Aalborg Universitet

AALBORG UNIVERSITY
DENMARK

On Lp boundedness of wave operators for four dimensional Schrödinger operators with threshold singularities

Jensen, Arne; Yajima, Kenji

Publication date:
2007

Document Version
Publisher's PDF, also known as Version of record

AALBORG UNIVERSITY

On L^p boundedness of wave operators for four dimensional Schrödinger operators with threshold singularities

by

Arne Jensen and Kenji Yajima

R-2007-01 January 2007

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Submitted exclusively to the London Mathematical Society
doi:10.1112/S0000000000000000

ON L^p BOUNDEDNESS OF WAVE OPERATORS FOR FOUR DIMENSIONAL SCHRÖDINGER OPERATORS WITH THRESHOLD SINGULARITIES

ARNE JENSEN AND KENJI YAJIMA

Abstract

Let $H = -\Delta + V(x)$ be a Schrödinger operator on $L^2(\mathbf{R}^4)$, $H_0 = -\Delta$. Assume that $|V(x)| + |\nabla V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 8$. Let $W_\pm = \text{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}$ be the wave operators. It is known that $W_\pm$ extend to bounded operators in $L^p(\mathbf{R}^4)$ for all $1 \le p \le \infty$, if 0 is neither an eigenvalue nor a resonance of H. We show that if 0 is an eigenvalue, but not a resonance of H, then the $W_\pm$ are still bounded in $L^p(\mathbf{R}^4)$ for all p such that $4/3 < p < 4$.

1. *Introduction*

Let $V(x)$ be a real-valued function satisfying $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 2$, where as usual $\langle x\rangle = (1 + |x|^2)^{1/2}$. Then it is well known that the Schrödinger operator $H = -\Delta + V$ on the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^m)$, $m \ge 1$, is selfadjoint with domain $D(H) = H^2(\mathbf{R}^m)$, the Sobolev space of order 2, and $C_0^\infty(\mathbf{R}^m)$ is a core. The spectrum $\sigma(H)$ of H consists of an absolutely continuous part $[0, \infty)$, and at most a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities. The singular continuous spectrum and positive eigenvalues are absent from $\sigma(H)$. We denote the point, the continuous, and the absolutely continuous subspaces for H by $\mathcal{H}_{\rm p}$, $\mathcal{H}_{\rm c}$, and $\mathcal{H}_{\rm ac}$, respectively, and the orthogonal projections onto the respective subspaces by $P_{\rm p}$, $P_{\rm c}$ and $P_{\rm ac}$. We have $\mathcal{H}_{\rm ac} = \mathcal{H}_{\rm c}$ and $P_{\rm ac} = P_{\rm c}$. $H_0 = -\Delta$ is the free Schrödinger operator. The wave operators $W_\pm$ are defined by the strong limits in $\mathcal{H}$:

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist and are complete in the sense that $\operatorname{Image} W_\pm = \mathcal{H}_{\rm ac}$. They satisfy the so called intertwining property, and the continuous part of H is unitarily equivalent to H_0 via $W_\pm$: For Borel functions f on $\mathbf{R}$, we have

$$f(H)P_{\rm ac}(H) = W_\pm f(H_0) W_\pm^*. \tag{1.1}$$

It follows that the mapping properties of $f(H)P_{\rm ac}(H)$ may be deduced from those of $f(H_0)$, once the corresponding properties of $W_\pm$ are known.

The mapping properties of $W_\pm$ have been studied for some time, and the following results have been proved under various smoothness and decay at infinity assumptions on V, see [**14**, **15**, **16**, **13**, **1**, **3**, **18**, **5**]. We say that 0 is a resonance of H, if there is a solution $\varphi(x)$ of $(-\Delta + V(x))\varphi(x) = 0$, such that $|\varphi(x)| \le C\min\{1, \langle x\rangle^{2-m}\}$, but $\varphi \notin \mathcal{H}$, and that H is of generic type, if 0 is neither an

---

2000 *Mathematics Subject Classification* 35J10 (primary), 47A40 (secondary).

eigenvalue nor a resonance of H, otherwise H is said to be of exceptional type. Note that there is no zero resonance, if $m \geq 5$.

(a) If H is of generic type, the $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$, if $m \geq 3$, and for all $1 < p < \infty$, if $m = 1$ or $m = 2$.

(b) If H is of exceptional type, the $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all p between $\frac{m}{2}$ and $\frac{m}{m-2}$, if $m \geq 5$ or $m = 3$, and for all $1 < p < \infty$, if $m = 1$.

Moreover, when H is of exceptional type, the $W_\pm$ are not bounded in $L^p(\mathbf{R}^m)$, if $p > m/2$ and $m \geq 5$, or if $p > 3$ and $m = 3$. This can be deduced from the results on the decay in time property of the propagator $e^{-itH}P_{ac}$ in the weighted L^2 spaces [**11**, **7**], or in L^p spaces [**4**, **17**]. We believe the same is true for p's on the other side of the interval given in (b), viz. $1 \leq p \leq m/(m-2)$ if $m \geq 5$ and $1 \leq p \leq 3/2$ if $m = 3$, though the proof is missing.

In the case when $m = 2$ or $m = 4$, and if 0 is a resonance of H, then the results of [**11**] and [**7**] mentioned above imply that the $W_\pm$ are not bounded in $L^p(\mathbf{R}^m)$ for $p > 2$ and, though proof is missing, we believe that this is the case for all p's except $p = 2$. In this paper we show, however, when $m = 4$, if 0 is a pure eigenvalue of H, and not a resonance, the $W_\pm$ are bounded in $L^p(\mathbf{R}^4)$ for $4/3 < p < 4$.

Theorem 1.1. *Let $|V(x)| + |\nabla V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 8$. Suppose that 0 is an eigenvalue of H, but not a resonance. Then the $W_\pm$ extend to bounded operators in the Sobolev spaces $W^{k,p}(\mathbf{R}^4)$ for any $0 \leq k \leq 2$ and $4/3 < p < 4$:*

$$\|W_\pm u\|_{W^{k,p}} \leq C_p \|u\|_{W^{k,p}}, \quad u \in W^{k,p}(\mathbf{R}^4) \cap L^2(\mathbf{R}^4). \tag{1.2}$$

Again, the results [**11**, **7**] imply that the $W_\pm$ are unbounded in L^p for $p > 4$ under the assumptions in the theorem, and we believe that this is the case also for $1 \leq p < 4/3$, though we do not have proofs.

When $f(\lambda) = e^{-it\lambda}$, (1.1) and (1.2) imply the so called L^p-L^q estimates for the propagator of the corresponding time dependent Schrödinger equations. The norm $\|u\|_{L^p(\mathbf{R}^m)}$ is often abbreviated as $\|u\|_p$.

Theorem 1.2. *Let V be as in* Theorem 1.1. *Then for any p and q such that $4/3 < q \leq 2 \leq p < 4$, and such that $1/p + 1/q = 1$,*

$$\|e^{-itH}P_c u\|_p \leq C_p |t|^{\frac{4}{p}-2}\|u\|_q, \tag{1.3}$$

for a constant C_p depending only on p.

For four dimensional Schrödinger operators H of generic type the estimate (1.3) has been proved for all $1 \leq q \leq 2 \leq p \leq \infty$ with $1/p + 1/q = 1$, via the L^p-boundedness of the wave operators ([**15**]), however, for H of exceptional type, this is a new result.

The intertwining property (1.1) and the boundedness results (1.2) may be applied to various other functions $f(H)P_c$ and can provide useful estimates. We shall not pursue this direction here and proceed directly to the proof of Theorem 1.1. We prove Theorem 1.1 only for W_-, which we denote by W for brevity. We shall mainly discuss the L^p boundedness, since the extension to Sobolev spaces is immediate, as in the last section of [**18**].

We write $\mathcal{H}_\gamma = L^2(\mathbf{R}^m, \langle x\rangle^{2\gamma}dx)$ for the weighted L^2 spaces, $R(z) = (H - z)^{-1}$ and $R_0(z) = (H_0 - z)^{-1}$ for the resolvents. We parametrize $z \in \mathbf{C} \setminus [0, \infty)$ by

$z = \lambda^2$ with $\lambda \in \mathbf{C}^+ = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ and define $G(\lambda) = R(\lambda^2)$ and $G_0(\lambda) = R_0(\lambda^2)$ for $\lambda \in \mathbf{C}^+$. They are $\mathbf{B}(\mathcal{H})$-valued meromorphic functions of $\lambda \in \mathbf{C}^+$, and the limiting absorption principle asserts that $G(\lambda)$ and $G_0(\lambda)$, when considered as functions with values in $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$, $\sigma > \frac{1}{2}$, have continuous extentions to $\overline{\mathbf{C}}^+ \setminus \{0\}$, $\overline{\mathbf{C}}^+ = \{z : \operatorname{Im} z \geq 0\}$ being the closure of $\mathbf{C}^+$.

The proof of the theorem is based on the stationary representation of wave operators, which expresses W via boundary values of the resolvents, see [**9**, **10**]:

$$Wu = u - \frac{1}{\pi i}\int_0^\infty G(\lambda)V\big(G_0(\lambda) - G_0(-\lambda)\big)u\,\lambda d\lambda, \tag{1.4}$$

where the integral $\int_0^\infty \cdots d\lambda$ should be understood as the strong limit of $\int_\varepsilon^\infty \cdots d\lambda$ in $L^2(\mathbf{R}^4)$ as $\varepsilon \downarrow 0$. As in [**18**], we decompose W into the high and the low energy parts,

$$W = W_> + W_< \equiv W\Psi(H_0)^2 + W\Phi(H_0)^2,$$

by using cut off functions $\Phi(\lambda)$ and $\Psi(\lambda)$, such that

$$\Phi(\lambda^2)^2 + \Psi(\lambda^2)^2 \equiv 1, \quad \Phi(\lambda^2) = 1 \text{ near } \lambda = 0, \quad \Phi(\lambda^2) = 0 \text{ for } |\lambda| > \lambda_0,$$

for a small constant $\lambda_0 > 0$ to be specified below.

Singularities at zero energy are irrelevant to the high energy behavior of the resolvent, and the following theorem has been proved in [**18**, Section 3.3], also for dimension four. We define the Fourier transform $\mathcal{F}u(\xi) = \hat{u}(\xi)$ by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^4} e^{-ix\xi}u(x)dx.$$

Proposition 1.3. *Let V satisfy*

$$\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{\frac{3}{2}}(\mathbf{R}^4) \quad \text{for some } \sigma > \tfrac{2}{3}, \tag{1.5}$$

and, in addition, $|V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 6$. Let $\Psi(\lambda) \in C^\infty(\mathbf{R})$ be such that $\Psi(\lambda^2) = 0$ for $|\lambda| < \lambda_0$ for some λ_0. Then $W_>$ is bounded in $L^p(\mathbf{R}^4)$ for all $1 \leq p \leq \infty$

Since $\|\hat{u}\|_{\frac{3}{2}} \leq C\|u\|_{H^1(\mathbf{R}^4)}$, (1.5) is satisfied by V of Theorem 1.1, and $W_>$ is bounded in $L^p(\mathbf{R}^4)$ for all $1 \leq p \leq \infty$. Thus we only have to study the low energy part $W_< = W\Phi(H_0)^2$. By using the intertwining property, we may write $W_<$ in the following form:

$$\Phi(H)\Phi(H_0) - \frac{1}{\pi i}\int_0^\infty \Phi(H)G(\lambda)V\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(H_0)\tilde{\Phi}(\lambda)\,\lambda d\lambda,$$

where $\tilde{\Phi} \in C_0^\infty(\mathbf{R})$ satisfies $\Phi(\lambda^2)\tilde{\Phi}(\lambda) = \Phi(\lambda^2)$. It is obvious that $\Phi(H_0)$ is a convolution with a function in $\mathcal{S}(\mathbf{R}^4)$, and it is well known (see [**15**]) that the integral kernel of $\Phi(H)$ is bounded by $C_N\langle x-y\rangle^{-N}$ for any N. Hence $\Phi(H)$ and $\Phi(H_0)$ are bounded in $L^p(\mathbf{R}^4)$ for all $1 \leq p \leq \infty$, and we only need to deal with the operator defined by the integral

$$\begin{aligned}\tilde{W} &= \int_0^\infty \Phi(H)G(\lambda)V\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(H_0)\tilde{\Phi}(\lambda)\,\lambda d\lambda \\ &= \int_0^\infty \Phi(H)G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(H_0)\tilde{\Phi}(\lambda)\,\lambda d\lambda.\end{aligned}$$

2. *Preliminaries*

We collect here some well known results, which will be used in the following sections. We define operators $\Omega_1, \Omega_2, \ldots$ by

$$\Omega_n u = \frac{1}{\pi i}\int_0^\infty (G_0(\lambda)V)^n\big(G_0(\lambda) - G_0(-\lambda)\big)u\,\lambda d\lambda\ ,$$

such that we have at least formally that $W = 1 - \Omega_1 + \Omega_2 - \cdots$. The following lemma is proved for any dimension $m \geq 3$. We let $m_* = \frac{m-1}{m-2}$.

Lemma 2.1 [**14**]. *Let $\sigma > 1/m_*$, $k = 0, 1, \ldots$, and $1 \leq p \leq \infty$. Then there exists a constant $C_k > 0$, which is independent of p, such that*

$$\|\Omega_1 u\|_{W^{k,p}} \leq C_k \sum_{|\alpha|\leq k} \|\mathcal{F}(\langle x\rangle^\sigma \partial^\alpha V)\|_{L^{m_*}(\mathbf{R}^m)}\|u\|_{W^{k,p}}, \tag{2.1}$$

$$\|\Omega_n u\|_{W^{k,p}} \leq C_k^n\Big(\sum_{|\alpha|\leq k} \|\mathcal{F}(\langle x\rangle^{2\sigma} \partial^\alpha V)\|_{L^{m_*}(\mathbf{R}^m)}\Big)^n\|u\|_{W^{k,p}}, \quad n = 2, \ldots.$$

Since $\|\hat f\|_{L^m_*(\mathbf{R}^m)} \leq C\|f\|_{H^\gamma(\mathbf{R}^m)}$, if $\gamma > \frac{m(m-3)}{2(m-1)}$, (2.1) implies

$$\|\Omega_1 u\|_{L^{\frac{3}{2}}(\mathbf{R}^4)} \leq C\|\langle x\rangle^\sigma V\|_{H^1(\mathbf{R}^4)}\|u\|_{L^p(\mathbf{R}^4)}. \tag{2.2}$$

Since the integral operator may be written in the form

$$Ku(x) = \int K(x,y)u(y)dy = \int K(x, x-y)u(x-y)dy = \int K_y(x)\tau_y u(x)dy,$$

where $K_y(x)$ is multiplication by $K(x, x-y)$, and τ_y is translation by y, the estimate (2.2) implies the following result, see [**15**].

Corollary 2.2. *Let K be an integral operator with kernel $K(x,y)$ satisfying*

$$\|K\|_{F_\sigma} \equiv \int_{\mathbf{R}^4} \|\langle x\rangle^\sigma K(x, x-y)\|_{H^1(\mathbf{R}^4_x)}dy < \infty \tag{2.3}$$

for some $\sigma > 2/3$. Then the operator $\Omega(K)$ defined by

$$\Omega(K)u = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)K\big(G_0(\lambda) - G_0(-\lambda)\big)u\,\lambda d\lambda$$

is bounded in $L^p(\mathbf{R}^4)$ for all $1 \leq p \leq \infty$. Moreover,

$$\|\Omega(K)\|_{\mathbf{B}(L^p)} \leq C\|K\|_{F_\sigma}, \quad 1 \leq p \leq \infty,$$

where the constant C is independent of p.

Definition 2.3. *We say that an operator-valued function $E(\lambda)$ defined for $\lambda \in (0, \lambda_0)$, and acting on functions on $\mathbf{R}^4$, is* moderate, *if the following condition is satisfied for a sufficiently small $\varepsilon > 0$, and some integer N: For all integers α, β, γ, with $0 \leq \alpha+\beta+\gamma \leq 3$, the function $\lambda \mapsto \langle x\rangle^{2+\beta+\varepsilon}E^{(\alpha)}(\lambda)\langle x\rangle^{2+\gamma+\varepsilon}$ is $\mathbf{B}(\mathcal{H})$-valued continuous on $(0, \lambda_0)$ and satisfies*

$$\|\langle x\rangle^{2+\beta+\varepsilon}E^{(\alpha)}(\lambda)\langle x\rangle^{2+\gamma+\varepsilon}\|_{\mathbf{B}(\mathcal{H})} \leq C|\lambda|^{2-\alpha}\langle\log\lambda\rangle^N. \tag{2.4}$$

Definition 2.4. *We say that the integral kernel $K(x,y)$ is* admissible*, if*

$$\sup_x \int_{\mathbf{R}^4} |K(x,y)|dy + \sup_y \int_{\mathbf{R}^4} |K(x,y)|dx < \infty. \tag{2.5}$$

It is a well known fact, due to Schur, that integral operators with admissible kernels are bounded in L^p for any $1 \le p \le \infty$.

Lemma 2.5. *Suppose that the operator-valued function $E(\lambda)$ defined on $(0,\lambda_0)$ is moderate, $\Phi(\lambda^2) \in C_0^\infty(\mathbf{R})$ is supported in $(-\lambda_0,\lambda_0)$, and $\tilde\Phi \in C_0^\infty(\mathbf{R})$ satisfies $\Phi(\lambda^2)\tilde\Phi(\lambda) = \Phi(\lambda^2)$. Then the operator defined by*

$$\Omega u = \int_0^\infty \Phi(H)G_0(\lambda)E(\lambda)\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(H_0)\tilde\Phi(\lambda)u\,\lambda d\lambda$$

has an admissible integral kernel.

Proof. Write $\Phi(x,y)$ and $\Phi_0(x,y)$ for the integral kernels of $\Phi(H)$ and $\Phi(H_0)$, respectively, and define $\Omega_\pm(x,y)$ by

$$\Omega_\pm(x,y) = \int_0^\infty \lambda\langle E(\lambda)G_0(\pm\lambda)\Phi_0(\cdot,y), G_0(-\lambda)\Phi(\cdot,x)\rangle\tilde\Phi(\lambda)d\lambda. \tag{2.6}$$

Then Lemma 4.3 of [**5**] implies that $\Omega_\pm(x,y)$ are continuous functions, and that the integral kernel of Ω is given by $\Omega_+(x,y) - \Omega_-(x,y)$. We define

$$G_{0l}(\lambda,\cdot,y) = e^{-i\lambda|y|}G_0(\lambda)\Phi_0(\cdot,y),\ G_{0r}(\lambda,\cdot,x) = e^{-i\lambda|x|}G_0(\lambda)\Phi(\cdot,x), \tag{2.7}$$

$$F_\pm(\lambda,x,y) = \lambda\langle E(\lambda)G_{0l}(\pm\lambda,\cdot,y), G_{0r}(\lambda,\cdot,x)\rangle\tilde\Phi(\lambda), \tag{2.8}$$

and we write (2.6) in the form

$$\Omega_\pm(x,y) = \int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm(\lambda,x,y)d\lambda. \tag{2.9}$$

We use the following lemma, which is Lemma 4.4 of [**5**]. It holds for all $m \ge 4$. We recall from [**5**] that, for a Banach space-valued function, $f \in C_*^\beta(\mathbf{R})$, $\beta \ge 0$ an integer, means that f is of class $C^{\beta-1}$ on $\mathbf{R}$, of class C^β outside 0 and it satisfies $\|f^{(\beta)}(\lambda)\| \le C\langle\log\lambda\rangle^N$ for constants $C > 0$ and $N > 0$, $\lambda \ne 0$. Here $f^{(\beta)}$ is the β-th derivative of f.

Lemma 2.6. *Let $\gamma > \frac12$, let $\beta \ge 0$ be an integer, and let $x,y \in \mathbf{R}^m$. Then we have the following results.*
(1) *As $\mathcal{H}$-valued functions of λ, $\langle\cdot\rangle^{-\beta-\gamma}G_{0l}(\lambda,\cdot,y)$ and $\langle\cdot\rangle^{-\beta-\gamma}G_{0r}(\lambda,\cdot,x)$ are of class $C^\beta(\mathbf{R})$ for $0 \le \beta \le m-3$, of class $C_*^\beta(\mathbf{R})$ for $\beta = m-2$, and of class $C^\beta(\mathbf{R}\setminus\{0\})$ for any $\beta \ge m-1$.*
(2) *For $0 \le \beta \le m-3$, $G_{0l}^{(\beta)}(\lambda,z,y)$ is continuous with respect to $\lambda \ge 0$ and satisfies the estimate*

$$|G_{0l}^{(\beta)}(0,z,y)| \le C\sum_{\beta_1+\beta_2=\beta}\frac{\langle z\rangle^{\beta_1}}{\langle z-y\rangle^{m-2-\beta_2}}. \tag{2.10}$$

(3) *Let $0 < \lambda_0 < 1/2$. For any $0 \le \beta$ and $\varepsilon > 0$, we have*

$$\|\langle\cdot\rangle^{-\beta-\varepsilon-\frac{m}{2}}G_{0l}^{(\beta)}(\lambda,\cdot,y)\| \le C\lambda^{\min\{0,\frac{m-3}{2}-\beta\}}\langle y\rangle^{-\frac{m-1}{2}},\quad 0<|\lambda|<\lambda_0. \tag{2.11}$$

(4) *With obvious modifications $G_{0r}(\lambda,z,x)$ satisfies* (2.11) *and* (2.10).

We continue the proof of Lemma 2.5. Define $\tilde{E}(\lambda) = \lambda E(\lambda)$. It is obvious that $\tilde{E}(\lambda)$ satisfies (2.4) with $C|\lambda|^{3-|\alpha|}\langle\log\lambda\rangle^N$ in place of $C|\lambda|^{2-|\alpha|}\langle\log\lambda\rangle^N$ on the right. We have

$$\begin{aligned}|\langle \tilde{E}^{(\alpha)}(\lambda)G_{0l}^{(\beta)}(\pm\lambda,\cdot,y), G_{0r}^{(\gamma)}(\lambda,\cdot,x)\rangle| \le \|\langle x\rangle^{\gamma+\varepsilon+2}\tilde{E}^{(\alpha)}(\lambda)\langle x\rangle^{\beta+\varepsilon+2}\|_{\mathbf{B}(\mathcal{H})}\\ \times \|\langle\cdot\rangle^{-\beta-\varepsilon-2}G_{0l}^{(\beta)}(\pm\lambda,\cdot,y)\|\,\|\langle\cdot\rangle^{-\gamma-\varepsilon-2}G_{0r}^{(\gamma)}(\lambda,\cdot,x)\|.\end{aligned}$$

It follows by virtue of Lemma 2.6 that $F_\pm(\lambda,x,y)$ are of class C^3 with respect to λ on $(0,\lambda_0)$, and they satisfies

$$\begin{aligned}&|F_\pm^{(j)}(\lambda,x,y)| \le C\lambda^{3-j}(\log\lambda)^N\langle x\rangle^{-\frac{3}{2}}\langle y\rangle^{-\frac{3}{2}}, \quad j=0,1,2,\\ &|F_\pm^{(3)}(\lambda,x,y)| \le C\langle\log\lambda\rangle^N\langle x\rangle^{-\frac{3}{2}}\langle y\rangle^{-\frac{3}{2}}.\end{aligned}$$

It follows that $|\Omega_\pm(x,y)| \le C\langle x\rangle^{-\frac{3}{2}}\langle y\rangle^{-\frac{3}{2}}$ and for $|x|\ne|y|$, we have by integration by parts with respect to λ that

$$\begin{aligned}|\Omega_\pm(x,y)| &= \left|\frac{1}{\{i(|x|\pm|y|)\}^3}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F_\pm^{(3)}(\lambda,x,y)d\lambda\right|\\ &\le \frac{C\langle x\rangle^{-\frac{3}{2}}\langle y\rangle^{-\frac{3}{2}}}{||x|\pm|y||^3}.\end{aligned}$$

Hence $|\Omega_\pm(x,y)| \le C\langle x\rangle^{-\frac{3}{2}}\langle y\rangle^{-\frac{3}{2}}\langle|x|\pm|y|\rangle^{-3}$ and the $\Omega_\pm(x,y)$ are admissible integral kernels. □

3. *Low energy asymptotics*

We write $D_0 = G_0(0)$ and, for $0<s<\delta$, define

$$\mathcal{N} = \{u \in \mathcal{H}_{-s} : (1 + D_0V)u = 0\}.$$

It is well known that D_0V is a compact operator in $\mathcal{H}_s$, the space $\mathcal{N}$ is finite dimensional, and independent of s for $0<s<\delta$ (assuming at least $\delta>2$), see for example [**7**]. $-(Vu,u)$ defines an inner product of $\mathcal{N}$, such that if $\{\varphi_j : j = 1,\dots,d\}$ is an orthonormal basis of $\mathcal{N}$ with respect to this inner product, then $Q = -\sum_{j=1}^d|\varphi_j\rangle\langle V\varphi_j|$ is the spectral projection of D_0V associated with the eigenvalue -1. All $\varphi\in\mathcal{N}$ satisfy the Schrödinger equation $-\Delta\varphi + V\varphi = 0$ in the sense of distributions, and $\varphi\in L^2(\mathbf{R}^4)$, if and only if

$$\int V(x)\varphi(x)dx = 0, \tag{3.1}$$

and in this case actually

$$|\varphi(x)| + \langle x\rangle|\nabla\varphi(x)| \le C\langle x\rangle^{-3}. \tag{3.2}$$

Assumption 3.1. *For any $\varphi\in\mathcal{N}$ (3.1) is satisfied, and $\mathcal{N}$ coincides with the eigenspace of H for the eigenvalue zero.*

Thus, in the terminology of [**7**], we are assuming that 0 is an exceptional point of the second kind for H. We define $\overline{Q} = 1 - Q$, such that $\overline{Q}(1+D_0V)\overline{Q}$ is invertible in $\overline{Q}\mathcal{H}_{-s}$, $0<s<\delta$. We write

$$K_0 = [\overline{Q}(1+D_0V)\overline{Q}]^{-1}. \tag{3.3}$$

We can also view K_0 as an operator on $\mathcal{H}_{-s}$. In that case this operator is denoted by K.

We let $\psi_1, \dots, \psi_d$ be an orthonormal basis of $\mathcal{N}$ with respect to the ordinary inner product in $\mathcal{H}$ and

$$P_0 = \sum_{j=1}^{d} |\psi_j\rangle\langle\psi_j|$$

is the orthogonal projection. We use the notation

$$\eta_j = V\psi_j, \quad j = 1, \dots, d, \tag{3.4}$$

in the following Lemma.

Lemma 3.2. *Let $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 8$. Suppose that* Assumption 3.1 *is satisfied. Then there exists λ_0, such that for $0 < \lambda < \lambda_0$ we have as operators in $\mathbf{B}(\mathcal{H}_{-s}, \mathcal{H}_{\delta-s})$, with s satisfying $\frac{1}{2} < s < \delta - \frac{1}{2}$,*

$$V(1 + G_0(\lambda)V)^{-1} = \frac{VP_0V}{\lambda^2} - (\log\lambda)L_1 + L_0 + L_{-1} + \mathcal{R}(\lambda), \tag{3.5}$$

with operators L_0, L_1, L_{-1}, and an operator-valued function $\mathcal{R}(\lambda)$ that satisfy the following properties:

(1) *The operator L_1 is of finite rank, and for suitable constants a_{jk} we have*

$$L_1 = \sum_{j,k=1}^{d} a_{jk}|\eta_j\rangle\langle\eta_k|.$$

(2) *The operator $L_0 - V$ has an integral kernel $L(x, y)$, which satisfies* (2.3) *for some $\sigma > 2/3$.*

(3) *The operator L_{-1} is of finite rank, and, with suitable constants b_{jk}, and with functions $\xi_1, \dots, \xi_d$, which satisfy $\langle x\rangle^{\delta-2-\varepsilon}\xi_j(x) \in H^1(\mathbf{R}^4)$ for any $\varepsilon > 0$, we have*

$$L_{-1} = \sum_{j,k=1}^{d} b_{jk}|\eta_j\rangle\langle\eta_k| + \sum_{j,k=1}^{d} (|\eta_j\rangle\langle\xi_k| + |\xi_j\rangle\langle\eta_k|).$$

(4) *$\mathcal{R}(\lambda)$ is moderate on the interval $0 < \lambda < \lambda_0$, see* Definition 2.3.

The proof of Lemma 3.2 is long and will be given in a series of lemmas. The assumption $\delta > 8$ is used only in the proof of Lemma 3.20 and the other results hold under the weaker assumption $\delta > 7$.

3.1. *Preliminaries*

We use the following elementary lemma from linear algebra.

Lemma 3.3. *Let $\mathcal{X} = \mathcal{X}_0 \dot{+} \mathcal{X}_1$ be a direct sum decomposition of a vector space $\mathcal{X}$. Suppose that a linear operator L in $\mathcal{X}$ is written in the form*

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

in this decomposition, and that L_{00} is invertible. Set $C = L_{11} - L_{10}L_{00}^{-1}L_{01}$. Then L is invertible, if and only if C is invertible. In this case

$$L^{-1} = \begin{pmatrix} L_{00}^{-1} + L_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1} & -L_{00}^{-1}L_{01}C^{-1} \\ -C^{-1}L_{10}L_{00}^{-1} & C^{-1} \end{pmatrix}. \tag{3.6}$$

We write $M(\lambda) = 1 + G_0(\lambda)V$. Using Q and $\overline{Q}$, we decompose $\mathcal{H}_{-\gamma} = \overline{Q}\mathcal{H}_{-\gamma}\dot{+}\mathcal{N}$ as a direct sum. With respect to this decomposition we write

$$M(\lambda) = \begin{pmatrix} \overline{Q}M(\lambda)\overline{Q} & \overline{Q}M(\lambda)Q \\ QM(\lambda)\overline{Q} & QM(\lambda)Q \end{pmatrix} \equiv \begin{pmatrix} L_{00}(\lambda) & L_{01}(\lambda) \\ L_{10}(\lambda) & L_{11}(\lambda) \end{pmatrix}, \tag{3.7}$$

where the right side is the definition.

We define the operator-valued function $\rho(\lambda)$ for $\lambda \in \mathbf{R}$ by

$$\rho(\lambda)u(\omega) = \hat{u}(\lambda\omega) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^4} e^{-i\lambda x\cdot\omega}u(x)dx, \quad \omega \in \Sigma.$$

Here Σ denotes the unit sphere in $\mathbf{R}^4$. The Sobolev embedding theorem implies that $\rho(\lambda)$ is a $\mathbf{B}(\mathcal{H}_{2+\sigma+\varepsilon}(\mathbf{R}^4), L^2(\Sigma))$-valued function of class C^σ for any $\sigma, \varepsilon > 0$. Furthermore, if $0 \le \sigma < 2$, then $\lambda^{\frac{3}{2}}\rho(\lambda)$ is of class C^σ as a $\mathbf{B}(\mathcal{H}_{\sigma+\frac{1}{2}+\varepsilon}(\mathbf{R}^4), L^2(\Sigma))$-valued function, see [**10**]. We define for $\lambda \in \mathbf{R}$

$$A(\lambda)u(x) = \rho(\lambda)^*\rho(\lambda)u(x) = \frac{1}{(2\pi)^4}\int_\Sigma\int_{\mathbf{R}^4} e^{i\lambda\omega(x-y)}u(y)dyd\omega. \tag{3.8}$$

We have $A(-\lambda) = A(\lambda)$, and $G_0(\lambda)$ may be expressed in terms of $A(\lambda)$:

$$G_0(\lambda) = D_0 + \lambda^2\int_0^\infty \frac{\mu A(\mu)}{\mu^2 - \lambda^2}d\mu, \quad \lambda \in \mathbf{C}^+. \tag{3.9}$$

The smoothness properties of $A(\lambda)$ are studied in [**5**]. We state some of the results from that paper in the case $m = 4$. Let D_1, D_2, and D_3 be the closed domains defined by

$$\begin{aligned} D_1 &= \{(k,\ell) : k,\ell \ge 0,\ k+\ell \le 3, \ell \le k\}, \\ D_2 &= \{(k,\ell) : k,\ell \ge 0,\ k \le \frac{3}{2}, \ell \ge k\}, \\ D_3 &= \{(k,\ell) : k,\ell \ge 0,\ k+\ell \ge 3, \frac{3}{2} \le k \le 3\}. \end{aligned}$$

They have disjoint interiors, and $D_1 \cup D_2 \cup D_3 = \{(k,\ell) : 0 \le k \le 3, 0 \le \ell\}$. Define the function $\sigma_0(k,\ell)$ for $0 \le k \le 3$ and $0 \le \ell$ by

$$\sigma_0(k,\ell) = \begin{cases} \frac{k+\ell+1}{2}, & (k,\ell) \in D_1, \\ \ell + \frac{1}{2}, & (k,\ell) \in D_2, \\ k+\ell-1, & (k,\ell) \in D_3. \end{cases} \tag{3.10}$$

The function $\sigma_0(k,\ell)$ is continuous, separately increasing with respect to k and ℓ and, on lines $k+\ell = c$ with fixed c, decreases with k. In the following lemma and below we use the notation $(a)_-$ for any real number $a' < a$.

LEMMA 3.4. *Let $\ell \ge 0$ be an integer and let $0 \le k \le 3$. Let $\sigma_0 = \sigma_0(k,\ell)$ be as above and $\sigma > \sigma_0$. Then $\lambda^{3-k}A^{(\ell)}(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in \mathbf{R}$ of class $C^{(\sigma-\sigma_0)_-}$.*

Corollary 3.5. *Let $\sigma > 1/2$. As a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in [0,\infty)$*
(1) *$\lambda^3 A(\lambda)$ is of class C^j, if $\sigma > j + \frac{1}{2}$,*
(2) *$\lambda^2 A(\lambda)$ is of class C^0, if $\sigma > 1$, and of class C^j, if $1 \le j < \sigma - \frac{1}{2}$.*
(3) *$\|\langle x\rangle^{-\sigma} \lambda^j A^{(j)}(\lambda) \langle x\rangle^{-\sigma}\| \le C$, for $\sigma > 2$, $j = 0,1$, and for $\sigma > j + \frac{1}{2}$, $j = 2,3$.*

Corollary 3.5 (2) has a slight improvement, see Lemma 2.4 of [**5**].

Lemma 3.6. *Let $\frac{1}{2} < \sigma, \tau < \frac{3}{2}$ be such that $\sigma + \tau > 2$, and define $\rho_0 = \tau + \sigma - 2$. Then as a $B(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function, $\lambda^2 A(\lambda)$ is of class C^ρ for any $\rho < \rho_0$ in $\mathbf{R}$, and of class $C^{(\min\{\sigma - \frac{1}{2}, \tau - \frac{1}{2}\})-}$.*

The following result is Proposition 2.6(2) from [**5**], in the case of dimension 4.

Proposition 3.7. *There exists an operator-valued function $F(\lambda)$ with the same smoothness properties as $A(\lambda)$, as stated in* Lemma 3.4, Corollary 3.5 *and* Lemma 3.6, *such that for $\lambda > 0$,*

$$G_0(\lambda) = D_0 + \lambda^2 \big(\tilde{F}(\lambda) - \log|\lambda|\, A(\lambda)\big), \tag{3.11}$$

where $\tilde{F}(\lambda) = F(\lambda) + \frac{i\pi}{2} A(\lambda)$.

We also use the following elementary lemma. Here $C^s_{0*}(\mathbf{R})$, $0 < s \le 1$, denotes functions of Hölder class C^s, vanishing at zero.

Lemma 3.8. *Suppose $f(x)$ is of class $C^s_{0*}(\mathbf{R})$, $0 < s \le 1$, then $f(x)\log(x)$ is of class $C^{(s)-}_{0*}(\mathbf{R})$.*

3.2. *Estimates for $L_{00}(\lambda)^{-1}$*

Since $K_0 = [\overline{Q}(1 + D_0 V)\overline{Q}]^{-1}$ exists in $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$, $0 < \gamma < \delta$, and since $\lambda^2 \tilde{F}(\lambda)$ satisfies Corollary 3.5(2), we can take λ_0 sufficiently small, such that both

$$N(\lambda) \equiv L_{00}(\lambda)^{-1} \quad \text{and} \quad X(\lambda) \equiv [\overline{Q}(1 + D_0 V + \lambda^2 \tilde{F}(\lambda) V)\overline{Q}]^{-1}$$

exist for $0 \le \lambda < \lambda_0$, and are continuous $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$-valued functions, for $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. In what follows we sometimes omit the variable $\lambda$and the operator $\overline{Q}$, if no confusion is to be feared.

Lemma 3.9. *Let $\frac{1}{2} < \tau, \sigma < \delta - \frac{1}{2}$ and $0 \le j < \min\{\sigma - \frac{1}{2}, \tau - \frac{1}{2}\}$. Then the following results hold.*
(1) *$X(\lambda) - \overline{Q}$ is a $\mathbf{B}(\mathcal{H}_{-\delta+\sigma}, \mathcal{H}_{-\tau})$-valued function of class C^j on $[0, \lambda_0)$.*
(2) *$N(\lambda) - \overline{Q}$ is a $\mathbf{B}(\mathcal{H}_{-\delta+\sigma}, \mathcal{H}_{-\tau})$-valued function of class C^j on $(0, \lambda_0)$. For $j = 0,1$, $N(\lambda) - \overline{Q}$ is of class C^j on $[0, \lambda_0)$. For $j = 2,3$, we have*

$$\|\langle x\rangle^{-\tau} N^{(j)}(\lambda) \langle x\rangle^{\delta - \sigma}\| \le \begin{cases} C\langle \log\lambda\rangle^2, & j = 2, \\ C|\lambda|^{-1}, & j = 3. \end{cases} \tag{3.12}$$

Proof. By virtue of Corollary 3.5, the $\mathbf{B}(\mathcal{H}_{-\sigma})$-valued function $X(\lambda)$ is continuous on $[0,\infty)$. If we write

$$X(\lambda) = \overline{Q} - \overline{Q}\big(D_0 + \lambda^2 \tilde{F}(\lambda)\big) V \overline{Q} X(\lambda) = \overline{Q} + X_1(\lambda), \tag{3.13}$$

then $X_1(\lambda)$ is $\mathbf{B}(\mathcal{H}_{-\sigma}, \mathcal{H}_{-\tau})$-valued continuous. We compute the derivative $X^{(j)}(\lambda)$ formally. The result is (omitting several $\overline{Q}$ factors) a linear combination of

$$X(\lambda)(\lambda^2\tilde{F}(\lambda))^{(j_1)}VX(\lambda)\dots(\lambda^2\tilde{F}(\lambda))^{(j_a)}VX(\lambda), \tag{3.14}$$

where $(j_1,\dots,j_a)$ is such that $j_1+\dots+j_a=j$ and $j_1,\dots,j_a\ge 1$. Let $\sigma_0=\tau$, $\sigma_{a+1}=\delta-\sigma$, and choose $\sigma_k>j_k+\frac{1}{2}$, $k=1,\dots,a$, sufficiently close to $j_k+\frac{1}{2}$. Then, since $j=j_1+\dots+j_a<\min\{\sigma-\frac{1}{2},\tau-\frac{1}{2}\}<\delta-1$, we may assume that $\sigma_k+\sigma_{k+1}<\delta$, $k=1,\dots,a$, and $\sigma_1<\sigma_0$. We write (3.14) in the form

$$\begin{aligned} X(\lambda)\langle x\rangle^{\sigma_1}\cdot\langle x\rangle^{-\sigma_1}(\lambda^2\tilde{F}(\lambda))^{(j_1)}\langle x\rangle^{-\sigma_1}\cdot\langle x\rangle^{\sigma_1}VX(\lambda)\langle x\rangle^{\sigma_2} \\ \times\langle x\rangle^{-\sigma_2}(\lambda^2\tilde{F}(\lambda))^{(j_2)}\langle x\rangle^{-\sigma_2}\cdot\langle x\rangle^{\sigma_3}VX(\lambda)\langle x\rangle^{\sigma_3}\cdots \\ \cdots\times\langle x\rangle^{-\sigma_a}(\lambda^2\tilde{F}(\lambda))^{(j_a)}\langle x\rangle^{-\sigma_a}\cdot\langle x\rangle^{\sigma_a}VX(\lambda) \end{aligned}$$

Then the factors

$$\langle x\rangle^{-\sigma_k}(\lambda^2\tilde{F}(\lambda))^{(j_k)}\langle x\rangle^{-\sigma_k},\quad k=1,\dots,a,$$

and

$$\langle x\rangle^{\sigma_k}VX(\lambda)\langle x\rangle^{\sigma_{k+1}},\quad k=0,\dots,a,$$

are $\mathbf{B}(\mathcal{H})$-valued continuous functions of $\lambda\in[0,\infty)$. Thus the operator-valued function (3.14) is $\mathbf{B}(\mathcal{H}_{-\delta+\sigma}, \mathcal{H}_{-\tau})$-valued continuous, and the proof of the statement (1) is completed.

To prove part (2) we first observe that the operator $N(\lambda)$ is obtained by replacing $\tilde{F}(\lambda)$ in $X(\lambda)$ with $\tilde{F}(\lambda)-\log|\lambda|A(\lambda)$. Since these operators have the same smoothness and mapping properties away from a neighborhood of zero, the proof above implies that $N(\lambda)-\overline{Q}$ is of class C^j outside 0. The analogue of (3.13) for $N(\lambda)-\overline{Q}$ and Lemma 3.8 imply that it is continuous up to 0. When $j=1,2,3$, we then differentiate $N(\lambda)-\overline{Q}$ j times, as above, and write it as a linear combination of terms of the structure (3.14), with $X(\lambda)$ and $\tilde{F}(\lambda)$ replaced by $N(\lambda)$ and $\tilde{F}(\lambda)-\log\lambda A(\lambda)$, respectively. By elementary computations we have

$$(\lambda^2\log\lambda)'=2\lambda\log\lambda+\lambda,\quad (\lambda^2\log\lambda)''=2\log\lambda+3,\quad (\lambda^2\log\lambda)'''=2\lambda^{-1}.$$

Thus factoring out all logarithmic terms (and λ^{-1}, when $j_k=3$), which appear from $(\lambda^2(\tilde{F}(\lambda)-\log\lambda A(\lambda)))^{(j_k)}$, $k=1,\dots,a$, and estimating as above, we obtain (3.12). That $N(\lambda)-\overline{Q}$ is C^1 on $[0,\lambda_0)$, if $j=1$, follows since $\langle x\rangle^{-\sigma_k}(\lambda^2\tilde{F}(\lambda)-\lambda^2\log\lambda A(\lambda))'\langle x\rangle^{-\sigma_k}$ is $\mathbf{B}(\mathcal{H})$-continuous up to 0, due to Lemma 3.8. □

LEMMA 3.10. *Let $Y(\lambda)$ be either one of*

$$N(\lambda)-X(\lambda),\quad X(\lambda)-K,\quad N(\lambda)-K.$$

Then the operator-valued function $VY(\lambda)$ is moderate, see Definition 2.3.

Proof. It suffices to prove the lemma for the first two operators. Since the proofs are similar, we prove the lemma only for $N(\lambda)-X(\lambda)$. The resolvent equation implies that

$$VY(\lambda)=VN(\lambda)(\lambda^2\log\lambda)A(\lambda)VX(\lambda).$$

It follows by choosing $\varepsilon>0$ such that $\sigma=\delta-5-\varepsilon>2$ that

$$\|\langle x\rangle^{5+\varepsilon}VY(\lambda)\langle x\rangle^{5+\varepsilon}\|\le\lambda^2|\log\lambda|\|\langle x\rangle^{5+\varepsilon}VN(\lambda)\langle x\rangle^{\sigma}\|$$

$$\cdot \|\langle x\rangle^{-\sigma}A(\lambda)\langle x\rangle^{-\sigma}\| \cdot \|\langle x\rangle^{\sigma}VX(\lambda)\langle x\rangle^{5+\varepsilon}\| \le C\lambda^2|\log\lambda|, \quad (3.15)$$

and (2.4) is satisfied, if $\alpha = 0$ and $0 \le \beta+\gamma \le 3$. We differentiate α-times $VY(\lambda)$ using Leibniz' rule. The result is a linear combination of the terms

$$VN^{(j_1)}(\lambda)(\lambda^2\log\lambda)^{(j_2)}A^{(j_3)}(\lambda)VX^{(j_4)}(\lambda), \quad j_1+\cdots+j_4=\alpha,\ 0\le j_1,\dots,j_4.$$

We show that for $1 \le \alpha+\beta+\gamma \le 3$

$$Q_{\alpha\beta\gamma} = \langle x\rangle^{2+\beta+\varepsilon}VN^{(j_1)}(\lambda)\{(\lambda^2\log\lambda)^{(j_2)}\lambda^{-j_3}\}\lambda^{j_3}A^{(j_3)}(\lambda)VX^{(j_4)}(\lambda)\langle x\rangle^{2+\gamma+\varepsilon}$$

satisfies $\|Q_{\alpha\beta\gamma}\|_{\mathbf{B}(\mathcal{H})} \le C|\lambda|^{2-|\alpha|}\langle\log\lambda\rangle^N$ for some N. We have

$$|(\lambda^2\log\lambda)^{(j_2)}\lambda^{-j_3}| \le C\lambda^{2-(j_2+j_3)}\langle\log\lambda\rangle, \quad \lambda\le\lambda_0.$$

(i) Let $j_1=j_4=0$ and $j_2+j_3=\alpha$.

$$\|\langle x\rangle^{2+\beta+\varepsilon}VN(\lambda)\langle x\rangle^{\delta-(2+\beta+\varepsilon)}\| \le C, \quad (3.16)$$

$$\|\langle x\rangle^{\delta-2-\gamma-\varepsilon}VX(\lambda)\langle x\rangle^{2+\gamma+\varepsilon}\| \le C \quad (3.17)$$

and, since $\max\{2, j_3+\frac12\} < \min\{\delta-2-\beta, \delta-2-\gamma\}$

$$\|\langle x\rangle^{-\delta+2+\beta+\varepsilon}\lambda^{j_3}A^{(j_3)}(\lambda)\langle x\rangle^{-\delta+2+\gamma+\varepsilon}\| \le C$$

by virtue of Corollary 3.5 (3). Thus, the desired estimate holds this case.
(ii) Next consider the case $j_1 \ge 1$ and $j_4 \ge 1$. Then $j_2+j_3 \le 1$. Choose $\varepsilon>0$, σ_1 and σ_4 such that

$$\delta-(2+\beta+\varepsilon) > j_1+\tfrac12, \qquad j_1+\tfrac12 < \sigma_1 < \delta-\max\{2, j_3+\tfrac12\}, \quad (3.18)$$

$$\delta-(2+\gamma+\varepsilon) > j_4+\tfrac12, \qquad j_4+\tfrac12 < \sigma_4 < \delta-\max\{2, j_3+\tfrac12\}. \quad (3.19)$$

Such a choice is clearly possible and, then, by virtue of Lemma 3.9,

$$\|\langle x\rangle^{2+\beta+\varepsilon}VN^{(j_1)}(\lambda)\langle x\rangle^{\delta-\sigma_1}\| \le C\begin{cases}\langle\log\lambda\rangle^2 & j_1=2,\\ |\lambda|^{-1}, & j_1=3,\end{cases} \quad (3.20)$$

$$\|\langle x\rangle^{\delta-\sigma_4}VX^{(j_4)}(\lambda)\langle x\rangle^{2+\gamma+\varepsilon}\| \le C, \quad (3.21)$$

and by virtue of Corollary 3.5(3),

$$\|\langle x\rangle^{-\delta+\sigma_1}\lambda^{j_3}A^{(j_3)}(\lambda)\langle x\rangle^{-\delta+\sigma_4}\| \le C.$$

Thus the desired estimate holds also this case.
(iii) Let $j_1 \ge 1$ and $j_4 = 0$. We choose $\varepsilon>0$ and σ_1 as in (3.18) so that (3.20) is satisfies. We also have (3.17). Choosing $\varepsilon>0$ smaller if necessary so that $\max\{2, j_3+\frac12\} < \delta-2-\gamma-\varepsilon$, we obtain that

$$\|\langle x\rangle^{-\delta+\sigma_1}\lambda^{j_3}A^{(j_3)}(\lambda)\langle x\rangle^{-\delta+2+\gamma+\varepsilon}\| \le C$$

Hence the desired estimate holds also this case.
(iv) The case $j_1=0$ and $j_4\ge1$ may be discussed in a way similar to the previous case (iii). We omit the details. □

3.3. *The behavior of $C^{-1}(\lambda)$*

We next study $C(\lambda) = L_{11}(\lambda) - L_{10}(\lambda)N(\lambda)L_{01}(\lambda)$. Recall that all $\varphi\in\mathcal{N}$ satisfy $V\varphi \in \mathcal{H}_{\delta+1-\varepsilon}$ for any $\varepsilon>0$, and that $\mathcal{F}(V\varphi)(0)=0$. It follows that for $\sigma>3$ we have

$$A(0)VQ = QA(0) = 0, \quad \langle x\rangle^{-\sigma}A'(0)\langle x\rangle^{-\sigma} = \langle x\rangle^{-\sigma}F'(0)\langle x\rangle^{-\sigma} = 0. \quad (3.22)$$

$$\rho(0)VQ = 0, \quad Q\rho^*(0) = 0, \quad \rho^*(0)\rho'(0) = 0. \tag{3.23}$$

The identities for the derivatives hold, since $A(\lambda)$ and $F(\lambda)$ are even functions of λ, and since $\rho'(0)u(\omega)$ is an odd function of $\omega \in \Sigma$.

LEMMA 3.11. *Let $\varepsilon > 0$, and let $\tilde{\alpha} = \max\{2, \alpha\}$, for $0 \le \alpha \le 3$. Then there exists a constant C, such that, for $\varphi \in \mathcal{N}$,*

$$\|\langle x\rangle^{-(2+\tilde{\alpha}+\varepsilon)} A^{(\alpha)}(\lambda)V\varphi\|_{\mathcal{H}} \le C|\lambda|^{\max\{2-\alpha,0\}}, \tag{3.24}$$

$$|\langle V\varphi|A^{(\alpha)}(\lambda)\langle x\rangle^{-(2+\tilde{\alpha}+\varepsilon)}u\rangle| \le C|\lambda|^{\max\{2-\alpha,0\}}\|u\|. \tag{3.25}$$

These estimates (3.24) *and* (3.25) *remain true if $A(\lambda)$ is replaced by $\tilde{F}(\lambda) - \tilde{F}(0)$.*

Proof. By virtue of (3.22), we have

$$A(\lambda)V\varphi = (A(\lambda) - A(0) - \lambda A'(0))V\varphi = \lambda^2 \int_0^1 (1-\theta)A''(\lambda\theta)V\varphi d\theta,$$

$$A'(\lambda)V\varphi = (A'(\lambda) - A'(0))V\varphi = \lambda \int_0^1 A''(\lambda\theta)V\varphi d\theta.$$

Since $\|\langle x\rangle^{-2-\alpha-\varepsilon} A^{(\alpha)}(\lambda)\langle x\rangle^{-2-\alpha-\varepsilon}\|_{\mathbf{B}(\mathcal{H})} \le C$, (3.24) follows. A duality argument implies the other result. The proof for $\tilde{F}(\lambda) - \tilde{F}(0)$ is similar. ☐

We define $A_2(\lambda)$ and $\tilde{F}_2(\lambda)$ by

$$\lambda^2 A_2(\lambda) = A(\lambda) - A(0) - \lambda A'(0), \quad \lambda^2 \tilde{F}_2(\lambda) = \tilde{F}(\lambda) - \tilde{F}(0) - \lambda \tilde{F}'(0).$$

By virtue of Lemma 3.4, $QA_2(\lambda)VQ$ and $Q\tilde{F}_2(\lambda)VQ$ are $\mathbf{B}(\mathcal{N})$-valued C^4, since $\delta + 1 > 8$. We may write

$$\begin{aligned} L_{11}(\lambda) &= \lambda^2 Q(\tilde{F}(0) + \lambda^2\tilde{F}_2(\lambda) - \lambda^2 \log\lambda A_2(\lambda))VQ, \\ L_{01}(\lambda) &= \lambda^2 \overline{Q}(\tilde{F}(\lambda) - \log\lambda A(\lambda))VQ, \\ L_{10}(\lambda) &= \lambda^2 Q(\tilde{F}(\lambda) - \log\lambda A(\lambda))V\overline{Q}. \end{aligned} \tag{3.26}$$

It is well known (see [**7**] or [**5**]) that

$$Q\tilde{F}(0)VQ = \int_0^\infty \frac{QA(\mu)VQ}{\mu} d\mu = Q(-\Delta)^{-2}VQ, \tag{3.27}$$

$Q(-\Delta)^{-2}VQ$ is invertible in $\mathcal{N}$,

$$(Q(-\Delta)^{-2}VQ)^{-1} = P_0V, \quad P_0VQ = P_0V, \quad \text{and} \quad VQP_0 = VP_0.$$

We define $E_2(\lambda)$ and $E_3(\lambda)$ by

$$E_2(\lambda) = Q(\log\lambda A_2(\lambda) - \tilde{F}_2(\lambda))VQ + E_3(\lambda), \tag{3.28}$$

$$E_3(\lambda) = Q(\tilde{F}(\lambda) - \log\lambda A(\lambda))V\overline{Q}N(\lambda)\overline{Q}(\tilde{F}(\lambda) - \log\lambda A(\lambda))VQ. \tag{3.29}$$

They are $\mathbf{B}(\mathcal{N})$-valued continuous, and $C(\lambda)$ may be written in the form

$$C(\lambda) = \lambda^2(Q(-\Delta)^{-2}VQ)\big(1 - \lambda^2 P_0VE_2(\lambda)\big). \tag{3.30}$$

It follows for small $0 < \lambda < \lambda_0$ that $C(\lambda)$ is invertible in $\mathcal{N}$ and

$$\begin{aligned} C(\lambda)^{-1} &= \lambda^{-2}\big(1 - \lambda^2 P_0VE_2(\lambda)\big)^{-1}P_0V \\ &= \frac{P_0V}{\lambda^2} + P_0VE_2(\lambda)P_0V \end{aligned}$$

$$+ \lambda^2 (P_0 V E_2(\lambda))^2 \big(1 - \lambda^2 P_0 V E_2(\lambda)\big)^{-1} P_0 V. \tag{3.31}$$

Lemma 3.12. *Let $K = \overline{Q} K_0 \overline{Q}$. We have*

$$E_3(\lambda) = Q(\tilde{F}(\lambda) - \log\lambda A(\lambda)) V K (\tilde{F}(\lambda) - \log\lambda A(\lambda)) V Q + R_1(\lambda), \tag{3.32}$$

where $R_1(\lambda)$ is such that $V P_0 V R_1(\lambda)$ is moderate.

Proof. Let $Y(\lambda) = N(\lambda) - K_0$ and write $Y(\lambda)$ again for $\overline{Q} Y(\lambda) \overline{Q}$. It suffices to show that

$$V P_0 V (\tilde{F}(\lambda) - \log\lambda A(\lambda)) V Y(\lambda) (\tilde{F}(\lambda) - \log\lambda A(\lambda)) V Q.$$

is moderate. We expand this and, out of various terms thus produced, we prove that

$$(\log\lambda)^2 V P_0 V A(\lambda) V Y(\lambda) A(\lambda) V Q$$

is moderate. That other terms are moderate may be proved in a similar fashion. Let $\alpha + \beta + \gamma \le 3$, $\alpha = j_1 + j_2 + j_3$ and let $\varepsilon > 0$ be sufficiently small. Then $\langle x\rangle^{2+\beta+\varepsilon} V P_0$ and $Q\langle x\rangle^{2+\gamma+\varepsilon}$ are bounded in $\mathcal{H}$, and Lemma 3.4 implies $V A^{(j_1)}(\lambda)\langle x\rangle^{-(2+j_1+\varepsilon)}$ and $\langle x\rangle^{-(2+j_3+\varepsilon)} A^{(j_3)}(\lambda) V$ are continuous on $\mathbf{R}$ as $\mathbf{B}(\mathcal{H})$-valued functions. Since

$$\|\langle x\rangle^{2+j_1+\varepsilon} V Y^{(j_2)}(\lambda) \langle x\rangle^{2+j_3+\varepsilon}\| \le C|\lambda|^{2-j_2} \langle \log\lambda\rangle^{j_2}$$

by virtue of Lemma 3.10, the lemma follows. ☐

Lemma 3.13. *Let $F_0 = QF(0) V K F(0) V Q$. Then*

$$E_3(\lambda) = F_0 + R_2(\lambda), \tag{3.33}$$

where $R_2(\lambda)$ is such that $V P_0 V R_2(\lambda)$ is moderate.

Proof. By Lemma 3.12 it suffices to prove that

$$V P_0 V \{(\tilde{F}(\lambda) - \log\lambda A(\lambda)) V K (\tilde{F}(\lambda) - \log\lambda A(\lambda)) - F(0) V K F(0)\} V Q$$

is moderate. Note that $F(0)$ may be replaced by $\tilde{F}(0)$, due to (3.22). We expand this expression

$$\begin{aligned} &V P_0 V (\tilde{F}(\lambda) - \tilde{F}(0)) V K \tilde{F}(\lambda) V Q + V P_0 V \tilde{F}(0) V K (\tilde{F}(\lambda) - \tilde{F}(0)) V Q \\ &\quad - (\log\lambda) V P_0 V \tilde{F}(\lambda) V K A(\lambda) V Q - (\log\lambda) V P_0 V A(\lambda) V K \tilde{F}(\lambda) V Q \\ &\qquad + (\log\lambda)^2 P_0 V A(\lambda) V K \overline{Q} A(\lambda) V Q. \end{aligned}$$

We only prove that $(\log\lambda) V P_0 V \tilde{F}(\lambda) V K A(\lambda) V Q$ is moderate, since the proof for other terms is similar. As in the proof of the previous lemma, $\langle x\rangle^{2+\beta+\varepsilon} V P_0$ and $Q\langle x\rangle^{2+\gamma+\varepsilon}$ are bounded in $\mathcal{H}$, and it suffices to show for $0 \le \alpha \le 3$ that

$$\|[(\log\lambda) P_0 V \tilde{F}(\lambda) V K A(\lambda) V Q]^{(\alpha)}\| \le C|\lambda|^{2-\alpha} \langle \log\lambda\rangle.$$

Differentiate using Leibniz's formula. We estimate for $0 \le \alpha = j_1 + j_2 + j_3 \le 3$ the norm

$$R = \|(\log\lambda)^{(j_1)} P_0 V \tilde{F}^{(j_2)}(\lambda) V K A^{(j_3)}(\lambda) V Q\|.$$

Consider first $j_3 = \alpha = 3$. Then, since $\delta - 2 - \varepsilon > 5 + \varepsilon$ for small $\varepsilon > 0$,

$$R \le C|\log\lambda| \|P_0 V \langle x\rangle^{-2-\varepsilon}\| \cdot \|\langle x\rangle^{-2-\varepsilon} \tilde{F}(\lambda) \langle x\rangle^{-2-\varepsilon}\|$$

$$\cdot \|\langle x\rangle^{2+\varepsilon} VK\langle x\rangle^{5+\varepsilon}\| \cdot \|\langle x\rangle^{-5-\varepsilon} A^{(3)}(\lambda)VQ\| \le C|\log\lambda|. \quad (3.34)$$

Next let $0 \le j_3 \le 2$. We have $\|\langle x\rangle^{-4-\varepsilon} A^{(j_3)}(\lambda)VQ\| \le C|\lambda|^{2-j_3}$ as a consequence of (3.24). Thus

$$R \le C|\lambda|^{-j_1}\langle\log\lambda\rangle \|\langle x\rangle^{-2-j_2-\varepsilon}\tilde{F}^{(j_2)}(\lambda)\langle x\rangle^{-2-j_2-\varepsilon}\| \cdot \|\langle x\rangle^{2+\varepsilon+j_2} VK\langle x\rangle^{3+\varepsilon}\|$$
$$\cdot \|\langle x\rangle^{-4-\varepsilon} A^{(j_3)}(\lambda)VQ\| \le C|\lambda|^{2-j_3-j_1}\langle\log\lambda\rangle \le C|\lambda|^{2-\alpha}\langle\log\lambda\rangle$$

for $|\lambda| \le 1$. This completes the proof. □

Lemma 3.14. *We have*

$$\big(\log\lambda A_2(\lambda) - \tilde{F}_2(\lambda)\big)VQ = \tfrac{1}{2}\big(\log\lambda A''(0) - \tilde{F}''(0)\big)VQ + R_3(\lambda),$$

where $R_3(\lambda)$ is such that $VP_0VR_3(\lambda)$ is moderate.

Proof. It suffices to show that for $0 \le \alpha \le 3$ we have

$$\|P_0V(A_2(\lambda) - \tfrac{1}{2}A''(0))^{(\alpha)}V\varphi\| \le C|\lambda|^{2-\alpha}, \quad (3.35)$$
$$\|P_0V(\tilde{F}_2(\lambda) - \tfrac{1}{2}\tilde{F}''(0))^{(\alpha)}V\varphi\| \le C|\lambda|^{2-\alpha}, \quad (3.36)$$

when $\varphi \in \mathcal{N}$. We only prove (3.35). We have $V\varphi \in \mathcal{H}_{8-\varepsilon}$. We have for $\alpha = 2, 3$, $\|\langle x\rangle^{-(4+\alpha+\varepsilon)} A^{(2+\alpha)}(\lambda)\langle x\rangle^{-(4+\alpha+\varepsilon)}\| \le C$ for any $\varepsilon > 0$, and (3.35) follows. Since $A(\lambda)$ is even, and $A'''(0)V\varphi = 0$, we have

$$\big(A_2(\lambda) - \tfrac{1}{2}A''(0)\big)V\varphi = \int_0^1 (1-\theta)\big(A''(\lambda\theta) - A''(0) - \lambda\theta A'''(0)\big)V\varphi d\theta.$$

Now the estimates

$$\|\langle x\rangle^{-6-\varepsilon}\big(A''(\lambda\theta) - A''(0) - \lambda\theta A'''(0)\big)\langle x\rangle^{-6-\varepsilon}\| \le C|\lambda\theta|^2,$$
$$\|\langle x\rangle^{-6-\varepsilon}\big(A'''(\lambda\theta) - A'''(0)\big)\langle x\rangle^{-6-\varepsilon}\| \le C|\lambda\theta|,$$

imply (3.35) for $\alpha = 0$, and for $\alpha = 1$, respectively. □

Combining Lemma 3.12 ∼ Lemma 3.14, we obtain the following lemma.

Lemma 3.15. *We have*

$$E_2(\lambda) = QF(0)VKF(0)VQ + \tfrac{1}{2}Q\big(\log\lambda A''(0) - \tilde{F}''(0)\big)VQ + R_4(\lambda), \quad (3.37)$$

where $R_4(\lambda)$ is such that $VP_0VR_4(\lambda)$ is moderate.

Lemma 3.16. *Let $R_5(\lambda) = \lambda^2(P_0VE_2(\lambda))^2\big(1 - \lambda^2P_0VE_2(\lambda)\big)^{-1}P_0V$. Then $VR_5(\lambda)$ is moderate.*

Before giving the proof we introduce the following definition.

Definition 3.17. *We say a function $p(\lambda)$ defined on $(0, \lambda_0)$, $\lambda_0 < 1$, is scalar-moderate (or, for brevity, s-moderate), if it satisfies*

$$|\partial_\lambda^\alpha p(\lambda)| \le C|\lambda|^{2-\alpha}\langle\log\lambda\rangle^N, \quad 0 \le \alpha \le 3, \quad (3.38)$$

for some N. Here the N may depend on the function.

Proof. Note that finite sums and products of s-moderate functions are again s-moderate. Since $R_4(\lambda)$ in (3.37) may be replaced by $QR_4(\lambda)Q$, we have

$$P_0VE_2(\lambda) = \sum d_{jk}(\lambda)|\varphi_j\rangle\langle V\varphi_k|, \quad d_{jk}(\lambda) = a_{jk} + (\log\lambda)b_{jk} + c_{jk}(\lambda),$$

with s-moderate $c_{jk}(\lambda)$. Let $e_{jk}(\lambda) = \lambda^2 d_{jk}(\lambda)$. Then these functions are s-moderate, and

$$\lambda^2 V(P_0VE_2(\lambda))^2 = \sum f_{jk}(\lambda)|V\varphi_j\rangle\langle V\varphi_k|,$$

with s-moderate functions $f_{jk}(\lambda)$. Since $VR_5^{(\alpha)}(\lambda)$ is a linear combination of terms of the form

$$f^{(\alpha_1)}_{j_1k_1}(\lambda)e^{(\alpha_2)}_{j_2k_2}(\lambda)\dots e^{(\alpha_a)}_{j_ak_a}(\lambda)|V\varphi_{j_1}\rangle\langle V\varphi_{k_1}|(1-\lambda^2P_0VE_2(\lambda))^{-1}|\varphi_{j_2}\rangle\langle V\varphi_{k_2}| \\ \dots|\varphi_{j_a}\rangle\langle V\varphi_{k_a}|(1-\lambda^2P_0VE_2(\lambda))^{-1}|\varphi_\ell\rangle\langle V\varphi_\ell|$$

with $\alpha = \alpha_1 + \dots + \alpha_a$, and since sums and products of s-moderate functions are again s-moderate, the Lemma follows. □

Combining Lemma 3.15 and Lemma 3.16 with (3.31), we obtain the following lemma.

Lemma 3.18. *There exists λ_0, $0<\lambda_0<1$, such that for $\lambda\in(0,\lambda_0)$, $C(\lambda)$ is invertible in $\mathcal{N}$, and $C(\lambda)^{-1}$ may be written in the form*

$$\frac{P_0V}{\lambda^2} + \tilde{L}_{-1} + (\log\lambda)\tilde{L}_1 + \sum_{j,k=1}^{d} c_{jk}(\lambda)|\varphi_j\rangle\langle V\varphi_k|, \tag{3.39}$$

where the $c_{jk}(\lambda)$ are s-moderate functions, and $\tilde{L}_{-1}$ and $\tilde{L}_1$ are given by

$$\tilde{L}_{-1} = P_0V\big(F(0)VKF(0) - \tfrac{1}{2}\tilde{F}''(0)\big)VP_0V,$$
$$\tilde{L}_1 = \tfrac{1}{2}P_0VA''(0)VP_0V.$$

3.4. *Other entries in $M^{-1}(\lambda)$*

We now fix the constant $\lambda_0>0$ as in Lemma 3.18. We write $C_r(\lambda)$ for $C(\lambda)^{-1} - \lambda^{-2}P_0V$. We have

$$C_r(\lambda) = \sum_{j,k=1}^{d}\big(a_{jk} + (\log\lambda)b_{jk} + c_{jk}(\lambda)\big)|\varphi_j\rangle\langle V\varphi_k|, \tag{3.40}$$

where the a_{jk}, b_{jk} are constants, and the $c_{jk}(\lambda)$ are s-moderate functions.

Lemma 3.19. *Both $VN(\lambda)L_{01}(\lambda)C_r(\lambda)$ and $VC_r(\lambda)L_{10}(\lambda)N(\lambda)$ are moderate operator-valued functions.*

Proof. We prove the lemma for $VN(\lambda)L_{01}(\lambda)C_r(\lambda)$. The proof for the other operator is similar. We substitute (3.40) for $C_r(\lambda)$. Then $VN(\lambda)L_{01}(\lambda)C_r(\lambda)$ is a linear combination of operator-valued functions of the form

$$A = p_{jk}(\lambda)VN(\lambda)F(\lambda)|V\varphi_j\rangle\langle V\varphi_k|,$$
$$B = \tilde{p}_{jk}(\lambda)VN(\lambda)A(\lambda)|V\varphi_j\rangle\langle V\varphi_k|,$$

where $p_{jk}(\lambda)$ and $\tilde{p}_{jk}(\lambda)$ are s-moderate functions and we have omitted harmless $\overline{Q}$ in front of $A(\lambda)$ and $F(\lambda)$. We write $A = C_1 + C_2$ by inserting $N(\lambda) = \overline{Q} + Z(\lambda)$ on the right in A. For $0 \le \alpha + \beta + \gamma \le 3$,

$$\begin{aligned}\langle x\rangle^{2+\beta+\varepsilon} V\overline{Q}F^{(\alpha)}(\lambda)|V\varphi_j\rangle\langle V\varphi_k|\langle x\rangle^{2+\gamma+\varepsilon} \\ = \langle x\rangle^{2+\beta+\varepsilon} V\overline{Q}\langle x\rangle^{2+\alpha+\varepsilon} \cdot \langle x\rangle^{-(2+\alpha+\varepsilon)} F^{(\alpha)}(\lambda)|V\varphi_j\rangle\langle V\varphi_k|\langle x\rangle^{2+\gamma+\varepsilon}\end{aligned}$$

is bounded, since $\delta > 7$ implies $\langle x\rangle^{2+\beta+\varepsilon} V\overline{Q}\langle x\rangle^{2+\alpha+\varepsilon} \in \mathbf{B}(\mathcal{H})$. Hence C_1 is moderate. To prove that C_2 is also moderate, we prove that

$$VZ(\lambda)F(\lambda)|V\varphi_j\rangle\langle V\varphi_k|$$

is moderate. If $\alpha = j_1 + j_2$ and $\alpha + \beta + \gamma \le 3$, then $j_1 + \frac{1}{2} < \min\{\delta - (2 + \beta + \varepsilon), \delta - (2 + j_2 + \varepsilon)\}$, and Lemma 3.9 implies that

$$\|\langle x\rangle^{2+\beta+\varepsilon} VZ^{(j_1)}(\lambda)\langle x\rangle^{2+j_2+\varepsilon}\| \le C|\lambda|^{\min\{2-j_1,0\}}.$$

Since $\|\langle x\rangle^{-(2+j_2+\varepsilon)} F^{(j_2)}(\lambda)|V\varphi_j\rangle\langle V\varphi_k|\langle x\rangle^{2+\gamma+\varepsilon}\|_{\mathbf{B}(\mathcal{H})} \le C$, the estimate above implies that $VZ(\lambda)F(\lambda)|V\varphi_j\rangle\langle V\varphi_k|$ is moderate. Thus A has been shown to be moderate. $\square$

We then consider

$$\begin{aligned}\lambda^{-2}N(\lambda)L_{01}(\lambda)P_0V &= N(\lambda)\big(\tilde{F}(\lambda) - \log\lambda A(\lambda)\big)VP_0V, \\ \lambda^{-2}P_0VL_{10}(\lambda)N(\lambda) &= P_0V\big(\tilde{F}(\lambda) - \log\lambda A(\lambda)\big)VN(\lambda)\end{aligned}$$

To prove the next lemma we remark that (3.9), (3.11) and (3.27) imply that

$$J(\lambda) = \big(\tilde{F}(\lambda) - \log\lambda A(\lambda) - F(0)\big)VP_0$$

is the boundary value from the upper half plane of

$$\begin{aligned}&\int_0^\infty \Big(\frac{\mu A(\mu)VP_0}{\mu^2 - \lambda^2} - \frac{A(\mu)VP_0}{\mu}\Big)d\mu \\ &\qquad = \frac{\lambda^2}{2}\Big(\int_0^\infty \frac{A(\mu)VP_0}{(\mu-\lambda)\mu^2}d\mu + \int_0^\infty \frac{A(\mu)VP_0}{(\mu+\lambda)\mu^2}d\mu\Big).\end{aligned} \tag{3.41}$$

We define $\rho_1(\lambda)$ and $\rho_2(\lambda)$ by

$$\begin{aligned}\rho(\lambda) - \rho(0) &= \lambda\int_0^1 \rho'(\lambda\theta)d\theta = \lambda\rho_1(\lambda), \\ \rho(\lambda) - \rho(0) - \lambda\rho'(0) &= \lambda^2\int_0^1 (1-\theta)\rho''(\lambda\theta)d\theta = \lambda^2\rho_2(\lambda)\end{aligned}$$

and, using (3.23), for $\varphi \in \mathcal{N}$ write

$$A(\mu)V\varphi = \mu^2\big(\rho(\mu)^*\rho_2(\mu) + \rho_1(\mu)^*\rho'(0)\big)V\varphi.$$

Now $\rho_2(\mu)V\varphi$ is a $L^2(\Sigma)$-valued function of class $C^{\delta-3-\varepsilon}$, and $\langle x\rangle^{-3-\sigma-\varepsilon}\rho_1(\mu)^*$ is a $\mathbf{B}(L^2(\Sigma), \mathcal{H})$-valued function of $\lambda \in [0, \infty)$ of class C^σ. It follows from the proof of Proposition 3.7 (see [**5**]) that $J(\lambda)$ may be written in the form

$$J(\lambda) = \lambda^2\big(J_1(\lambda) - \log\lambda J_2(\lambda)\big)$$

with $J_1(\lambda)$ and $J_2(\lambda)$ such that $\langle x\rangle^{-(3+\sigma+\varepsilon)}J_1(\lambda)$ and $\langle x\rangle^{-(3+\sigma+\varepsilon)}J_2(\lambda)$ are $\mathbf{B}(\mathcal{N}, \mathcal{H})$-valued functions of class C^σ on $[0, \lambda_0)$ as long as $\sigma < \delta - 3$. Thus, as was in the proof of Lemma 3.19, we see that, because $\delta > 8$,

$$VN(\lambda)\big(\tilde F(\lambda) - \log\lambda A(\lambda) - F(0)\big)VP_0V$$

and

$$VP_0V\big(\tilde F(\lambda) - \log\lambda A(\lambda) - F(0)\big)VN(\lambda)$$

are moderate.

Lemma 3.20. *We have the following equations:*

$$N(\lambda)L_{01}(\lambda)C(\lambda)^{-1} = KF(0)VP_0V + E_8(\lambda)VP_0V, \tag{3.42}$$

$$C(\lambda)^{-1}L_{10}(\lambda)N(\lambda) = P_0VF(0)VK + P_0VE_9(\lambda), \tag{3.43}$$

where $VE_8(\lambda)VP_0V$ and $VP_0VE_9(\lambda)$ are moderate.

Proof. By Lemma 3.19 and the argument above, it suffices to show that $V\big(N(\lambda)-K\big)F(0)VP_0V$ and $VP_0VF(0)V\big(N(\lambda)-K\big)$ are moderate. But this follows immediately from Lemma 3.10. □

Lemma 3.21. *The operator-valued function*

$$VN(\lambda)L_{01}(\lambda)C(\lambda)^{-1}L_{10}(\lambda)N(\lambda)$$

is moderate.

Proof. Using equation (3.43), we may write the operator in the form

$$\lambda^2VN(\lambda)\big(\tilde F(\lambda) - \log\lambda A(\lambda)\big)\big(VP_0VF(0)VK + VP_0VE_9(\lambda)\big).$$

An argument similar to the one above shows that it is moderate. □

3.5. *Completion of the proof of Lemma 3.2*

We combine Lemma 3.10, Lemma 3.18, Lemma 3.20, and Lemma 3.21 by means of Lemma 3.3. The result is that

$$\begin{aligned}(1+G_0(\lambda)V)^{-1} = \frac{P_0V}{\lambda^2} + \tilde L_{-1} + (\log\lambda)\tilde L_1 + K\\ - KF(0)VP_0V - P_0VF(0)VK + R_r(\lambda),\end{aligned} \tag{3.44}$$

where $VR_r(\lambda)$ is moderate. We define $\mathcal{R}(\lambda) = VR_r(\lambda)$ and

$$\begin{aligned}L_1 &= V\tilde L_1(\lambda) = \tfrac12 VP_0VA''(0)VP_0V,\\ L_0 &= VK,\\ L_{-1} &= V\big(\tilde L_{-1} - KF(0)VP_0V - P_0VF(0)VK\big).\end{aligned}$$

Then statements (1), (3), and (4) in Lemma 3.2 follow, if we define a_{jk} and b_{jk} by

$$\begin{aligned}a_{jk} &= \tfrac12\langle V\psi_j, A''(0)V\psi_k\rangle,\\ b_{jk} &= \langle V\psi_j, \big(F(0)VKF(0) - \tfrac12F''(0)\big)V\psi_k\rangle,\end{aligned}$$

and the functions ξ_j by

$$\xi_j(x) = (VKF(0)V\psi_j)(x), \quad j = 1,\dots,d.$$

We remark that $\langle x\rangle^{\delta-2-\varepsilon}\xi_j \in H^1(\mathbf{R}^4)$ because $1+D_0V$ is invertible also in $H^1_{-\sigma}(\mathbf{R}^4) = \langle x\rangle^{\sigma} H^1(\mathbf{R}^4)$ for any $0<\sigma<\delta$

We now prove statement (2). Recall that $K = \overline{Q}K_0\overline{Q}$, considered as an operator in $\mathcal{H}_{-s}$. Define $T = \overline{Q}D_0V\overline{Q}$, and consider it as an operator on $\mathcal{H}_{-s}$. Since $\overline{Q}\mathcal{H}_{-s}$ is D_0V-invariant, $Q\overline{Q}=0$, and $D_0VQ=-Q$, we have

$$(D_0V)^n = T^n + (-1)^n Q, \quad n = 1,2,\dots, \tag{3.45}$$

such that $K_0\dot{+}Q = (1+T)^{-1} = 1 - T + T^2 - T^3 + T^2(1+T)^{-1}T^2$. Thus $K = (1+T)^{-1} - Q$, and then

$$\begin{aligned} L_0 - V = V(K-1) &= -V\big[Q + T - T^2 + T^3 - T^2(1+T)^{-1}T^2\big] \\ &= -4VQ + \sum_{j=1}^{3} V(-1)^j (D_0V)^j \\ &\quad + V\big((D_0V)^2 - Q\big)(1+T)^{-1}\big((D_0V)^2 - Q\big). \end{aligned} \tag{3.46}$$

The integral kernel $k_0(x,y)$ of Q is given by

$$k_0(x,y) = -\sum_{j=1}^{d} V(y)\varphi_j(x)\varphi_j(y),$$

and the integral kernel of $-4VQ$, $V(x)k_0(x,y)$, satisfies the condition (2.3) for some $\sigma > 2/3$, since $\varphi \in \mathcal{N}$ satisfies (3.2).

The operator D_0V has the integral kernel

$$k_1(x,y) = \frac{1}{(2\pi)^2}\frac{V(y)}{|x-y|^2}$$

and, as $\|\langle x\rangle^{\sigma} V(x)V(x-y)\|_{H^1} \le \langle y\rangle^{-\delta+\sigma}$, the integral kernel of VD_0V, $V(x)k_1(x,y)$, also satisfies the property (2.3) for some $\sigma > 2/3$.

The integral kernels of $(D_0V)^2$ and $(D_0V)^3$ are given, respectively, by $k_2(x,y)V(y)$ and $k_3(x,y)V(y)$, where

$$\begin{aligned} k_2(x,y) &= \frac{1}{(2\pi)^4}\int_{\mathbf{R}^4} \frac{V(z)dz}{|x-z|^2|z-y|^2}, \\ k_3(x,y) &= \frac{1}{(2\pi)^6}\iint_{\mathbf{R}^8} \frac{V(z_1)V(z_2)dz_1dz_2}{|x-z_1|^2|z_1-z_2|^2|z_2-y|^2}. \end{aligned}$$

The kernel of $T^2(1+T)^{-1}T^2$ is given by $k_4(x,y) = \sum_{j=0}^{3} k_{4,j}(x,y)$, where

$$k_{4,0}(x,y) = \langle k_2(x,\cdot)V(\cdot), (1+T)^{-1}k_2(\cdot,y)V(y)\rangle, \tag{3.47}$$

$k_{4,1}(x,y)$ and $k_{4,2}(x,y)$ are obtained from (3.47) by replacing $k_2(x,\cdot)V(\cdot)$ by $-k_0(x,\cdot)$ and $k_2(\cdot,y)V(y)$ by $-k_0(\cdot,y)$ respectively, and $k_{4,3}(x,y)$ is obtained by performing both replacements.

Lemma 3.22. *The functions $k_2(x,y)$ and $k_3(x,y)$ satisfy*

$$|k_2(x,y)| \le C\frac{\langle[\log\gamma_{x,y}]_-\rangle}{\langle x\rangle^2\langle y\rangle^2}, \quad \gamma_{x,y} = \frac{|x-y|}{\sqrt{\langle x\rangle\langle y\rangle}}, \tag{3.48}$$

$$|k_3(x,y)| \le C\langle x\rangle^{-2}\langle y\rangle^{-2}, \tag{3.49}$$

where $[a]_- = \max\{-a, 0\}$, *and their derivatives satisfy*

$$|\nabla_x k_2(x,y)| \le \frac{C\langle x-y\rangle}{\langle x\rangle^3\langle y\rangle^2|x-y|}, \qquad |\nabla_y k_2(x,y)| \le \frac{C\langle x-y\rangle}{\langle x\rangle^2\langle y\rangle^3|x-y|}, \tag{3.50}$$

$$|\nabla_x k_3(x,y)| \le \frac{C}{\langle x\rangle^3\langle y\rangle^2}, \qquad |\nabla_y k_3(x,y)| \le \frac{C}{\langle x\rangle^2\langle y\rangle^3}. \tag{3.51}$$

Proof. Since $k_j(x,y) = k_j(y,x)$, $j = 2,3$, we may assume that $|y| \le |x|$. We first prove (3.48). Let $|x| \ge 10$. We have

$$|k_2(x,y)| \le \Big(\int_{|x-z|\ge\frac{|x|}{4}} + \int_{|x-z|<\frac{|x|}{4}}\Big)\frac{|V(z)|dz}{|x-z|^2|z-y|^2} \equiv k_{21}(x,y) + k_{22}(x,y).$$

It is obvious that have

$$k_{21}(x,y) \le \frac{C}{\langle x\rangle^2}\int_{\mathbf{R}^4}\frac{|V(z)|dz}{|z-y|^2} \le \frac{C}{\langle x\rangle^2\langle y\rangle^2}$$

If $|z-x| \le \frac{|x|}{4}$, then $|z| \ge \frac{3|x|}{4}$, and $k_{22}(x,y)$ is bounded by

$$\begin{aligned} k_{22}(x,y) &\le \int_{|z-x|\le\frac{|x|}{4}}\frac{C\langle x\rangle^{-\delta}dz}{|z-x|^2|z-y|^2} \le \int_{|w|\le\frac{|x|}{4}}\frac{C\langle x\rangle^{-\delta}dw}{|w|^2|w-(x-y)|^2} \\ &\le \frac{C}{\langle x\rangle^\delta}\int_{|\zeta|\le\frac14}\frac{d\zeta}{|\zeta|^2|\zeta-a|^2}, \quad a = \frac{x-y}{|x|}. \end{aligned} \tag{3.52}$$

If $|a| > \frac12$, viz. if $|x-y| \ge \frac12|x|$, it follows that

$$k_{22}(x,y) \le C\langle x\rangle^{-\delta} \le C\langle x\rangle^{-2}\langle y\rangle^{-2}.$$

If $|a| < \frac12$, then $\frac12 < \frac{|y|}{|x|} \le 1$, hence $|y| \ge 5$, and

$$\int_{|\zeta|\le\frac14}\frac{d\zeta}{|\zeta|^2|\zeta-a|^2} = \int_{|\zeta|\le\frac{1}{4|a|}}\frac{d\zeta}{|\zeta|^2|\zeta-\hat a|^2} \le C\langle[\log|a|]_-\rangle, \quad \hat a = \frac{a}{|a|}.$$

Since $\frac{1}{10} < |a|/\gamma_{x,y} < 10$, we obtain (3.48) for $|x| \ge 10$. Next, let $|y| \le |x| \le 10$. Then

$$|k_2(x,y)| \le C + \int_{|z|\le 20}\frac{Cdz}{|z-x|^2|z-y|^2} \le C + \int_{|w|\le 30}\frac{Cdw}{|w|^2|w-(x-y)|^2}.$$

If we write $b = x-y$ and $\hat b = b/|b|$, then the integral may be estimated by

$$\int_{|w|\le 30/|b|}\frac{dw}{|w|^2|w-\hat b|^2} \le C\langle[\log|b|]_-\rangle,$$

and (3.48) is satisfied also when $|x| \le 10$. By differentiation,

$$\nabla_x k_2(x,y) = C\int_{\mathbf{R}^4}\frac{(x-z)V(z)dz}{|x-z|^4|z-y|^2},$$

$$\nabla_y k_2(x,y) = C\int_{\mathbf{R}^4}\frac{(y-z)V(z)V(y)dz}{|x-z|^2|z-y|^4},$$

and an estimate somewhat simpler than the one for $k_2(x,y)$ yields (3.50). We omit the details.

We break up the integral for $k_3(x,y)$ as follows

$$k_3(x,y)=\left(\int_{\frac{1}{2}<\frac{|x|}{|z|}\le 2}+\int_{\frac{|x|}{|z|}\notin[\frac{1}{2},2]}\right)\frac{k_2(x,z)V(z)}{|z-y|^2}dz=k_{31}(x,y)+k_{32}(x,y).$$

If $\frac{|x|}{|z|}\notin[\frac{1}{2},2]$, then $|\gamma_{x,z}|\ge C>0$. Thus (3.48) implies

$$|k_{32}(x,y)|\le C\int_{\mathbf{R}^4}\frac{|V(z)|}{\langle x\rangle^2\langle z\rangle^2|z-y|^2}dz\le\frac{C}{\langle x\rangle^2\langle y\rangle^2}. \tag{3.53}$$

If $2^{-1}\le\frac{|x|}{|z|}\le 2$, and if $|x|<5$, then $|z|<10$ and $|\gamma_{x,z}|\le C\langle\log|z-x|\rangle$. Hence, for $|x|\le 5$,

$$|k_{31}(x,y)|\le\int_{|z|<10}\frac{C\langle\log|z-x|\rangle}{|z-y|^2}dz\le\frac{C}{\langle y\rangle^2}. \tag{3.54}$$

If $|x|>5$, then on the domain of integration of $k_{31}(x,y)$,

$$|\log\gamma_{x,z}|\le C\big(1+[\log|z-x|]_-+\log|z|+\log|x|\big).$$

Then for any $\varepsilon>0$,

$$\begin{aligned}|k_{31}(x,y)|\le\frac{C_\varepsilon}{\langle x\rangle^{4-\varepsilon}}&\int_{\mathbf{R}^4}\frac{|V(z)|dz}{|z-y|^2}\\&+\frac{C_\varepsilon}{\langle x\rangle^4}\int_{|x-z|<1}\frac{|V(z)||\log|z-x||dz}{|z-y|^2}\le\frac{C}{\langle x\rangle^2\langle y\rangle^2}.\end{aligned} \tag{3.55}$$

Combining (3.53), (3.54), and (3.55), we obtain (3.49). The proof of (3.51) for the derivatives is similar, and we omit it. ☐

We continue the the proof of Lemma 3.2. Lemma 3.22 clearly implies that $V(x)k_2(x,y)V(y)$ and $V(x)k_3(x,y)V(y)$ satisfy the property (2.3) for some $\sigma>2/3$. Applying the estimates (3.48) and (3.50), and the fact that $(1+T)^{-1}\in\mathbf{B}(L^2_{-s})$ for $s>1/2$, it is easy to see that $V(x)k_4(x,y)V(y)$ also satisfies the property (2.3) for some $\sigma>2/3$. This completes the proof of Lemma 3.2.

4. *Low energy estimate*

In this section we show that the operator $\tilde{W}$ defined by

$$\int_0^\infty\Phi(H)G_0(\lambda)V(1+G_0(\lambda)V)^{-1}\big(G_0(\lambda)-G_0(-\lambda)\big)\Phi(H_0)\tilde{\Phi}(\lambda)\lambda d\lambda \tag{4.1}$$

is bounded in $L^p(\mathbf{R}^4)$, if $3/4<p<4$. We take $\lambda_0>0$ as in Lemma 3.2 and substitute (3.5) for $V(1+G_0(\lambda)V)^{-1}$ in (4.1). This yields

$$\tilde{W}=W_s-W_1+W_0+W_{-1}+W_r,$$

where W_s,W_1,W_0,W_{-1}, and W_r, respectively, are the terms obtained from (4.1), when replacing $V(1+G_0(\lambda)V)^{-1}$ by PV_0/λ^2, $(\log\ \lambda)L_1$, L_0, L_{-1}, and $\mathcal{R}(\lambda)$, respectively.

Lemma 4.1. *The operators W_0,W_{-1}, and W_r are bounded in $L^p(\mathbf{R}^4)$ for all $1\le p\le\infty$.*

Proof. The integral kernels of $L_0 - V$ and L_1 satisfy the estimate (2.3) for some $\sigma > 2/3$ by Lemma 3.2(2) and (3). Hence Lemma 2.1 and Corollary 2.2 imply that W_0 and W_{-1} are bounded in $L^p(\mathbf{R}^4)$ for all $1 \le p \le \infty$. The same holds for W_r, because $\mathcal{R}(\lambda)$ is moderate by Lemma 3.2(4), see Lemma 2.5. $\square$

Next we study the operator

$$W_s = \Phi(H)\Big(\int_0^\infty G_0(\lambda) V P_0 V \big(G_0(\lambda) - G_0(-\lambda)\big)\tilde{\Phi}(\lambda)\lambda^{-1} d\lambda\Big)\Phi(H_0).$$

By computing both sides using the Parseval identity, we have

$$\langle u, \big(G_0(\lambda) - G_0(-\lambda)\big)h(\lambda)v\rangle = \langle u, \big(G_0(\lambda) - G_0(-\lambda)\big)h(|D|)v\rangle \tag{4.2}$$

for $u, v \in \mathcal{S}(\mathbf{R}^4)$ and $\lambda > 0$, where $h(|D|)$ is the Fourier multiplier given by

$$h(|D|)u(x) = \frac{1}{(2\pi)^2}\int_{\mathbf{R}^4} e^{ix\xi} h(|\xi|)\hat{u}(\xi) d\xi.$$

In what follows we ignore $\Phi(H)$ and $\Phi(H_0)$ and denote $\tilde{\Phi}(\lambda)$ by $\Phi(\lambda)$:

$$\begin{aligned} W_s &= \sum_{j=1}^d \int_0^\infty G_0(\lambda)(V\psi_j \otimes V\psi_j)\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(\lambda)\lambda^{-1} d\lambda \\ &= \sum_{j=1}^d \int_0^\infty G_0(\lambda)(V\psi_j \otimes |D|^{-1} V\psi_j)\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(\lambda) d\lambda \\ &= \sum_{j=1}^d W_{sj}. \end{aligned}$$

We recall that $G_0(\lambda)$ is the convolution with

$$G_0(\lambda, x) = \frac{C e^{i\lambda|x|}}{|x|^2}\int_0^\infty e^{-t} t^{\frac{1}{2}} (t - 2i\lambda|x|)^{\frac{1}{2}} dt, \tag{4.3}$$

where the branch of the square root is such that $z^{\frac{1}{2}} > 0$, when $z > 0$. For a function u on $\mathbf{R}^4$, we define

$$M(r, u) = \frac{1}{|\Sigma|}\int_\Sigma u(r\omega) d\omega, \quad r \in \mathbf{R}.$$

It is an even function of $r \in \mathbf{R}$. The following lemma is Lemma 5.3 from [**5**].

Lemma 4.2. *Let $f \in L^1(\mathbf{R}^4)$ be real-valued, and let $u \in \mathcal{S}(\mathbf{R}^4)$. Then*

$$\begin{aligned} &\langle f, (G_0(\lambda) - G_0(-\lambda))u\rangle \\ &\qquad = C\int_0^\infty e^{-t} t^{\frac{1}{2}} \Big(\int_{\mathbf{R}} e^{-i\lambda r}(t + 2i\lambda r)^{\frac{1}{2}} r M(r, f * \check{u}) dr\Big) dt, \end{aligned} \tag{4.4}$$

where $\check{u}(x) = u(-x)$ and $\operatorname{Re}(t + 2i\lambda r)^{\frac{1}{2}} > 0$ for $t > 0$ and $\lambda \in \mathbf{R}$.

Thus, if we define $M_j(r) = M(r, |D|^{-1} V\psi_j * \check{u})$ and

$$\begin{aligned} (K_j u)(\rho) = \int_0^\infty & e^{i\lambda\rho}\Phi(\lambda)\Big\{\int_0^\infty\int_0^\infty e^{-(t+s)} t^{\frac{1}{2}} s^{\frac{1}{2}} \\ &\times (s - 2i\lambda\rho)^{\frac{1}{2}}\Big(\int_{\mathbf{R}} e^{-i\lambda r}(t + 2i\lambda r)^{\frac{1}{2}} r M_j(r) dr\Big) dt ds\Big\} d\lambda, \end{aligned} \tag{4.5}$$

we have as in lemma 5.4 of [**5**] that

$$W_{sj}u(x) = C\int_{\mathbf{R}^4} \frac{V(y)\psi_j(y)(K_j u)(|x-y|)}{|x-y|^2}dy \tag{4.6}$$

We remark that the integrand of (4.5) is integrable with respect to the variables t, s, r, λ, and that the order of the integrations may be freely changed.

In what follows we omit the index j, and we prove $W_s \in \mathbf{B}(L^p(\mathbf{R}^4))$, whenever $4/3 < p < 4$. We use the following two lemmas. The first one may be found in [**12**].

Lemma 4.3. (1) *The function* $|r|^a$ *on* $\mathbf{R}$ *is an* $(A)_p$ *weight, if and only if* $-1 < a < p-1$. *The Hilbert transform* $\tilde{\mathcal{H}}$, *and the Hardy-Littlewood maximal operator* $\mathcal{M}$, *are bounded in* $L^p(\mathbf{R}, w(r)dr)$ *for* $(A)_p$ *weights* $w(r)$.
(2) *Let a function* $h(x)$ *on* $\mathbf{R}^m$ *have a spherically symmetric decreasing integrable majorant, then*

$$|(h*g)(x)| \le C\mathcal{M}g(x), \quad x \in \mathbf{R}^m$$

for a constant depending only on h.

Lemma 4.4. (1) *The function* $(|D|^{-1}V\psi)(x)$ *is of* C^2 *class, and for large* x *we have*

$$\Big|(|D|^{-1}V\psi)(x) - \sum_{j=1}^{4} \frac{c_j x_j}{|x|^5}\Big| \le C\langle x\rangle^{-5}.$$

(2) *The convolution operator with* $(|D|^{-1}V\psi)(x)$ *is bounded in* $L^p(\mathbf{R}^4)$ *for any* $1 < p < \infty$.

Proof. We have

$$|D|^{-1}V\psi(x) = C\int_{\mathbf{R}^4} \frac{V(y)\psi(y)}{|x-y|^3}dy.$$

Since

$$|V(x)\psi(x)| + \langle x\rangle|\nabla(V\psi)(x)| \le C\langle x\rangle^{-\delta-3}$$

by (3.2), and since $\int V(y)\psi(y) = 0$ by Assumption 3.1, (1) follows immediately. Statement (2) is a consequence of (1) and the Calderón-Zygmund theory. $\square$

In definition (4.5) we substitute

$$\begin{aligned}(s-2i\lambda\rho)^{\frac12}(t+2i\lambda r)^{\frac12} \\ = s^{\frac12}t^{\frac12} + ((s-2i\lambda\rho)^{\frac12} - s^{\frac12})t^{\frac12} + (s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12} - t^{\frac12})\end{aligned}$$

and write $S_j(\lambda)$ for the function produced by the j-th summand, such that $K(\rho) = S_1(\rho) + S_2(\rho) + S_3(\rho)$. We then let $W_s = Z_1 + Z_2 + Z_3$ be the corresponding decomposition of W_s.

Lemma 4.5. *Let* $4/3 < p < 4$. *Then* Z_1 *is bounded in* $L^p(\mathbf{R}^4)$.

Proof. Integrating out with respect to t and s, we obtain

$$S_1(\rho) = \int_0^\infty e^{i\lambda\rho}\Phi(\lambda)\Big\{\int_0^\infty\int_0^\infty e^{-(t+s)}ts\Big(\int_{\mathbf{R}} e^{-i\lambda r}rM(r)dr\Big)dtds\Big\}d\lambda$$

$$= \int_0^\infty e^{i\lambda\rho}\Phi(\lambda)\Big(\int_{\mathbf{R}} e^{-i\lambda r} rM(r)dr\Big)d\lambda = \mathcal{H}(\check{\Phi} * (rM(r)))(\rho) \quad (4.7)$$

where $\mathcal{H} = c(1+\tilde{\mathcal{H}})/2$ with $\tilde{\mathcal{H}}$ the Hilbert transform and c a constant.

First consider $4/3 < p < 2$. Then $|r|^{3-2p}$ is an $(A)_p$ weight, and by Lemma 4.3 we have, with $f(x) = (|D|^{-1}V\psi)(x)$, that

$$\|Z_1 u\|_p \le C\|V\psi\|_1\Big(\int_0^\infty \rho^{3-2p}|S_1(\rho)|^p d\rho\Big)^{\frac{1}{p}} \le C\|V\psi\|_1\Big(\int_{\mathbf{R}} r^{3-p}|M(r)|^p dr\Big)^{\frac{1}{p}} \le C\|V\psi\|_1\Big(\int_{\mathbf{R}^4} \frac{|(f*u)(x)|^p}{|x|^p}dx\Big)^{\frac{1}{p}}.$$

The last integral may be split into the parts $|x| \le 1$ and $|x| \ge 1$ and estimated by

$$C\|V\psi\|_1\big(\||x|^{-p}\|^{\frac{1}{p}}_{L^1(|x|<1)}\|f*u\|_\infty + \|f*u\|_p\big) \le C\|u\|_p,$$

where in the last step we estimated $\|f*u\|_p \le C\|u\|_p$ by using Lemma 4.4(2) and $\|f*u\|_\infty \le \|f\|_{p'}\|u\|_p$, by Hölder's inequality, where $p' = p/p-1$.

Next consider $2 < p < 4$. We take $\frac{4}{3} < q < 2$ such that $\frac{1}{q} - \frac{1}{p} < \frac{1}{4}$ and then r such that $\frac{3}{4} < \frac{1}{r} = 1 - \Big(\frac{1}{q} - \frac{1}{p}\Big) < 1$. Using Young's inequality and the weighted inequality we get

$$\|Z_1 u\|_p \le C\|V\psi\|_r\Big(\int_0^\infty \rho^q|S_1(\rho)|^{3-2q}d\rho\Big)^{\frac{1}{q}} \le C\|V\psi\|_1\Big(\int_{\mathbf{R}} r^{3-q}|M(r)|^q dr\Big)^{\frac{1}{q}} \le C\|V\psi\|_1\Big(\int_{\mathbf{R}^4} \frac{|(f*u)(x)|^q}{|x|^q}dx\Big)^{\frac{1}{q}}.$$

Let $\theta = p/q$ and $\theta' = p/(p-q)$, such that $q\theta' > 4$ by the choice of q, and estimate the right hand side by

$$C\big(\||x|^{-q}\|^{\frac{1}{q}}_{L^1(|x|<1)}\|f*u\|_\infty + \||x|^{-1}\|_{L^{q\theta'}(|x|>1)}\|f*u\|_p\big) \le C\|u\|_p.$$

Thus $\|Z_1 u\|_p \le C\|u\|_p$ also for $2 < p < 4$, and the lemma follows by interpolation. □

To deal with $Z_2 u$ and $Z_3 u$, we need to estimate $S_2(\rho)$ and $S_3(\rho)$. We write them in the following form:

$$S_j(\rho) = C_j\int_{\mathbf{R}} T_j(\rho,r)rM(r)dr, \quad j = 2,3,$$

where $T_2(\rho,r)$ and $T_3(\rho,r)$ are given by

$$T_2(\rho,r) = \int_0^\infty e^{-s}s^{\frac{1}{2}}\Big(\int_0^\infty e^{i\lambda(\rho-r)}\Phi(\lambda)\big((s-2i\lambda\rho)^{\frac{1}{2}} - s^{\frac{1}{2}}\big)d\lambda\Big)ds \quad (4.8)$$

and

$$T_3(\rho,r) = \int_0^\infty\int_0^\infty e^{-(t+s)}t^{\frac{1}{2}}s^{\frac{1}{2}} \times \times\Big(\int_0^\infty e^{i\lambda(\rho-r)}\Phi(\lambda)(s-2i\lambda\rho)^{\frac{1}{2}}\big((t+2i\lambda r)^{\frac{1}{2}} - t^{\frac{1}{2}}\big)d\lambda\Big)dt\,ds. \quad (4.9)$$

Lemma 4.6. *There exist constants C_2 and C_3 such that*

$$|T_2(r,\rho)| \le \frac{C_2\rho}{\langle \rho - r\rangle^2}, \tag{4.10}$$

$$|T_3(r,\rho)| \le \frac{C_3(\langle \rho\rangle + \langle r\rangle)}{\langle \rho - r\rangle^2}. \tag{4.11}$$

Proof. First we estimate $T_2(\rho,r)$. Since $\mathrm{Re}(s-2i\lambda\rho)^{\frac12} > 0$ for $s>0$, we have

$$|(s-2i\lambda\rho)^{\frac12} - s^{\frac12}| \le \frac{2\lambda\rho}{|(s-2i\lambda\rho)^{\frac12}+s^{\frac12}|} \le \frac{2\lambda\rho}{s^{\frac12}}, \tag{4.12}$$

and then $|T_2(\rho,\rho)| \le C_2\rho$. Therefore (4.10) holds for $|\rho-r|\le 1$. When $|\rho-r|>1$, we apply integration by parts twice to the inner integral in (4.8). The boundary term yields the term

$$\frac{i}{(\rho-r)^2}\int_0^\infty e^{-s}\rho ds = \frac{i\rho}{(\rho-r)^2},$$

and the other terms yield

$$\frac{-1}{(\rho-r)^2}\int_0^\infty e^{i\lambda(\rho-r)}\Phi''(\lambda)\Big(\int_0^\infty e^{-s}s^{\frac12}((s-2i\lambda\rho)^{\frac12}-s^{\frac12})ds\Big)d\lambda$$
$$+\frac{-2}{(\rho-r)^2}\int_0^\infty e^{i\lambda(\rho-r)}\Phi'(\lambda)\Big(\int_0^\infty \frac{-ie^{-s}s^{\frac12}\rho}{(s-2i\lambda\rho)^{\frac12}}ds\Big)d\lambda$$
$$+\frac{-1}{(\rho-r)^2}\int_0^\infty e^{i\lambda(\rho-r)}\Phi(\lambda)\Big(\int_0^\infty \frac{e^{-s}s^{\frac12}\rho^2}{(s-2i\lambda\rho)^{\frac32}}ds\Big)d\lambda.$$

The first two terms are bounded in modulus by $C\rho/|\rho-r|^2$ due to (4.12) and $|(s-2i\lambda\rho)^{\frac12}| \ge s^{\frac12}$. Let

$$\ell(\mu) = \int_0^\infty \frac{e^{-s}s^{\frac12}}{(s+\mu)^{\frac32}}ds. \tag{4.13}$$

Fubini's theorem implies that ℓ is integrable, and then the third term above is bounded by

$$\frac{C\rho^2}{(\rho-r)^2}\int_0^\infty |\Phi(\lambda)\ell(\lambda\rho)|d\lambda \le \frac{C\rho^2}{(\rho-r)^2}\int_0^\infty |\ell(\lambda\rho)|d\lambda \le \frac{C\rho^2}{(\rho-r)^2}.$$

Thus (4.10) is satisfied. We next prove (4.11). For $|\rho-r|<1$ we have

$$|T_3(\rho,r)| \le \int_0^\infty\int_0^\infty e^{-(t+s)}t^{\frac12}s^{\frac12}\times$$
$$\times\Big(\int_0^\infty |\Phi(\lambda)|(s^{\frac12}+\rho^{\frac12})(t^{\frac12}+r^{\frac12})d\lambda\Big)dt\,ds \le C\langle\rho\rangle^{\frac12}\langle r\rangle^{\frac12},$$

and (4.11) is satisfies, when $|\rho-r|<1$. When $|\rho-r|\ge 1$, we apply integration by parts twice to the inner integral in (4.9). Then $T_3(\rho,r)$ is the sum of

$$\frac{ir}{(\rho-r)^2}\int_0^\infty\int_0^\infty e^{-(t+s)}sdt\,ds = \frac{ir}{(\rho-r)^2},$$

which comes from the boundary term, and $-(\rho-r)^{-2}$ times the sum

$$\sum_{j=1} T_{3,j}(\rho,r) \equiv \sum_{j=1}^3 \int_0^\infty\int_0^\infty e^{-(t+s)}t^{\frac12}s^{\frac12}J_j(t,s,r,\rho)dt\,ds, \tag{4.14}$$

where the $J_j(t,s,r,\rho)$ are given by

$$J_1 = \int_0^\infty e^{i\lambda(\rho-r)}\Phi''(\lambda)(s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12}-t^{\frac12})d\lambda,$$
$$J_2 = 2\int_0^\infty e^{i\lambda(\rho-r)}\Phi'(\lambda)\big((s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12}-t^{\frac12})\big)'d\lambda,$$
$$J_3 = \int_0^\infty e^{i\lambda(\rho-r)}\Phi(\lambda)\big((s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12}-t^{\frac12})\big)''d\lambda.$$

It is obvious that $|J_1| \le C(s^{\frac12}+\rho^{\frac12})(t^{\frac12}+r^{\frac12})$ and

$$|T_{3,1}(r,\rho)| \le C\langle r\rangle^{\frac12}\langle\rho\rangle^{\frac12}.$$

The derivative $\big((s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12}-t^{\frac12})\big)'$ is computed and estimated on the support of Φ' as follows:

$$\Big|\frac{-i\rho}{(s-2i\lambda\rho)^{\frac12}}((t+2i\lambda r)^{\frac12}-t^{\frac12}) + (s-2i\lambda\rho)^{\frac12}\frac{ir}{(t+2i\lambda r)^{\frac12}}\Big|$$
$$\le C\Big(\frac{\rho}{(s+\rho)^{\frac12}}(t+r)^{\frac12} + (s+\rho)^{\frac12}\frac{r}{(t+r)^{\frac12}}\Big)$$
$$\le C(\rho^{\frac12}(t+r)^{\frac12}+(s+\rho)^{\frac12}r^{\frac12}).$$

Thus we again obtain

$$|T_{3,2}(r,\rho)| \le \langle r\rangle^{\frac12}\langle\rho\rangle^{\frac12}.$$

Finally, we compute $\big((s-2i\lambda\rho)^{\frac12}((t+2i\lambda r)^{\frac12}-t^{\frac12})\big)''$. The result is

$$\frac{(sr+\rho t)^2}{(s-2i\lambda\rho)^{\frac32}(t+2i\lambda r)^{\frac32}} - \frac{\rho^2 t^{\frac12}}{(s-2i\lambda\rho)^{\frac32}},$$

which is bounded in modulus by a constant times

$$\tilde J = \frac{s^2r^2+\rho^2t^2}{(s+\lambda\rho)^{\frac32}(t+\lambda r)^{\frac32}} + \frac{\rho^2t^{\frac12}}{(s+\lambda\rho)^{\frac32}} \le \frac{s^{\frac12}r^2}{(t+\lambda r)^{\frac32}} + \frac{\rho^2t^{\frac12}}{(s+\lambda\rho)^{\frac32}}.$$

If we integrate with respect to t,s first, we have

$$\int_0^\infty\int_0^\infty e^{-(t+s)}t^{\frac12}s^{\frac12}\tilde J(t,s,\lambda,r,\rho)dt\,ds$$
$$\le r^2\int_0^\infty e^{-t}\frac{t^{\frac12}dt}{(t+\lambda r)^{\frac32}} + \rho^2\int_0^\infty e^{-s}\frac{s^{\frac12}ds}{(s+\lambda\rho)^{\frac32}} = r^2\ell(\lambda r)+\rho^2\ell(\lambda\rho), \tag{4.15}$$

where ℓ is defined by (4.13) and is integrable, as was seen previously. It follows that $|T_{3,3}(r,\rho)| \le C(r+\rho)$, and the proof of the lemma is completed. □

Lemma 4.7. *Let $4/3 < p < 4$. Then Z_2 and Z_3 are bounded in $L^p(\mathbf{R}^4)$.*

Proof. Due to the estimates (4.10) and (4.11) it suffices to show that

$$Zu(x) = \int_{\mathbf{R}^4}\frac{|V(y)\psi(y)|S(|x-y|)}{|x-y|^2}dy$$

satisfies $\|Zu\|_p \le C_p\|u\|_p$, where $M(r) = M(r, f*\check u)$, $f(x) = (|D|^{-1}V\psi)(x)$, and

$$S(\rho) = \int_{\mathbf{R}} A(r,\rho)|rM(r)|dr, \quad A(r,\rho) = \frac{\langle\rho\rangle+\langle r\rangle}{\langle\rho-r\rangle^2}. \tag{4.16}$$

We decompose $Zu(x)$ into two terms:

$$\Big(\int_{|y|\le 1} + \int_{|y|\ge 1}\Big) \frac{|V(x)\psi(x-y)|S(|y|)}{|y|^2} dy \equiv Z^{(1)}u(x) + Z^{(2)}u(x),$$

and estimate $\|Z^{(1)}u\|_p$ first. As $A(r,\rho) \le C\langle r\rangle^{-1}$ for $0 \le \rho < 1$, we have

$$|S(\rho)| \le \int_{\mathbf{R}} \langle r\rangle^{-1} |rM(r)| dr \le C\int_{\mathbf{R}^4} \frac{|f * \check{u}(x)|}{\langle x\rangle |x|^2} dx. \tag{4.17}$$

Let $q = p/p-1$. We estimate the right hand side of (4.17) by

$$C\begin{cases} \|\langle x\rangle^{-1}|x|^{-2}\|_q \|f * \check{u}\|_p \le C\|u\|_p, & \text{when } 2 < p < 4, \\ \|\langle x\rangle^{-1}|x|^{-2}\|_p \|f * \check{u}\|_q \le C\|f\|_r \|u\|_p, & \text{when } 4/3 < p < 2, \end{cases}$$

where $0 < \frac{1}{r} = 1 + \big(\frac{1}{q} - \frac{1}{p}\big) < 1$. It follows that for any $4/3 < p < 4$, $p \ne 2$,

$$\|Z^{(1)}u\|_p \le \|V\psi\|_p \int_{|y|\le 1} \frac{S(|y|)}{|y|^2} dy = C\int_0^1 \rho S(\rho) d\rho \le C\|u\|_p,$$

and by interpolation for $p = 2$ as well.

We next estimate $\|Z^{(2)}u\|_p$, first for $4/3 < p < 2$. Take $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{3p-4}{2},$$

which implies that $|r|^{3-2p}$ and $|r|^{3-(2-\varepsilon)p}$ both are one dimensional $(A)_p$ weights. Since $\langle r-\rho\rangle \le \langle r\rangle + \langle \rho\rangle$, we have

$$A(r,\rho) = \frac{\langle\rho\rangle + \langle r\rangle}{\langle \rho - r\rangle^2} \le \frac{2\langle r\rangle}{\langle r-\rho\rangle^2} + \frac{1}{\langle \rho - r\rangle} \equiv B(r,\rho).$$

It follows that

$$A(r,\rho) \le A^{\varepsilon}(r,\rho) B^{1-\varepsilon}(r,\rho) \le C\Big(\frac{\langle r\rangle + \langle r\rangle^{1-\varepsilon}\langle\rho\rangle^{\varepsilon}}{\langle r-\rho\rangle^2} + \frac{\langle r\rangle^{\varepsilon} + \langle \rho\rangle^{\varepsilon}}{\langle r-\rho\rangle^{1+\varepsilon}}\Big),$$

and if we use this estimate, we have

$$|S(\rho)| \le C\mathcal{M}(\langle r\rangle rM)(\rho) + C\langle\rho\rangle^{\varepsilon}\mathcal{M}(\langle r\rangle^{1-\varepsilon} rM)(\rho).$$

Then the weighted inequality for the maximal functions implies

$$\begin{aligned} \|Z^{(2)}u\|_p &\le \|V\psi\|_1 \Big(\int_1^\infty \rho^{3-2p}|S(\rho)|^p d\rho\Big)^{\frac{1}{p}} \\ &\le C\Big(\int_0^\infty \rho^{3-p}\langle\rho\rangle^p |M(\rho)|^p d\rho\Big)^{\frac{1}{p}} + \Big(\int_0^\infty \rho^{3-(1+\varepsilon)p}\langle\rho\rangle^{(1-\varepsilon)p}|M(\rho)|^p d\rho\Big)^{\frac{1}{p}} \\ &\le C\Big(\int_{|x|<1} \frac{|(f*\check{u})(x)|^p}{|x|^p}\Big)^{\frac{1}{p}} + C\|f*\check{u}\|_p \le C(\|f\|_q\|u\|_p + \|u\|_p), \end{aligned} \tag{4.18}$$

where q is the dual exponent of p.

We next let $2 < p < 4$ and take $\varepsilon > 0$, such that $\varepsilon < (4-p)/4$, which implies that $|r|^{3-p}$ and $|r|^{3-(1+\varepsilon)p}$ both are $(A)_p$ weights. We then estimate

$$A(r,\rho) \le \frac{2\langle\rho\rangle}{\langle r-\rho\rangle^2} + \frac{1}{\langle r-\rho\rangle} = \tilde{B}(r,\rho),$$

and therefore

$$A(r,\rho) \le A^{\varepsilon}(r,\rho)\tilde{B}^{1-\varepsilon}(r,\rho) \le C\Big(\frac{\langle\rho\rangle + \langle r\rangle^{\varepsilon}\langle\rho\rangle^{1-\varepsilon}}{\langle r-\rho\rangle^2} + \frac{\langle r\rangle^{\varepsilon} + \langle\rho\rangle^{\varepsilon}}{\langle r-\rho\rangle^{1+\varepsilon}}\Big).$$

If $A(r,\rho)$ in (4.16) is estimated by this right hand side, we have

$$|S(\rho)| \le C\langle\rho\rangle\mathcal{M}(rM)(\rho) + C\langle\rho\rangle^{1-\varepsilon}\mathcal{M}(r\langle r\rangle^{\varepsilon})(\rho).$$

We then proceed as before. Since $3-p$ and $3-(1+\varepsilon)p$ both are $(A)_p$ weights,

$$\begin{aligned}\|Z^{(2)}u\|_p &\le \|V\psi\|_1\Big(\int_1^\infty \rho^{3-2p}|S(\rho)|^p d\rho\Big)^{\frac{1}{p}} \\ &\le C\Big(\int_0^\infty \rho^3|M(\rho)|^p d\rho\Big)^{\frac{1}{p}} + C\Big(\int_0^\infty \rho^{3-\varepsilon p}\langle\rho\rangle^{\varepsilon p}|M(\rho)|^p d\rho\Big)^{\frac{1}{p}} \\ &\le C\|f*u\|_p + C\Big(\int_{|x|<1}\frac{|(f*\check{u})(x)|^p}{|x|^{p\varepsilon}}dx\Big)^{\frac{1}{p}} \le C(\|u\|_p + \|f\|_q\|u\|_p). \quad (4.19)\end{aligned}$$

This completes the proof. $\square$

Lemma 4.8. *The operator W_1 is bounded in $L^p(\mathbf{R}^4)$ for all $4/3 < p < 4$.*

Proof. Due to (4.2) W_1 is a linear combination of terms of the form

$$\Phi(H)\int_0^\infty G_0(\lambda)(V\psi_j) \otimes \big(\Phi(H_0)|D|\log|D|V\psi_k\big)\big(G_0(\lambda) - G_0(-\lambda)\big)\Phi(\lambda)d\lambda.$$

This is identical to W_{sj}, if $\Phi(H_0)|D|\log|D|V\psi_k$ is replaced by $|D|^{-1}V\psi_j$. Since $\mathcal{F}(V\psi_k)(0)=0$, it is easy to see that $f(x)=\Phi(H_0)|D|\log|D|V\psi_k(x)$ is integrable on $\mathbf{R}^4$. Thus we may proceed as in the proof for W_s by applying the obvious L^p boundedness of convolution operators with integrable functions, instead of applying Lemma 4.4(2), and show that W_1 is bounded in L^p for $4/3 < p < 4$. We may safely omit the details. $\square$

Acknowledgement. This work was initiated while the second author visited Department of Mathematical Sciences, Aalborg University. It was completed while the first author visited Department of Mathematics, Tokyo Institute of Technology. The hospitality of the two Departments is gratefully acknowledged. The stay of the first author at Tokyo Institute of Technology was made possible by the financial support from the Interactive Research Center of Science, Tokyo Institute of Technology. The support is gratefully acknowledged.

The first author was partially supported by the Danish Natural Science Research Council, under the grant Mathematical Physics and Partial Differential Equations.

The second author was supported by JSPS grant in aid for scientific research No. 14340039.

References

1. G. Artbazar and K. Yajima, *The L^p-continuity of wave operators for one dimensional Schrödinger operators*, J. Math. Sci. Univ. Tokyo **7** (2000), 221-240.

2. J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York 1976.

3. P. D'Ancona and L. Fanelli, *L^p-boundedness of the wave operator for the one dimensional Schrödinger operator*, Comm. Math. Phys. **268** (2006), 415–438.

4. M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three I.* Dyn. Part. Diff. Equ., **1** (2004), 359–379.

5. D. Finco and K. Yajima, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities II. Even dimensional case*, to appear in J. Math. Sci. Univ. Tokyo.

6. M. Goldberg and M. Visan, *A counterexample to dispersive estimates for Schrödinger operators in higher dimensions*, Comm. Math. Phys. **266** (2006), 211–238.

7. A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^4)$*, J. Math. Anal. Appl. **101** (1984), 397–422.

8. A. Jensen and K. Yajima, *A remark on L^p-boundedness of wave operators for two dimensional Schrödinger operators*, Comm. Math. Phys. **225** (2002), 633–637.

9. T. Kato, *Wave operators and similarity for non-selfadjoint operators*, Math. Ann. **162** (1966), 258–279.

10. S. T. Kuroda, *Introduction to Scattering Theory*, Lecture Notes Series, 51, Aarhus University 1978.

11. M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal., **49** (1982), 10–56.

12. E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ. (1993).

13. R. Weder, *L^p-$L^{p'}$ estimates for the Schrödinger equations on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170** (2000), 37–68.

14. K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan **47** (1995), 551-581.

15. K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators. III. Even dimensional cases*, J. Math. Sci. Univ. Tokyo **2** (1995), 311–346.

16. K. Yajima, *The L^p-boundedness of wave operators for two dimensional Schrödinger operators*, Comm. Math. Phys. **208** (1999), 125–152.

17. K. Yajima, *Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue*, Comm. Math. Phys. **259** (2005), 475-509.

18. K. Yajima, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities I, Odd dimensional case*, J. Math. Sci. Univ. Tokyo **13** (2006), 43–93.

Arne Jensen
Department of Mathematical Sciences
Aalborg University, Fr. Bajers Vej 7G
DK-9220 Aalborg Ø
Denmark

matarne@math.aau.dk

Kenji Yajima
Department of Mathematics
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo 171-8588
Japan

kenji.yajima@gakushuin.ac.jp