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Original Article

Some conformal vector fields and conformal Ricci solitons on $N(k)$-contact metric manifolds

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ABSTRACT: The target of this paper is to study $N(k)$-contact metric manifolds with some types of conformal vector fields like ϕ-holomorphic planar conformal vector fields and Ricci biconformal vector fields. We also characterize $N(k)$-contact metric manifolds allowing conformal Ricci almost soliton. Obtained results are supported by examples.

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1. Introduction

Some intrinsic properties of contact metric manifolds can be explained beautifully by the existence of conformal vector fields on a contact metric manifold. For example, it is known that [26] if an m dimensional Riemannian manifold admits a maximal, i.e., $\frac{(m+1)(m+2)}{2}$-parameter group of conformal motions, then it is conformally flat. It is also known that [16] a conformally flat Sasakian manifold is of constant curvature 1. Again, it is to be noted that a complete and connected Sasakian manifold of dimension greater than three is isometric to sphere if it admits a conformal motion. This result was determined by Okumura in 1962 [15]. Later study of contact manifolds admitting conformal motions was extended to $N(k)$-contact metric manifolds by Sharma [23]. Sharma also introduced the notion of holomorphic planar conformal vector fields in Hermitian manifolds [20]. Conformal vector fields are alternatively known as conformal motions or conformal transformations in differential geometry of contact manifolds ([24], [21], [22]). Planar conformal vector fields has also been studied in ([11], [20]).

We say that a vector field V on a contact manifold M is an infinitesimal contact transformation if $\pounds_V \omega = f\omega$ for a function f, where $\pounds$ denote the Lie derivative operator and ω the contact form of the manifold. We also say

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that a vector field V on M is an automorphism of the contact metric structure if V leaves all the structure tensors of the manifold invariant [25].

In 1992, Baikoussis, Blair and Koufogiorgos [1] introduced a new type of contact metric manifolds known as $N(k)$-contact metric manifolds. $N(k)$-contact metric manifolds are such manifolds whose characteristic vector field ρ belongs to k-nullity distribution. Further properties of $N(k)$-contact manifolds were determined by Blair et al. [4]. More characterizations of $N(k)$-contact metric manifolds can be found in the papers ([6],[7],[13],[17]).

In the present paper, we would like to study $N(k)$-contact metric manifolds admitting planar conformal vector fields.

The notion of biconformal vector fields has been introduced in [10]. In that paper, the authors have also given the geometric interpretation of such vector fields. In the present paper we have introduced a new notion namely Ricci biconformal vector fields.

The theory of Ricci flow was developed by Hamilton [12] and it was applied by Perelman [18] to prove the century long well known open problem Poincare conjecture. After these works a major thrust has been seen in the study of Ricci flow. A Ricci soliton is a fixed solution of Hamilton's Ricci flow equation up to diffeomorphism and scaling. Pigola et al. [19] generalized Ricci soliton to almost Ricci soliton. The notion of conformal Ricci solitons can be found in the papers ([2], [9],[14]). For details about Ricci flow we refer [5].

The present paper is organized as follows:

After the introduction in Section 1, we recall required preliminaries in Section 2. Section 3 contains the study of ϕ-holomorphic planar conformal vector fields on $N(k)$-contact metric manifolds. In Section 4, we introduce the definition of Ricci biconformal vector fields and study them on $N(k)$-contact metric manifolds. Section 5 is devoted to study conformal Ricci almost soliton on $N(k)$-contact metric manifolds. Finally, we give an example of a three-dimensional $N(k)$-contact metric manifold to verify some results.

2. Preliminaries

An odd dimensional differentiable manifold M is called an almost contact manifold [3] if it satisfies

$$\phi^2 U = -U + \omega(U)\rho, \quad \omega(\rho) = 1, \tag{1}$$

where ϕ is a (1,1) tensor field, ρ is a unit vector field, ω is a 1-form and U is a smooth vector field on the manifold. Such a manifold is called almost contact metric manifold if there exists a Riemannian metric g such that

$$g(\phi U, \phi V) = g(U, V) - \omega(U)\omega(V) \tag{2}$$

for any smooth vector field U, V on M.
An almost contact metric manifold is called contact metric manifold when

$$d\omega(U, V) = g(U, \phi V) \tag{3}$$

is satisfied.
An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(U, f\frac{d}{dt}) = (\phi U - f\rho, \omega(U)\frac{d}{dt})$$

is integrable, where t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.
A normal contact metric manifold is called Sasakian manifold. If the characteristic vector field ρ is Killing, the manifold is known as K-contact manifold. Every Sasakian manifold is K-contact but the converse is true only for three-dimension.
It is familiar that the tangent sphere bundle of a flat Riemannian manifold allows a contact metric structure satisfying $R(U, V)\rho = 0$. On the other hand for Sasakian case

$$R(U, V)\rho = \omega(V)U - \omega(U)V,$$

where R is the Riemann curvature of the manifold. As a generalization of the above cases, Baikoussis et al. [1] introduced the notion of contact manifolds with ρ belonging to k-nullity distribution. Contact manifolds with ρ belonging to k-nullity distribution is known as $N(k)$-contact metric manifolds.
On a contact metric manifold, the (1,1)-tensor field h is defined as $h = \frac{1}{2}\pounds_\rho \phi$, where $\pounds$ denotes the Lie-derivative. The tensor field h satisfies

$$h\phi = -\phi h, \quad \text{trace}(h) = \text{trace}(\phi h) = 0, \quad h\rho = 0. \tag{4}$$

Also on a contact metric manifold, we know

$$\nabla_U \rho = -\phi U - \phi h U, \tag{5}$$

where ∇ is the Levi-Civita connection on M.
The k-nullity distribution on a Riemannian manifold is given by

$$N(k) = \{W \in TM : R(U,V)W = k[g(V,W)U - g(U,W)V]\},$$

k being a real number and TM is the set of all vector fields on M.

For an $N(k)$-contact metric manifold, we know

$$h^2 = (k-1)\phi^2, \tag{6}$$

$$(\nabla_U \omega)V = g(U + hU, \phi V), \tag{7}$$

$$(\nabla_U \phi)V = g(U + hU, V)\rho - \omega(V)(U + hU), \tag{8}$$

$$S(U,\rho) = 2nk\omega(U). \tag{9}$$

Again on a $(2n+1)$-dimensional $N(k)$-contact metric manifold the Ricci tensor S and the Riemannian curvature R are given by

$$\begin{aligned} S(U,V) &= 2(n-1)g(U,V) + 2(n-1)g(hU,V) \\ &\quad +[2nk - 2(n-1)]\omega(U)\omega(V), \end{aligned} \tag{10}$$

$$R(U,V)\rho = k[\omega(V)U - \omega(U)V]. \tag{11}$$

In a three-dimensional Riemannian manifold, we have

$$\begin{aligned} R(U,V)W &= g(V,W)QU - g(U,W)QV + S(V,W)U - S(U,W)V \\ &\quad -\frac{r}{2}[g(V,W)U - g(U,W)V], \end{aligned} \tag{12}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold. Putting $W = \rho$ in (12) and using (9) and (11), we get

$$\begin{aligned} k[\omega(V)U - \omega(U)V] &= \omega(V)QU - \omega(U)QV \\ &\quad +(2k - \frac{r}{2})[\omega(V)U - \omega(U)V]. \end{aligned} \tag{13}$$

Replacing V by ρ in the above equation, we get

$$QU = [\frac{r}{2} - k]U + [3k - \frac{r}{2}]\omega(U)\rho. \tag{14}$$

From above it also follows that

$$Q\rho = 2k\rho. \tag{15}$$

3. $N(k)$-contact metric manifold with ϕ-holomorphic planar conformal vector fields

Definition 3.1. *Let (M,g) be a $(2n+1)$-dimensional $N(k)$-contact metric manifold. A smooth vector field X of M is called a conformal vector field or conformal motion if*

$$\pounds_X g = 2fg \tag{16}$$

for a smooth function f on M. In view of Koszul's formula, a vector field X on M is called conformal if

$$\nabla_U X = fU + \phi U \tag{17}$$

for a smooth function f on M.

Suppose the characteristic vector field ρ of an $N(k)$-contact metric manifold is conformal. Then in view of (5) and (17)

$$fU + \phi U = -\phi U - \phi hU.$$

For $U = \rho$ the above equation gives $f = 0$. Hence $\pounds_\rho g = 0$. Consequently, ρ is Killing. Thus, we get

Proposition 3.2. *If the characteristic vector field ρ of an $N(k)$-contact metric manifold is conformal, then the manifold is K-contact.*

Remark 3.3. *In [23], $N(k)$-contact metric manifolds admitting conformal vector fields was characterized for dimension greater than three and it was proved that if an $N(k)$-contact metric manifold of dimension greater than three admits a conformal vector field X, then the manifold is either Sasakian or X is Killing. If, in particular, $X = \rho$ then by Proposition 3.1, the result is true for any dimension.*

Since any three-dimensional K-contact manifold is Sasakian, by the above discussion and Proposition 3.1, we can state the following:

Corollary 3.4. *If the characteristic vector field ρ of an $N(k)$-contact metric manifold is conformal, then the manifold is either Sasakian or K-contact.*

Definition 3.5. *A vector field X on an $N(k)$-contact metric manifold will be called ϕ-holomorphic planar conformal vector field if [20]*

$$\nabla_U X = lU + m\phi U \tag{18}$$

for two smooth functions l and m on M and any smooth vector field U on M.

If $m = 1$, X becomes conformal.

Suppose an $N(k)$-contact metric manifold admits a ϕ-holomorphic planar conformal vector field X. Now we shall discuss the following two cases:

(i) $\phi X = 0$
(ii) $\phi X \neq 0$.

Case i: Suppose an $N(k)$-contact metric manifold admits a conformal vector field X such that $\phi X = 0$. Then we have

$$X = \omega(X)\rho. \tag{19}$$

Differentiating the above equation with respect to U and using (5),(7) and (18), we get

$$lU + m\phi U = g(U + hU, \phi X)\rho - \omega(X)(\phi U + \phi hU).$$

Since $\phi X = 0$, the above equation takes the form

$$lU + m\phi U = -\omega(X)(\phi U + \phi hU). \tag{20}$$

By inner product with hU, we get from above

$$lg(U, hU) + mg(\phi U, hU) = -\omega(X)g(\phi U, hU). \tag{21}$$

Taking inner product in both sides of (20) with U, we have

$$lg(U, U) = -\omega(X)g(\phi hU, U). \tag{22}$$

From (19), we observe that $\omega(X)$ is non-zero. So From the above equation, we note that

$$g(hU, \phi U) = \frac{l}{\omega(X)}g(U, U). \tag{23}$$

Using (23) in (21), we see that

$$lg(U, hU) + \frac{ml}{\omega(X)}g(U, U) + lg(U, U) = 0. \tag{24}$$

From (24), we infer that $l = 0$. If not, from (24) , we observe that

$$hU = -(\frac{m}{\omega(X)} + 1)U.$$

Having taken inner product with ϕU, we conclude from above $g(hU, \phi U) = 0$. Hence, by (22), $g(U, U) = 0$, a contradiction. Thus we obtain the following:

Lemma 3.6. *If an $N(k)$-contact metric manifold admits a ϕ-holomorphic planar conformal vector field X given by $\nabla_U X = lU + m\phi U$ such that $\phi X = 0$, then $l = 0$.*

In view of (23)

$$g(\phi hU, U) = -\frac{l}{\omega(X)} g(U, U).$$

The above equation gives $\phi hU = -\frac{l}{\omega(X)} U$. Thus, by the above lemma, $\phi hU = 0$. Consequently, $hU = 0$, where U is arbitrary. Hence $h = 0$. It is known that [23], if in an $N(k)$-contact metric manifold $h = 0$, then the manifold is K-contact.
Thus, we can state the following:

Theorem 3.7. *If an $N(k)$-contact metric manifold admits a ϕ-holomorphic planar conformal vector field X such that $\phi X = 0$, then the manifold is K-contact.*

Case ii: Let us consider a ϕ-holomorphic conformal vector field on an $N(k)$-contact metric manifold with $\phi X \neq 0$. Then in view of (3.3), we get after simplification

$$\begin{aligned} R(U,V)X &= (Ul)V - (Vl)U + (Um)\phi V - (Vm)\phi U \\ &\quad + m[(\nabla_U \phi)V - (\nabla_V \phi)U]. \end{aligned}$$

By virtue of (8), the above equation gives

$$\begin{aligned} R(U,V)X &= (Ul)V - (Vl)U + (Um)\phi V - (Vm)\phi U \\ &\quad + m[\omega(U)(V + hV) - \omega(V)(U + hU)]. \end{aligned} \tag{25}$$

Replacing U by ϕU and V by ϕV in (25), we have

$$\begin{aligned} R(\phi U, \phi V)X &= (\phi Ul)\phi V - (\phi Vl)\phi U - (\phi Um)V + (\phi Um)\omega(V)\rho \\ &\quad + (\phi Vm)U - (\phi Vm)\omega(U)\rho. \end{aligned} \tag{26}$$

Adding (25) and (26), we get

$$\begin{aligned} R(U,V)X + R(\phi U, \phi V)X &= (Ul)V - (Vl)U + (Um)\phi V - (Vm)\phi U \\ &\quad + m[\omega(U)(V + hV) - \omega(V)(U + hU)] \\ &\quad + (\phi Ul)\phi V - (\phi Vl)\phi U - (\phi Um)V \\ &\quad + (\phi Um)\omega(V)\rho + (\phi Vm)U - (\phi Vm)\omega(U)\rho. \end{aligned}$$

By inner product with X, we get

$$\begin{aligned} &(Ul)g(V,X) - (Vl)g(U,X) + (Um)g(\phi V, X) - (Vm)g(\phi U, X) \\ &+ m[\omega(U)g(V + hV, X) - \omega(V)g(U + hU, X)] + (\phi Ul)g(\phi V, X) \\ &- (\phi Vl)g(\phi U, X) - (\phi Um)g(V, X) + (\phi Um)\omega(V)\omega(X) \\ &+ (\phi Vm)g(U, X) - (\phi Vm)\omega(U)\omega(X) = 0. \end{aligned} \tag{27}$$

Replacing U by ϕU and V by ϕV, we get from the above equation

$$\begin{aligned} &(\phi Ul)g(\phi V, X) - (\phi Vl)g(\phi U, X) \\ &+ (\phi Um)g(\phi^2 V, X) - (\phi Vm)g(\phi^2 U, X) \\ &+ (\phi^2 Ul)g(\phi^2 V, X) - (\phi^2 Vl)g(\phi^2 U, X) \\ &- (\phi^2 Um)g(\phi V, X) + (\phi^2 Vm)g(\phi U, X) = 0. \end{aligned}$$

Putting $\phi V = X$ in the above equation, we have

$$\begin{aligned} &(\phi Ul)g(X,X) - (Xl)g(\phi U, X) - (Xm)g(\phi^2 U, X) \\ &- (\phi Xl)g(\phi^2 U, X) - (\phi^2 Um)g(X,X) + (\phi Xm)g(\phi U, X) = 0. \end{aligned} \tag{28}$$

In (27) putting $U = \rho$, we get

$$\begin{aligned} &(\rho l)g(V,X) - (Vl)\omega(X) + (\rho m)g(\phi V, X) \\ &+ m[g(V + hV, X) - \omega(V)\omega(X)] = 0. \end{aligned}$$

By the substitution $V = \phi X$ in the above equation, we infer that

$$(\rho m)g(\phi X, \phi X) - mg(\phi X, hX) = 0. \tag{29}$$

In (27) taking $V = \rho$, we see that

$$\begin{aligned}(Ul)\omega(X) - (\rho l)g(U, X) - (\rho m)g(\phi U, X)\\ +m[\omega(U)\omega(X) - g(U + hU, X)] = 0.\end{aligned}$$

Using $U = \phi X$ in the above equation, we get

$$(\phi Xl)\omega(X) + (\rho m)g(\phi X, \phi X) - mg(\phi X, hX) = 0. \tag{30}$$

By virtue of (29) and (30), we get

$$(\phi Xl)\omega(X) = 0.$$

If X is not orthogonal to ρ, we get from the above

$$\phi Xl = 0. \tag{31}$$

Next, suppose X is orthogonal to ρ. Then putting $U = X$ in (28), we have

$$\phi Xl + Xm = 0. \tag{32}$$

Thus, we are in a position to state the following:

Lemma 3.8. *If an $N(k)$-contact metric manifold admits a non-null ϕ-holomorphic planar conformal vector field X described by $\nabla_U X = lU + m\phi U$, then*
(i) $\phi Xl = 0$, when X is not orthogonal to ρ
(ii) $\phi Xl + Xm = 0$, when X is orthogonal to ρ.
Here l and m are certain smooth functions.

Suppose X is not orthogonal to ρ. So, $\phi Xl = 0$. This means

$$g(\text{grad}l, \phi X) = 0. \tag{33}$$

Since g is metric connection, we know

$$(\nabla_W g)(U, V) = 0$$

for any U and V.
Taking $U = \text{grad}l$ and $V = \phi X$, we get

$$\nabla_W g(\text{grad}l, \phi X) - g(\nabla_W \text{grad}l, \phi X) - g(\text{grad}l, \nabla_W \phi X) = 0. \tag{34}$$

By virtue of (33) and (34), we infer

$$g(\nabla_W \text{grad}l, \phi X) + g(\text{grad}l, (\nabla_W \phi)X) + g(\text{grad}l, \phi(\nabla_W X)) = 0. \tag{35}$$

Putting $W = \phi X$ and using (8) and (18), we get

$$\begin{aligned}&g(\nabla_{\phi X}\text{grad}l, \phi X) + g(h\phi X, X)(\rho l) + \omega(X)\phi Xl\\ &+\omega(X)h\phi Xl - \phi(l\phi X + m\phi^2 X)l = 0.\end{aligned} \tag{36}$$

Suppose $\text{grad}l$ is constant. Then

$$g(\nabla_{\phi X}\text{grad}l, \phi X) \quad = \quad 0. \tag{37}$$

Using (37) in (36), we infer

$$g(h\phi X, X) = 0.$$

Since X is non-zero and not necessarily ρ, $h\phi X = 0$. So, $h = 0$.

From [23], it is known that if in an $N(k)$-contact metric manifold $h = 0$, then the manifold is K-contact. So, in this case, the manifold is K-contact.

Thus, we are in a position to state the following:

Theorem 3.9. *If an $N(k)$ -contact metric manifold admits a ϕ-holomorphic planar conformal vector field X given by $\nabla_U X = lX + m\phi X$ such that $\phi X \neq 0$ and X is not orthogonal to ρ, then the manifold is K-contact, provided* $\text{grad}l$ *is constant.*

On the remaining case $\phi Xl + Xm = 0$ and X is orthogonal to ρ. Proceeding in the similar way, we get $h = 0$, provided $\text{grad}l + \phi\text{grad}m$ is constant.
Hence, we are in a position to state the following:

Theorem 3.10. *If an $N(k)$-contact metric manifold admits a ϕ-holomorphic planar conformal vector field X given by $\nabla_U X = lX + m\phi X$ such that $\phi X \neq 0$ and X is orthogonal to ρ, then the manifold is K-contact, provided* $\text{grad}l + \phi\text{grad}m$ *is constant.*

4. $N(k)$-contact metric manifold admitting Ricci biconformal vector fields

In [10], the authors have defined biconformal vector fields using two (0,2) tensor fields. They also have given the geometric importance of the study of such vector fields on a Riemannian manifold. Here, we define Ricci biconformal vector fields by taking the metric tensor field g and the Ricci tensor field S as the two (0,2) tensor fields. Let us introduce the following:

Definition 4.1. *A vector field X on a Riemannian manifold will be called Ricci biconformal vector field if it satisfies the following equations*

$$(\pounds_X g)(U,V) = \alpha g(U,V) + \beta S(U,V) \tag{38}$$

and

$$(\pounds_X S)(U,V) = \alpha S(U,V) + \beta g(U,V) \tag{39}$$

for arbitrary non-zero smooth functions α and β.

Let us consider a Riemannian manifold with Ricci biconformal vector field.
Replacing U by QU in (38), and subtracting it from (4.2) we get

$$g(\pounds_X QU,V) + g(QU,\pounds_X V) - S(\pounds_X U,V) - S(U,\pounds_X V) = \beta g(U - Q^2U,V). \tag{40}$$

Putting $V = U$ in the above equation, we get

$$U - Q^2U = (\pounds_X Q)U. \tag{41}$$

Thus, we are in a position to state the following:

Lemma 4.2. *Let X be a biconformal vector field in a Riemannian manifold. Then the operator $\pounds_X$ annihilates the Ricci operator Q if and only if the square of the Ricci operator is identity operator.*

Let us consider a three-dimensional $N(k)$-contact metric manifold. Then in view of (14) and (15)

$$(\pounds_X Q)\rho = (3k - \frac{r}{2})\pounds_X \rho. \tag{42}$$

In view of (41) and (42), we conclude the following:

Lemma 4.3. *If a three-dimensional $N(k)$-contact metric manifold admits a Ricci biconformal vector field X then $Q^2\rho = \rho$ if and only if $\pounds_X\rho = 0$ or $r = 6k$, where r is the scalar curvature of the manifold and Q is the Ricci operator.*

Suppose $\pounds_X\rho = 0$ or $r = 6k$. Then by virtue of (2.15), $(4k^2 - 1)\rho = 0$. Which yields $k = \pm\frac{1}{2}$. The above situation leads us to state the following:

Theorem 4.4. *If a three-dimensional $N(k)$-contact metric manifold admits Ricci biconformal vector field X such that $\pounds_X\rho = 0$ or $r = 6k$, then $k = \pm\frac{1}{2}$, where r is the scalar curvature of the manifold.*

Since every three-dimensional K-contact manifold is Sasakian, as direct consequences of the above theorem, we note the following:

Corollary 4.5. *If a three-dimensional $N(k)$-contact metric manifold admits a Ricci biconformal vector field X such that $\pounds_X\rho = 0$ or $r = 6k$, then it is not K-contact and hence not Sasakian, where r is the scalar curvature of the manifold.*

Corollary 4.6. *No three-dimensional Sasakian manifold admits Ricci biconformal vector field X such that $\pounds_X\rho = 0$ or $r = 6k$, where r is the scalar curvature of the manifold.*

5. Conformal Ricci almost soliton

Definition 5.1. *The metric of an $N(k)$-contact metric manifold will be called conformal Ricci almost soliton [9] if it satisfies*

$$(\pounds_X g)(U,V) + 2S(U,V) = [2\lambda - (p + \frac{2}{2n+1})]g(U,V), \tag{43}$$

where λ and p are smooth functions called soliton functions.

Let us consider an $N(k)$-contact metric manifold admits conformal Ricci almost soliton. In view of (10) after a straight forward computation, we get the following:

Lemma 5.2. *In an $N(k)$-contact metric manifold, the following holds:*

$$\begin{aligned} & (\nabla_U S)(V,W) + (\nabla_V S)(W,U) - (\nabla_W S)(U,V) \\ = \; & 2(n-1)[g((\nabla_U h)V,W) + g((\nabla_V h)W,U) - g((\nabla_W h)U,V)] \\ & +2[2nk - 2(n-1)][g(V,\phi W)\omega(U) + g(hU,\phi V)\omega(W) \\ & +g(U,\phi W)\omega(V)]. \end{aligned} \tag{44}$$

Differentiating (43) with respect to W, we get

$$(\nabla_W \pounds_X g)(U,V) = [2d\lambda(W) - dp(W)]g(U,V) - 2(\nabla_W S)(U,V). \tag{45}$$

By a well known computation formula

$$\begin{aligned} g((\pounds_X \nabla)(U,V),W) \; = \; & \frac{1}{2}(\nabla_U \pounds_X g)(V,W) + \frac{1}{2}(\nabla_V \pounds_X g)(U,W) \\ & -\frac{1}{2}(\nabla_W \pounds_X g)(U,V). \end{aligned}$$

Using (45) in the above equation, we have

$$\begin{aligned} g((\pounds_X \nabla)(U,V),W) \; = \; & d\lambda(U)g(V,W) + d\lambda(V)g(U,W) - d\lambda(W)g(U,V) \\ + \; & \frac{1}{2}[dp(U)g(V,W) + dp(V)g(U,W) - dp(W)g(U,V)] \\ - \; & [(\nabla_U S)(V,W) + (\nabla_V S)(U,W) - (\nabla_W S)(U,V)]. \end{aligned} \tag{46}$$

Using Lemma 5.1 in the above equation, we get

$$\begin{aligned} g((\pounds_X \nabla)(U,V),W) \; = \; & d\lambda(U)g(V,W) + d\lambda(V)g(U,W) - d\lambda(W)g(U,V) \\ + \; & \frac{1}{2}[dp(U)g(V,W) + dp(V)g(U,W) - dp(W)g(U,V)] \\ - \; & 2(n-1)[g((\nabla_U h)V,W) + g((\nabla_V h)W,U) - g((\nabla_W h)U,V)] \\ - \; & 2[2nk - 2(n-1)][g(V,\phi W)\omega(U) + g(hU,\phi V)\omega(W) \\ + \; & g(U,\phi W)\omega(V)]. \end{aligned} \tag{47}$$

Putting $V = \rho$ and $n = 1$ in the above equation, we get for a three-dimensional $N(k)$-contact metric manifold

$$\begin{aligned} g((\pounds_X \nabla)(U,\rho),W) \; = \; & d\lambda(U)\omega(W) + d\lambda(\rho)g(U,W) - d\lambda(W)\omega(U) \\ + \; & \frac{1}{2}[dp(U)\omega(W) + dp(\rho)g(U,W) - dp(W)\omega(U)] \\ - \; & 4kg(U,\phi W). \end{aligned} \tag{48}$$

Putting $W = U$ in the above equation, after a straight forward computation, we get

$$g((\pounds_X \nabla)(U,\rho),U) = g(\text{grad}\lambda + \frac{\text{grad}p}{2},\rho)g(U,U). \tag{49}$$

If $\text{grad}\lambda + \frac{\text{grad}p}{2}$ is orthogonal to ρ, then we have

$$(\pounds_X \nabla)(U,\rho) = 0. \tag{50}$$

Differentiating (50), we get

$$\nabla_V(\pounds_X\nabla)(U,\rho)=0.$$

The above equation implies

$$(\nabla_V\pounds_X\nabla)(U,\rho)+(\pounds_X\nabla)(\nabla_V U,\rho)+(\pounds_X\nabla)(U,\nabla_V\rho)=0. \tag{51}$$

Using (5) in the above equation, we get

$$(\nabla_V\pounds_X\nabla)(U,\rho)=(\pounds_X\nabla)(U,\phi V)+(\pounds_X\nabla)(U,\phi hV). \tag{52}$$

Now, we state the following:

Lemma 5.3. *If a three-dimensional $N(k)$-contact metric manifold admits Ricci almost soliton and* $\mathrm{grad}\lambda+\frac{\mathrm{grad}p}{2}$ *is orthogonal to ρ, then*

$$(\nabla_V\pounds_X\nabla)(U,\rho)=(\pounds_X\nabla)(U,\phi V)+(\pounds_X\nabla)(U,\phi hV).$$

By a well known computation, we get

$$(\pounds_X R)(U,V)W=(\nabla_U\pounds_X\nabla)(V,W)-(\nabla_V\pounds_X\nabla)(U,W).$$

Putting $V=W=\rho$ and using Lemma 5.2, we get

$$(\pounds_X R)(U,\rho)\rho=0. \tag{53}$$

On the other hand by direct computation from (11), we get

$$\begin{aligned}(\pounds_X R)(U,\rho)\rho \quad = \quad & \pounds_X(k[U-\omega(U)]\rho)-k[\pounds_X U-\omega(\pounds_X U)\rho]\\ & -k[\omega(\pounds_X\rho)U-\omega(U)\pounds_X\rho]-R(U,\rho)\pounds_X\rho.\end{aligned} \tag{54}$$

Comparing (53) and (54), we get

$$k(\pounds_X\omega(U))\rho+k(\omega(\pounds_X\rho)U-\omega(U)\pounds_X\rho)-R(U,\rho)\pounds_X\rho=0.$$

By inner product and using (11), we get

$$k(\pounds_X\omega)U-kg(U-\omega(U)\rho,\pounds_X\rho)=0.$$

For $U=\rho$, the above equation gives $k=0$.

It is known that [23] a three-dimensional $N(k)$-contact metric manifold with $k=0$ is flat. Thus we are in a position to state the following:

Theorem 5.4. *If a three-dimensional $N(k)$-contact metric manifold admits Ricci almost soliton with soliton functions λ and p such that* $\mathrm{grad}\lambda+\frac{\mathrm{grad}p}{2}$ *is orthogonal to ρ, then the manifold is flat.*

6. Example

Example of a three-dimensional $N(k)$-contact metric manifold has been given in the paper of Sharma [23]. For the purpose of illustration, we use the example here.

Let $M=\{(x,y,z)\in\mathbb{R}^3,(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standard coordinates of $\mathbb{R}^3$. Let us consider three linearly independent vector fields e_1,e_2,e_3 such that

$$[e_1,e_2]=(1+a)e_3,\quad [e_2,e_3]=2e_1\quad \textit{and}\quad [e_3,e_1]=(1-a)e_2,$$

where $a=\pm\sqrt{1-k}$, a=constant.

Let g be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1$$

and

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0.$$

Let ω be a 1-form defined by $\omega(Z) = g(Z, e_1)$ for any vector field Z on M. Let ϕ be (1,1) tensor field defined by

$$\phi e_1 = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2.$$

Then M is an $N(k)$-contact metric manifold [23] for $k = (1 - a^2)$. Using Koszul's formula we have

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0,$$

$$\nabla_{e_2} e_1 = -(1+a)e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = (1+a)e_1,$$
$$\nabla_{e_3} e_1 = (1-a)e_2, \quad \nabla_{e_3} e_2 = -(1-a)e_1, \quad \nabla_{e_3} e_3 = 0.$$

We can also calculate the components of the Ricci tensor as

$$S(e_1, e_1) = 2(1 - a^2), \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0. \tag{55}$$

If we take $a = 0$, we note that $X = e_1$ is a ϕ-holomorphic planar conformal vector field for $l = 0$ and $m = -1$ such that $\phi e_1 = 0$. Hence the example fits for Lemma 3.1. Then for $a = 0$, $k = 1$ and the manifold is K-contact and hence Sasakian. This agrees with Theorem 3.1. The vector fields e_2 and e_3 are not ϕ-holomorphic planar conformal.

For $a = 0$, the vector field e_2 is not Ricci biconformal. We see that if e_2 would Ricci biconformal, then

$$(\pounds_{e_2} g)(e_1, e_1) = \alpha g(e_1, e_1) + \beta S(e_1, e_1)$$

and

$$(\pounds_{e_2} S)(e_1, e_1) = \alpha S(e_1, e_1) + \beta g(e_1, e_1)$$

gives $\alpha = 0 = \beta$ but

$$(\pounds_{e_2} g)(e_1, e_3) = \alpha g(e_1, e_3) + \beta S(e_1, e_3)$$

gives $\alpha = 1$. So e_2 is not Ricci biconformal. Similarly e_3 and e_1 are not Ricci biconformal. This validates Corollary 4.1 and Corollary 4.2.

For $a = \pm 1$, the manifold is conformal Ricci almost soliton for $\lambda = \frac{p}{2} + \frac{2}{3} - a^2$. In that case $r = 0$ and the manifold is flat. This argues that Theorem 5.1 is authentic.

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