation schema as operating on two distinct but interconnected levels, namely, (i) the endorsement of a non-reifying perspective, and (ii) the use of harmonious and normalizable rules.

(i) Read contends that those perspectives which conflate semantic and metaphysical concerns, specifically those that overlap questions of meaning with denotational and existential issues, are misguided, and deems the latter concerns as irrelevant for determining the truth of inferentialism. Accordingly, it would be mistaken to hold the view that the meaning (semantics) of expressions is solely based on a reference to a realm of objects as it happens, e.g., in a model theory.[25] Instead, he claims that the meaning of an expression is determined by the rules that govern its use in deductions, and

[23] This idea originated in Gentzen's doctoral dissertation: "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'" (Gentzen, 1969, 80).

[24] The details can be checked in Read (2008, 15-18) and Read (2015, 653-654).

[25] This interpretation of model theory is not novel among philosophers of logic: "Model-theoretic semantics is often described as being realistic; it establishes a kind of correspondence between linguistic expressions

that the question of whether the expressions in these rules serve to denote something is a separate matter, mainly pertaining metaphysics, rather than semantics:

> "[Quoting] Leibniz with approval: "There is no need to let mathematical analysis depend on metaphysical controversies." But the infinitesimal calculus (to which Leibniz was referring) is still meaningful whether or not infinitesimal quantities exist. Semantics should not be identified with model theory."[26] (Read, 2015, 656)

Furthermore, Read advocates the idea that one should avoid being misled by heuristic or metaphorical interpretations of logical expressions, and supports this argument by drawing another analogy with mathematical analysis.

> "We are setting up a formal system, in which labels and 'R' are auxiliary symbols whose meaning, if any, is conferred by the rules. [...] so the reading of iRj is at best a useful metaphor. A similar point is often made about such an expression as $\lim_{n \mapsto \infty} a_n$, where '$\infty$' does not refer to a value of n, but indicates rather that there is no greatest value of n." (Read, 2008, 15, notation adapted)

In summary, Read argues that expressions like xRy, which are employed in modal rules to characterize formulas such as $\Box\varphi$ and $\Diamond\varphi$, lack inherent meaning. Instead, they function as auxiliary tools based on Leibniz's metaphor of possible worlds and Kripke's idea of relative possibility. Indeed, relational atoms such as xRy are never actually asserted, but rather serve to limit the range of labels, which are simply syntactic devices used in constructing proofs. For example, the above rules are sufficient for the basic modal logic **K** and to obtain stronger logics, additional relational rules can be added to allow for a wider range of labels to occur in proofs. For instance, the following relational rules may be used (Read, 2008, 18), (Read, 2015, 654):

$$\dfrac{\begin{array}{c}(xRx)\\ \vdots \\ i:\varphi\end{array}}{i:\varphi}\,{\scriptstyle T} \qquad \dfrac{xRy \quad yRz \quad \begin{array}{c}(xRz)\\ \vdots \\ i:\varphi\end{array}}{i:\varphi}\,{\scriptstyle 4} \qquad \dfrac{xRy \quad \begin{array}{c}(yRx)\\ \vdots \\ i:\varphi\end{array}}{i:\varphi}\,{\scriptstyle B} \qquad \dfrac{xRy \quad xRz \quad \begin{array}{c}(yRz)\\ \vdots \\ i:\varphi\end{array}}{i:\varphi}\,{\scriptstyle 5}$$

As such, the meaning of modalities is determined solely by the shape of their introduction and elimination rules, rather than by expressions that may be mistakenly reified due to the seductive nature of Leibniz's metaphor.[27]
(ii) Read maintains that the labelled rules displayed above for $\Box$ and $\Diamond$ are harmonious, that is, the introduction rules encapsulate the whole meaning of the modal operators. Moreover, since "harmony guarantees normalization", and given that the elimination rules represent the consequences of the corresponding introduction rules, immediate normalization results for $I\Box$ and $E\Box$ (resp. $I\Diamond$ and $E\Diamond$) can be obtained. (The normalization procedures can be found in Read (2008, 15-17) and Read (2015, 650-654).) Overall, Read's investigation has concluded that labelled deductive systems for modal logics are wrongly regarded as unsuitable for proof-theoretic and philosophical research

and elements of formal structures which are either thought of as a part of reality or as representing a part of reality" (Wansing, 2000, 3).

[26] On a similar note, also Boretti argues that "the risk of a metaphysical commitment is somehow neutralised, [...] since [in labelled calculi] the relational semantics [is] only a helpful formal tool, [...] and not [...] an 'interpretation' in the sense of a metaphysical explanation" (Boretti, 2009, 34).

[27] See Read (2015, 659-660).

due to the intrusion of semantic elements into the syntax.[28] He instead maintains that labelled rules for $\Box$ and $\Diamond$ are transparent in displaying their meaning by simply specifying their use in inferences, and this aligns with the perspective of proof-theoretic inferentialism. Indeed, as suggested by Gentzen, the rules displayed above precisely articulate the grounds for asserting modal formulas and their justifiable consequences, and thus, rather than "being semantically polluted, [they] wear their meaning on their sleeves" (Read, 2015, 660).[29]

3 Limits of labelled calculi and solution strategies

To conclude the analysis, let me finally consider some of the limitations faced by labelled frameworks and their relation to the practice of proof theory. Indeed, although it is true that labelled calculi have an intuitive construction method – involving the conversion of semantic clauses and frame properties into specific inference rules – and allow for the proofs of general theorems related to fundamental proof-theoretic properties, it is also true that there are features that can make labelled frameworks unsuitable for certain proof-theoretic enterprises. To provide a more comprehensive explanation of how labelled systems are practically used, I'll focus on a group of limitations of labelled calculi – extensively addressed and discussed by working proof theorists[30] – and consider how the possible solution strategies relate to the four perspectives on labelled proof systems discussed so far.

Concerning terminating proof-search and decidability, labelled calculi are usually regarded as not ideal, precisely due to the use of extra-logical rules encoding some relational semantics. This additional complexity in their syntax renders labelled systems not *analytic* in a strong sense, which is a fundamental characteristic of proof systems designed for practical implementation. Roughly, a derivation is considered analytic if it contains only subformulas of the formulas in the conclusion. Due to the presence of relational atoms, a labelled derivation might include elements like xRy that are not subformulas of formulas belonging to some endsequent. Achieving the subformula property in labelled systems essentially requires adapting the notion of subformula to include relational atoms and considering weaker formulations of it. Nevertheless, simply having a suitably formulated subformula property is insufficient to establish decidability results. Indeed, also the presence of relational rules in which relational atoms vanish from the premise to the conclusion is not an ideal tool in a calculus intended to achieve provable terminating proof search. Intuitively, the problem with such rules is that the possibility of applying them results in duplications of relational atoms, significantly increasing the complexity of sequents. For example, systems including rules with a reflexivity atom xRx give rise to a loop. Furthermore, labelled rules formulated in such a way that their application in derivations requires the introduction of fresh variables, i.e., ones not previously used, as well as rules including

[28] "It has been claimed that [...] labelled deductive systems for modal logic are unsuitable on the grounds that they are semantically polluted and suffer from an untoward intrusion of semantics into syntax. The charge is shown to be mistaken" (Read, 2015, 649).

[29] Another approach similar to Read's, which requires some further machinery and additional explanations, was made in Negri and von Plato (2015). Despite the relevance of this work for our discussion, we won't examine it for the sake of brevity.

[30] See e.g., Lyon (2021c) and Negri and von Plato (2011).

copies of the end formula in the premises, are sources of non-terminating proof search. The issues I have introduced have found extensive discussion and formal treatment in structural proof theory. Therefore, it is instructive for our philosophical analysis to consider also how proof theorists have approached the limitations of labelled calculi. To be precise, I will propose a systematization of the various solutions that have been concretely adopted based on the four-perspectives analysis of attitudes towards the use of labels in proof systems. Specifically, I will focus on the *solution strategies* that are typically employed when addressing the practical limitations of labelled calculi.

Firstly, proof theorists who adopt convinced-no and relaxed-no attitudes can find reinforcement for their position not to work with labelled systems. The introduction of an expanded vocabulary not only violates both strong and weak principles of syntactic purity but also presents challenges in obtaining fundamental results straightforwardly. Hence, respondents with convinced-no and relaxed-no attitudes strategically propose a radical shift towards non-labelled frameworks. This shift allows them to avoid working with semantically polluted calculi and to overcome the technical limitations of labelled calculi.

In another sense, the exploration of the limitations of labelled calculi becomes more subtle and logically interesting when viewing these limits as challenges to overcome, rather than insurmountable obstacles and reasons for abandoning labelled frameworks. Practitioners endorsing an opportunistic-yes perspective, by considering semantic pollution as an initial step towards comprehending the proof theory of specific logics and leveraging the opportunities provided by labelled calculi without being constrained by strict or weak principles of syntactic purity, are more inclined to advocate for *external solution strategies* when facing limitations in the practical use of labelled calculi. In the case we examined, the use of such external solution strategies is evident through the various bridges established between labelled calculi and other frameworks. Intuitively, these external strategies offer solutions to recover results that might be too complex to be achieved within a labelled system by relying on some non-labelled framework. For example, since proof search algorithms for labelled sequents, due to the large amount of data they include, tend to be difficult to prove terminating, employing sequents with simpler structures (e.g., hypersequents or nested sequents) can result in a system with smaller-sized proofs.[31] This makes proof search more manageable, and decidability relatively easy to prove. In this case, similar to the negative attitudes, the suggestion is to transition from labelled to non-labelled notation. However, it's worth noting that the limitation is often perceived as a more "circumscribed" issue. Hence, the need to shift to a simpler framework is not seen as a general strategy to address issues with labelled calculi, but rather it is the strategy to adopt when the limits of labelled notations become apparent and hinder certain proof-theoretic endeavours. Indeed, as mentioned earlier, in opportunistic-yes responses the transition between notations is primarily driven by practical considerations rather than concerns about the impurity of the labelled notation. In fact, there may be practical scenarios where labelled calculi

[31] The work by Lyon and Ostropolski-Nalewaja (2023) confirms that nested sequent systems generally yield shorter proofs with syntactically simpler sequents compared to labelled systems. This result was established within the framework of so-called *abstract sequent calculi* – systems consisting of a set of **g**-sequents, i.e., binary graphs representing ordinary Gentzen-style sequents, along with a finite set of operations applied to them.

provide more suitable frameworks to work with, as discussed, for example, by Lyon (2021c). Indeed, the strategies falling under the opportunistic-yes responses, such as those we have showcased in Section 2.3, generally driven by the idea that certain tasks can be more easily achieved in certain frameworks rather than others, tend to provide fine-grained translation maps between labelled and non-labelled systems, allowing proofs to be inter-translatable.

Finally, proof theorists endorsing an enthusiastic-yes attitude towards the use of labels, by considering labelled calculi as valuable tools for structural proof theory and for applications in philosophical contexts, might not opt for an external solution that relies on translation to another non-labelled framework, but rather they may seek for *internal solution strategies*. Intuitively, certain works within the context of labelled calculi have chosen to address issues such as termination while retaining the labelled framework. They achieve this by refining notions and methods in such a way that, without changing the framework, solutions to the limitations of labelled calculi can be found. Naturally, these types of works may entail subtle and mathematically complex solutions. To provide concrete examples, we will describe some works and how they have successfully addressed the challenges of termination and decidability through internal solutions. Concerning the calculi discusses in the case study on pp. 16ff, in Negri (2005); Negri and von Plato (2011), an internal solution was devised to achieve terminating proof search and establish decidability proofs for various labelled calculi. Intuitively, decidability cannot be proven unless a bound is placed on the number of fresh variables that can occur in a derivation of a given sequent. To establish such a bound, one needs to consider minimal derivations, i.e., derivations in which shortenings are not possible. This means that a derivation in which a rule instance produces root-first a duplication of an atom like xRy, should not be allowed to be shortened by the application of height-preserving admissibility of contraction at the rule instance. Additionally, to avoid loops generated by atoms like xRx, one must keep track of the labels used in applications of reflexivity rules, requiring all labels in atoms of the form xRx added root-first in a minimal derivation of a sequent of the form $\Gamma \Rightarrow \Delta$ to be labels occurring in Γ and Δ. This, in turn, allows one to maintain the subterm property, meaning that all terms (labels, worlds) in a derivation are either fresh variables or terms in the conclusion. Finally, as mentioned earlier, another potential source of non-terminating proof searches arises from the repetition of the principal formulas in the premises. To address this, it is essential to ensure that rules of that kind cannot be applied more than once to the same pair of principal formulas on any branch.

This strategy has proven successful for various labelled frameworks constructed following Negri's methodology, such as those designed for modal logics. However, in cases involving more intricate labelled calculi, more nuanced solution strategies must be employed, and the methods for proving termination and decidability become mathematically complex. For instance, consider Garg et al. (2012), which explores termination and decidability in labelled calculi for intuitionistic multi-modal logics. In their paper, the authors achieve the desired decidability results through a method that builds a proof-theoretic version of semantic filtration on top of the proof-or-countermodels methodology. Their method guarantees termination in labelled sequent

calculi, provides an approach to extract countermodels when the search process terminates, and allows for a proof of the finite model property for the logics under consideration.
However, besides the several different formal solutions adopted, what's worth noting for the present investigation is that advocates of enthusiastic-yes attitudes may still find ways to address the limitations of labelled frameworks, even if it means dealing with added complexities.
In summary, although the recognition of limitations in labelled calculi is common in proof theoretic works, the reactions of practitioners may vary depending on their preferences for syntactic purity and the practical scenarios they encounter. We have identified that, based on these factors, practitioners may (i) reject the suitability of labelled calculi and opt for a radical change in framework, (ii) employ external solution strategies by considering labels a drawback in certain practical scenarios and *opportunistically* relying on non-labelled frameworks, or (iii) utilize internal solution strategies by *enthusiastically* continuing to work directly within the labelled framework to find solutions without abandoning it.

4 Conclusions and further research

We have examined to what degree a proof system is accepted based on the presence of semantic elements, which are viewed as extraneous and as affecting its pure syntactic character.
Two negative answers to this question have been proposed in the literature, but both have questionable or partial readings. The first answer, based on full-blooded (or strong) syntactic purity, posits that everything is semantically polluted. The second answer, based on weak syntactic purity, draws an explicit-implicit distinction that can be problematic based on the considerations expressed above. I would like to highlight that, in both cases, the notion of semantic element appears too vague in the arguments' formulations. Specifically, with respect to convinced-no answers, when specifying that a calculus should not reveal any semantic information, we observed that standard elements in the vocabulary of sequent calculi, such as the symbols "," and "$\Rightarrow$", or even the shape of the rules for operators, might be seen as semantically polluted. Embracing such a position could exclude a wide variety of calculi from developing structural proof theory, including Gentzen's original calculi from the 1930s. With respect to the relaxed-no answer, we observed that the distinction between explicit and implicit internalization of semantic elements can be problematic. This is because what it mainly matters is not the *explicitness* of the additional elements but rather the *way* they are included in the formulation of the calculus. We have observed this by following Read's argument and by comparing two simple derivations in which R and "/", although stated differently, accomplish the same job. All of these answers share a negative attitude towards labelled systems, and, as I have argued, do not take into account practical considerations.
By expanding the scope of our investigation to include the actual uses and applications of labelled systems, I have identified two other responses to concerns about semantic pollution. The primary distinction here is that instead of relying on theoretical

principles, these responses are grounded in concrete examples of actual practices. The proposed systematization is intentionally general because practitioners may have used various specific approaches while still sharing common practical points. Our observations suggest that practicing proof theorists tend to have a more positive attitude towards labelled systems and may adopt either an opportunistic-yes or enthusiastic-yes stance. However, in both cases, acceptance levels vary and depend mainly on goals and practical applications. When it comes to opportunistic-yes answers, semantic elements are not regarded as decisive in rejecting frameworks but rather as useful tools for gaining further insights into the relationships between frameworks. The example on p. 11 shows that tree-hypersequents, Poggiolesi's and Restall's proposed alternative framework, are more closely connected to labelled calculi than previously assumed. In sum, the third response assumes methodological connotations, urging a shift in attitude towards syntactic purity and semantic pollution by focusing on *what can be gained from labelled calculi*. On the other hand, when enthusiastic-yes answers are given, the importance of labelled deductive systems becomes apparent in terms of their proof-theoretic power as stand-alone systems and philosophical applicability. In such instances, our case studies, as well as other articles, indicate that labelled calculi are endorsed without concerns about impurity. Moreover, some argue, based on inferentialist grounds, that the claim that labelled calculi fail to provide a good framework for determining the meaning of operators in terms of their use in inferences is misguided. In the case study that was examined, Read proposed that understanding the value of labelled deductive systems does not require relying on our "reification attitudes".[32] Indeed, by leaving denotational and existential concerns aside, we can comprehend the enriched formalism of labelled calculi solely in terms of proof-theoretic concepts. Relational labels, based on their function in labelled deductions, should be seen as tools (like instructions) for constructing proofs, as they determine which atoms may or may not occur, thereby determining the provability of different results in systems with different relational rules.

Overall, our observations highlight the need to adopt a more practical and nuanced approach in philosophical reflections on labelled proof systems, based on their actual contributions to achieving goals and objectives of some proof-theoretic or philosophical enterprise. We have acknowledged that these systems offer more than what is *prima facie* apparent. By examining how working proof theorists have implemented and utilized labelled calculi, we gained insights into the potential value and usefulness of these systems, and how they have impacted the development of structural proof theory. Our reflection has focused on the issue of semantic pollution, and we have aimed at analysing such an issue by trying to bridge the gap between philosophical theorizing and the actual use of labelled calculi in practice. Indeed, as we have observed above:

> "One of the significant benefits of [our] bottom-up method, in comparison to traditional top-down approaches, is that it offers the opportunity of greater progress on established questions within the philosophy of logic." (Martin, 2022, 275)

[32] See also Dutilh Novaes (2012a, 91-94).

To broaden the scope of our investigation, it would be important to consider also the role of logical communities[33] and the establishment of research programs in proof theory.[34] The field is characterized by collaborative efforts between researchers that facilitate the development of more sophisticated and effective proof systems, as well as the dissemination of research findings and the promotion of a broader or alternative understanding of proof-theoretic concepts. Efforts to build and support logical communities, along with practical considerations, can be seen as important steps towards advancing our philosophical understandings of logical research programs – their establishment, development and potential relationships. A practical approach to proof theory, informed by considerations on research programs, and which takes into account collaborative efforts in logical communities, can pave the way towards more effective and impactful philosophical insights into what we do when we engage in logical practices and activities.

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References

Aberdein, A. (2009). Communities of logical practices (abstract). *Workshop on Practice-Based Philosophy of Logic and Mathematics.* University of Amsterdam: Institute for Logic, Language and Computation. (Link)

Aberdein, A., & Read, S. (2009). The philosophy of alternative logics. L. Haaparanta (Ed.), *The Development of Modern Logic* (pp. 613–723). Oxford (UK): Oxford University Press.

Anderson, A.R., & Belnap, N.D. (1975). *Entailment: The Logic of Relevance and Necessity* (Vol. 1). Princeton: Princeton University Press.

Avron, A. (1996). The method of hypersequents in the proof theory of propositional non-classical logics. W. Hodges, M. Hyland, C. Steinhorn, & J. Truss (Eds.), *Logic: from Foundations to Applications. European Logic Colloquium* (p. 1-32). Oxford (UK): Clarendon Press.

Basin, D., D'Agostino, M., Gabbay, D.M., Matthews, S., & Viganò, L. (Eds.). (2000). *Labelled Deduction* (Vol. 17). Dordrecht: Springer.

Basin, D., Matthews, S., Viganò, L. (1997). Labelled propositional modal logics: Theory and practice. *Journal of Logic and Computation*, *7*(6), 685–717, https://

[33] Ideas on such a notion have been sketched e.g. in Aberdein (2009) and Martin (2021). Additionally, Rosental (2008) presents a sociological analysis of logic, emphasizing that logical discoveries and their recognition are not solely the product of individual work but also a social process. His contribution is intriguing because it examines this social process through a case study on the development and acceptance of a theorem about the foundations of fuzzy logic. He focuses on the mechanisms that foster collective discussions and corroboration of proofs by observing how students are trained and how researchers interact and debate emerging ideas.

[34] The idea is not novel to science or mathematics. One can consider the work by Lakatos (1978). In the philosophy of logics one can consider the reflections offered in Aberdein and Read (2009).

doi.org/10.1093/logcom/7.6.685

Basin, D., Matthews, S., Viganò, L. (1998). Natural deduction for non-classical logics. *Studia Logica*, *60*, 119–160, https://doi.org/10.1023/A:1005003904639

Bimbó, K. (2014). *Proof Theory. Sequent Calculi and Related Formalisms.* Boca Raton: CRC Press.

Boretti, B. (2009). *Proof Analysis in Temporal Logic* (Doctoral dissertation). Università degli Studi di Milano. Faculty of Arts and Philosophy, Milan, Italy.

Boretti, B., & Negri, S. (2009). Decidability for Priorean linear time using a fixed-point labelled calculus. M. Giese & A. Waaler (Eds.), *Automated Reasoning with Analytic Tableaux and Related Methods* (Vol. 5607, pp. 108–122). Heidelberg: Springer.

Brünnler, K. (2006). Deep sequent systems for modal logic. G. Governatori, I. Hodkinson, & Y. Venema (Eds.), *Advances in Modal Logic* (Vol. 6, pp. 107–119). London: College Publications.

Brünnler, K. (2009). Deep sequent systems for modal logic. *Archive for Mathematical Logic*, *48*(6), 551–577, https://doi.org/10.1007/s00153-009-0137-3

Bull, R.A. (1992). Cut elimination for propositional dynamic logic without $*$. *Mathematical Logic Quarterly*, *38*(1), 85–100, https://doi.org/10.1002/malq.19920380107

Ciabattoni, A., Lyon, T.S., Ramanayake, R., Tiu, A. (2021). Display to labeled proofs and back again for tense logics. *ACM Transactions on Computational Logic (TOCL)*, *22*(3), 1–31, https://doi.org/10.1145/3460492

Ciabattoni, A., Maffezioli, P., Spendier, L. (2013). Hypersequent and labelled calculi for intermediate logics. D. Galmiche & D. Larchey-Wendling (Eds.), *Automated Reasoning with Analytic Tableaux and Related Methods* (Vol. 8123, pp. 81–96). Heidelberg: Springer.

Ciabattoni, A., Straßburger, L., Terui, K. (2009). Expanding the realm of systematic proof theory. E. Grädel & R. Kahle (Eds.), *Computer Science Logic: CSL 2009* (Vol. 5771, pp. 163–178). Berlin: Springer.

De Martin Polo, F. (2023). Modular labelled calculi for relevant logics. *The Australasian Journal of Logic*, *20*(1), 47-87, https://doi.org/10.26686/ajl.v20i1

.7990

De Martin Polo, F. (202x). Fusion, fission, and Ackermann's truth constant in relevant logics: A proof-theoretic investigation. I. Sedlár, S. Standefer, & A. Tedder (Eds.), *New Directions in Relevant Logic.* Cham: Springer. (Forthcoming. Preprint here)

Dragalin, A. (1988). *Mathematical Intuitionism: Introduction to Proof Theory* (Vol. 67; B. Silver & E. Mendelson, Eds.). Providence: American Mathematical Society.

Dutilh Novaes, C. (2012a). *Formal Languages in Logic: A Philosophical and Cognitive Analysis.* Cambridge (UK): Cambridge University Press.

Dutilh Novaes, C. (2012b). Towards a practice-based philosophy of logic: Formal languages as a case study. *Philosophia Scientiæ. Travaux d'histoire et de philosophie des sciences*, *16*(1), 71–102, https://doi.org/10.4000/philosophiascientiae.719

Dyckhoff, R. (1992). Contraction-free sequent calculi for intuitionistic logic. *The Journal of Symbolic Logic*, *57*(3), 795–807, https://doi.org/10.2307/2275431

Dyckhoff, R. (1997). *Dragalin's proof of cut-admissibility for the intuitionistic sequent calculi G3i and G3i′.* (University of St. Andrews Report CS/97/8)

Dyckhoff, R., & Negri, S. (2000). Admissibility of structural rules for contraction-free systems of intuitionistic logic. *The Journal of Symbolic Logic*, *65*(4), 1499–1518, https://doi.org/10.2307/2695061

Dyckhoff, R., & Negri, S. (2001). Admissibility of structural rules for extensions of contraction-free sequent calculi. *Logic Journal of the IGPL*, *9*(4), 541–548, https://doi.org/10.1093/jigpal/9.4.541

Dyckhoff, R., & Negri, S. (2012). Proof analysis in intermediate logics. *Archive for Mathematical Logic*, *51*(1-2), 71–92, https://doi.org/10.1007/s00153-011-0254-7

Dyckhoff, R., & Negri, S. (2016). A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel–McKinsey-Tarski embedding. *Journal of Logic and Computation*, *26*(1), 169–187, https://doi.org/10.1093/logcom/ext036

Fitting, M. (1983). *Proof Methods for Modal and Intuitionistic Logics* (Vol. 169). Dordrecht: Springer.

Gabbay, D.M. (1996). *Labelled Deductive Systems* (Vol. 1). Oxford (UK): Oxford University Press.

Garg, D., Genovese, V., Negri, S. (2012). Countermodels from sequent calculi in multi-modal logics. *2012 27th Annual IEEE Symposium on Logic in Computer Science* (pp. 315–324). Dubrovnik, Croatia: IEEE.

Gentzen, G. (1969). Investigations into logical deductions (1935). M. Szabo (Ed.), *Collected Papers of Gerhard Gentzen* (pp. 68–131). Amsterdam: North Holland.

Giambrone, S., & Urquhart, A. (1987). Proof theories for semilattice logics. *Mathematical Logic Quarterly*, *33*(5), 433–439, https://doi.org/10.1002/malq.19870330507

Giardino, V., Moktefi, A., Mois, S., van Bendegem, J.P. (2012). Introduction: From practice to results in mathematics and logic. *Philosophia Scientiæ. Travaux d'histoire et de philosophie des sciences*, *1*(16-1), 5–11, https://doi.org/10.4000/philosophiascientiae.706

Girlando, M. (2019). *On the Proof Theory of Conditional Logics* (Doctoral dissertation). University of Helsinki & Aix Marseille Université, Helsinki, Finland & Marseille, France.

Girlando, M., Kuznets, R., Marin, S., Morales, M., Straßburger, L. (2023). Intuitionistic S4 is decidable. *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (pp. 1–13). Boston (MA): IEEE.

Girlando, M., Negri, S., Sbardolini, G. (2019). Uniform labelled calculi for conditional and counterfactual logics. R. Iemhoff, M. Moortgat, & R. de Queiroz (Eds.), *Logic, Language, Information, and Computation. WoLLIC 2019.* (Vol. 11541, pp. 248–263). Heidelberg: Springer.

Goré, R., & Ramanayake, R. (2012). Labelled tree sequents, tree hypersequents and nested (deep) sequents. T. Bolander, T. Braüner, S. Ghilardi, & L. Moss (Eds.), *Advances in Modal Logic* (Vol. 9, pp. 279–299). London: College Publications.

Hein, R. (2005). *Geometric Theories and Modal Logic in the Calculus of Structures* (Master's thesis). Technische Universität Dresden. Faculty of Computer Science, Dresden, Germany.

Humberstone, L. (2011). *The Connectives.* Cambridge (Mass.): MIT Press.

Indrzejczak, A. (2021). *Sequents and Trees. An Introduction to the Theory and Applications of Propositional Sequent Calculi.* Cham: Birkhäuser.

Kashima, R. (1994). Cut-free sequent calculi for some tense logics. *Studia Logica*, *53*(1), 119–135, https://doi.org/10.1007/BF01053026

Kashima, R. (2003). On semilattice relevant logics. *Mathematical Logic Quarterly*, *49*(4), 401–414, https://doi.org/10.1002/malq.200310043

Kleene, S.C. (1952). *Introduction to Metamathematics.* Groningen: Wolters-Noordhoff Publisher.

Kurokawa, H., & Negri, S. (2020). Relevant logics: from semantics to proof systems. A. Rezus (Ed.), *Contemporary Logic and Computing* (pp. 259–306). London: College publications.

Lakatos, I. (1978). *The Methodology of Scientific Research Programmes* (Vol. 1; J. Worrall & G. Currie, Eds.). Cambridge (UK): Cambridge University Press.

Lellmann, B., & Poggiolesi, F. (202x). Nested sequents or tree-hypersequents – A survey. R. Padro & Y. Weiss (Eds.), *Saul Kripke on Modal Logic.* New York: Springer. (Forthcoming)

Lyon, T.S. (2021a). Nested sequents for intuitionistic modal logics via structural refinement. A. Das & S. Negri (Eds.), *Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2021* (Vol. 12842, pp. 409–427). Cham: Springer.

Lyon, T.S. (2021b). On the correspondence between nested calculi and semantic systems for intuitionistic logics. *Journal of Logic and Computation*, *31*(1), 213–265, https://doi.org/10.1093/logcom/exaa078

Lyon, T.S. (2021c). *Refining Labelled Systems for Modal and Constructive Logics with Applications* (Doctoral dissertation). Technische Universität Wien. Faculty of Computer Science, Wien, Austria. (arXiv: 2107.14487)

Lyon, T.S., & Ostropolski-Nalewaja, P. (2023). *Foundations for an abstract proof theory in the context of Horn rules.* (arXiv: 2304.05697)

Lyon, T.S., & van Berkel, K. (2019). Automating agential reasoning: Proof-calculi and syntactic decidability for STIT logics. M. Baldoni, M. Dastani, B. Liao, Y. Sakurai, & R. Zalila Wenkstern (Eds.), *PRIMA 2019: Principles and Practice of Multi-Agent Systems* (Vol. 11873, pp. 202–218). Cham: Springer.

Maddirala, N. (2014). *Philosophy of Logical Practice: A Case Study in Formal Semantics* (Master's thesis). University of Amsterdam. Institute for Logic, Language and Computation, Amsterdam, Netherlands.

Maffezioli, P., Naibo, A., Negri, S. (2013). The Church–Fitch knowability paradox in the light of structural proof theory. *Synthese*, *190*(14), 2677–2716, https://doi.org/10.1007/s11229-012-0061-7

Mancosu, P. (Ed.). (2008). *The Philosophy of Mathematical Practice*. Oxford (UK): Oxford University Press.

Marin, S., Morales, M., Straßburger, L. (2021). A fully labelled proof system for intuitionistic modal logics. *Journal of Logic and Computation*, *31*(3), 998–1022, https://doi.org/10.1093/logcom/exab020

Martin, B. (2021). Identifying logical evidence. *Synthese*, *198*(10), 9069–9095, https://doi.org/10.1007/s11229-020-02618-y

Martin, B. (2022). The philosophy of logical practice. *Metaphilosophy*, *53*(2-3), 267-283, https://doi.org/10.1111/meta.12552

Martin, B. (2023). The philosophy of logical practice. Y. Shan (Ed.), *Examining Philosophy Itself* (pp. 151–174). New York: Wiley Online Library.

Martinot, R. (202x). *Towards a formal analysis of semantic pollution of proof systems.* (Submitted to *The Logica Yearbook 2022*)

Massacci, F. (1998). *Single step tableaux for modal logics: methodology, computations.* (Technical Report TR-04, University of Rome "La Sapienza")

Massacci, F. (2005). Strongly analytic tableaux for normal modal logics. A. Bundy (Ed.), *Automated Deduction – CADE-12* (Vol. 814, pp. 723–737). Heidelberg: Springer.

Mints, G. (1997). Indexed systems of sequents and cut-elimination. *Journal of Philosophical Logic*, *26*(6), 671–696, https://doi.org/10.1023/a:1017948105274

Negri, S. (2003). Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Archive for Mathematical Logic*, *42*(4), 389–401, https://doi.org/10.1007/s001530100124

Negri, S. (2005). Proof analysis in modal logic. *Journal of Philosophical Logic*, *34*(5), 507–544, https://doi.org/10.1007/s10992-005-2267-3

Negri, S. (2007). Proof analysis in non-classical logics. C. Dimitracopoulos, L. Newelski, D. Normann, & J.R. Steel (Eds.), *Logic Colloquium 2005* (Vol. 28, pp. 107–128). Cambridge (UK): Cambridge University Press.

Negri, S. (2009). Kripke completeness revisited. G. Primiero (Ed.), *Acts of Knowledge: History, Philosophy and Logic. Essays Dedicated to Göran Sundholm* (pp. 247–282). London: College Publications.

Negri, S., & Sbardolini, G. (2016). Proof analysis for Lewis counterfactuals. *The Review of Symbolic Logic*, *9*(1), 44–75, https://doi.org/10.1017/S1755020315000295

Negri, S., & von Plato, J. (1998). Cut elimination in the presence of axioms. *Bulletin of Symbolic Logic*, *4*(4), 418–435, https://doi.org/10.2307/420956

Negri, S., & von Plato, J. (2001). *Structural Proof Theory.* Cambridge (UK): Cambridge University Press.

Negri, S., & von Plato, J. (2011). *Proof Analysis. A Contribution to Hilbert's Last Problem.* Cambridge (UK): Cambridge University Press.

Negri, S., & von Plato, J. (2015). Meaning in use. H. Wansing (Ed.), *Dag Prawitz on Proofs and Meaning* (pp. 239–257). Cham: Springer.

Nerode, A. (1991). Some lectures on modal logic. F. Bauer (Ed.), *Logic, Algebra, and Computation.* NATO ASI Series (Vol. F79, p. 281-334). Springer.

Orlandelli, E. (2021). Labelled calculi for quantified modal logics with definite descriptions. *Journal of Logic and Computation*, *31*(3), 923–946, https://doi.org/10.1093/logcom/exab018

Orlowska, E. (1991). Relational proof systems for some AI logics. *Fundamentals of Artificial Intelligence Research. FAIR 1991* (Vol. 535, pp. 33–47). Berlin, Heidelberg: Springer.

Orlowska, E. (1992). Relational proof system for relevant logics. *The Journal of Symbolic Logic*, *57*(4), 1425–1440, https://doi.org/10.2307/2275375

Poggiolesi, F. (2009a). The method of tree-hypersequents for modal propositional logic. D. Makinson, J. Malinowski, & H. Wansing (Eds.), *Towards Mathematical Philosophy. Papers from the Studia Logica Conference Trends in Logic IV* (pp. 31–51). Dordrecht: Springer.

Poggiolesi, F. (2009b). A purely syntactic and cut-free sequent calculus for the modal logic of provability. *The Review of Symbolic Logic*, *2*(4), 593–611, https://doi.org/10.1017/S1755020309990244

Poggiolesi, F. (2010). *Gentzen Calculi for Modal Propositional Logic*. New York: Springer.

Poggiolesi, F. (2016). Natural deduction calculi and sequent calculi for counterfactual logics. *Studia Logica*, *104*, 1003–1036, https://doi.org/10.1007/s11225-016-9662-3

Poggiolesi, F., & Restall, G. (2012). Interpreting and applying proof theories for modal logic. G. Russell & G. Restall (Eds.), *New Waves in Philosophical Logic* (pp. 39–62). London: Palgrave Macmillan.

Read, S. (2008). Harmony and modality. C. Dégremont, L. Kieff, & H. Rückert (Eds.), *Dialogues, Logics and Other Strange Things: Essays in Honour of Shahid Rahman* (pp. 285–303). London: College Publications. (Published online: link)

Read, S. (2015). Semantic pollution and syntactic purity. *The Review of Symbolic Logic*, *8*(4), 649–661, https://doi.org/10.1017/S1755020315000210

Rescher, N., & Urquhart, A. (1971). *Temporal Logic*. New York: Springer.

Restall, G. (2006). *Comparing modal sequent systems.* (Published online: link)

Rittberg, C.J. (2019). On the contemporary practice of philosophy of mathematics. *Acta Baltica Historiae et Philosophiae Scientiarum*, *7*(1), 5–26, https://doi.org/10.11590/abhps.2019.1.01

Rosental, C. (2008). *Weaving Self-Evidence. A Sociology of Logic*. Princeton: Princeton University Press. (Translated by C. Porter)

Simpson, A.K. (1994). *The Proof Theory and Semantics of Intuitionistic Modal Logic* (Doctoral dissertation). University of Edinburgh. College of Science and Engineering. School of Informatics, Edinburgh, Scotland.

Soler, L., Zwart, S., Lynch, M., & Israel-Jost, V. (Eds.). (2014). *Science after the Practice Turn in the Philosophy, History, and Social Studies of Science* (1st ed.). London: Routledge.

Tesi, M. (2021). Labelled sequent calculi for Lewis' non-normal propositional modal logics. *Studia Logica*, *109*(4), 725–757, https://doi.org/10.1007/s11225-020-09924-z

Tesi, M., & Negri, S. (2023). The Gödel-McKinsey-Tarski embedding for infinitary intuitionistic logic and its extensions. *Annals of Pure and Applied Logic*, *174*(8), 103285, https://doi.org/10.1016/j.apal.2023.103285

Tiu, A., Ianovski, E., Goré, R. (2012). Grammar logics in nested sequent calculus: Proof theory and decision procedures. T. Bolander, T. Braüner, S. Ghilardi, & L.S. Moss: (Eds.), *Advances in Modal Logic* (Vol. 9, pp. 516–537). London: College Publications.

Troelstra, A.S., & Schwichtenberg, H. (2000). *Basic Proof Theory*. Cambridge (UK): Cambridge University Press.

Viganò, L. (2000). *Labelled Non-Classical Logics*. New York: Springer.

Wansing, H. (2000). The idea of a proof-theoretic semantics and the meaning of the logical operations. *Studia Logica*, *64*(1), 3–20, https://doi.org/10.1023/A:1005217827758