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Research article

On some weak contractive mappings of integral type and fixed point results in b-metric spaces

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Abstract: In the article, we considered the fixed point problem for contractive mappings of integral type in the setting of b-metric spaces for the first time. First, we introduced the concepts of θ-weak contraction and θ-ψ-weak contraction. Second, the existence and uniqueness of fixed points of contractive mappings of integral type in b-metric spaces were studied. Meanwhile, two examples were given to prove the feasibility of our results. As an application, we proved the solvability of a functional equation arising in dynamic programming.

Keywords: fixed point; contractive mapping of integral type; θ-weak contraction; θ-ψ-weak contraction; b-metric
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1. Introduction

In 1922, Banach first presented the Banach contraction principle [1] in metric spaces, which is a powerful and classical means to solve problems about fixed point. Subsequently, it has been generalized in many aspects. One vital generalization is to promote the concept of metric spaces. b-metric spaces is regarded as a well-known generalization of metric spaces. In 1993, Czerwik [2] first introduced the concept of b-metric spaces by modifying the third condition of the metric function. The author also provided fixed point results for contraction conditions in this type space. In the sequel, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see [3–6]).

In 1969, Boyd and Wong [7] gave a definition of ϕ-contraction in metric spaces for the first time. Afterward, Alber and Guerre [8] defined the concept of weak contraction and got some fixed point results in Hilbert space. In [9], Rhoades generalized Alber and Guerre's results to more general forms. Alutn [10] proved the common fixed point theorem for weakly contraction mappings of integral type. Later, more scholars [11–14] presented some fixed point theorems for weakly contractive mappings in different spaces.

In particular, Perveen [15] obtained the θ^*-weak contraction principle in metric spaces as follows:
Theorem 1.1. [15] Suppose $(\Omega, \hbar)$ is a complete metric space and $\mathbb{S} : \Omega \to \Omega$ is a θ^*-weak contraction. If θ is continuous, then

(a) $\mathbb{S}$ has unique fixed point (say, $z^* \in \Omega$),

(b) $\lim\limits_{n\to+\infty} \mathbb{S}^n z = z^*$, $\forall z \in \Omega$.

Motivated and inspired by results in [15], in this paper we give some fixed point theorems for contractive mappings of the integral type in b-metric spaces. Furthermore, two examples are given to prove the feasibility of the theorems. Also, the solvability of a functional equation arising in dynamic programming is considered by means of obtained results.

2. Preliminaries

We introduce the following definitions and lemmas, which will be used to obtain our main results.
Definition 2.1. [2] Let $\aleph$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $\varpi : \aleph \times \aleph \to [0, +\infty)$ is said to be a b-metric if, and only if, for all $\kappa, \lambda, \mu \in \aleph$, the following conditions are satisfied:

(i) $\varpi(\kappa, \lambda) = 0$ if, and only if, $\kappa = \lambda$;

(ii) $\varpi(\kappa, \lambda) = \varpi(\lambda, \kappa)$;

(iii) $\varpi(\kappa, \lambda) \leq s(\varpi(\kappa, \mu) + \varpi(\lambda, \mu))$.

In general, $(\aleph, \varpi)$ is called a b-metric space with parameter $s \geq 1$.
Remark 2.2. Visibly, every metric space is a b-metric space with $s = 1$. There are several examples of b-metric spaces that are not metric spaces (see [16]).
Example 2.3. [17] Let $(\aleph, d)$ be a metric space, and $\varpi(\kappa, \lambda) = (d(\kappa, \lambda))^p$, where $p > 1$ is a real number, then $\varpi(\kappa, \lambda)$ is a b-metric with $s = 2^{p-1}$.
Definition 2.4. [18] Let $(\aleph, \varpi)$ be a b-metric space with parameter $s \geq 1$, then a sequence $\{\kappa_\iota\}_{\iota=1}^{+\infty}$ in $\aleph$ is said to be:

(i) b-convergent if there exists $\kappa \in \aleph$ such that $\varpi(\kappa_\iota, \kappa) \to 0$ as $\iota \to +\infty$;

(ii) a Cauchy sequence if $\varpi(\kappa_\iota, \kappa_\upsilon) \to 0$ when $\iota, \upsilon \to +\infty$.

As usual, a b-metric space is called complete if, and only if, each Cauchy sequence in this space is b-convergent.

The following lemma plays a key role in our conclusion.
Lemma 2.5. [17] Let $(\aleph, \varpi)$ be a b-metric space with parameter $s \geq 1$. Assume that $\{\kappa_\iota\}_{\iota=1}^{+\infty} \subset \aleph$ and $\{\lambda_\iota\}_{\iota=1}^{+\infty} \subset \aleph$ are b-convergent to κ and λ, respectively, then we have

$$\frac{1}{s^2}\varpi(\kappa, \lambda) \leq \liminf_{\iota\to+\infty} \varpi(\kappa_\iota, \lambda_\iota) \leq \limsup_{\iota\to+\infty} \varpi(\kappa_\iota, \lambda_\iota) \leq s^2\varpi(\kappa, \lambda).$$

In particular, if $\kappa = \lambda$, then we have $\lim_{\iota\to+\infty} \varpi(\kappa_\iota, \lambda_\iota) = 0$. Moreover, for each $\mu \in \aleph$, we have

$$\frac{1}{s}\varpi(\kappa, \mu) \leq \liminf_{\iota\to+\infty} \varpi(\kappa_\iota, \mu) \leq \limsup_{\iota\to+\infty} \varpi(\kappa_\iota, \mu) \leq s\varpi(\kappa, \mu).$$

Lemma 2.6. [19] Let $\varphi \in \Im$ and $\{\kappa_\iota\}_{\iota\in\mathbb{N}}$ be a nonnegative sequence with $\lim\limits_{n\to+\infty} \kappa_\iota = \kappa$, then

$$\lim_{n\to+\infty} \int_0^{\kappa_\iota} \varphi(\omega)\, d\omega = \int_0^{\kappa} \varphi(\omega)\, d\omega.$$

Lemma 2.7. [19] Let $\varphi \in \Im$ and $\{\kappa_\iota\}_{\iota\in\mathbb{N}}$ be a nonnegative sequence, then

$$\lim_{\iota\to+\infty}\int_0^{\kappa_\iota}\varphi(\omega)\,d\omega = 0$$

if, and only if, $\lim_{\iota\to+\infty}\kappa_\iota = 0$.

Throughout this paper, we assume that $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, where $\mathbb{N}$ stands for the set of positive integers,

$$\Im = \{\xi|\xi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies that } \xi \text{ is Lebesgue integrable, and } \int_0^\delta \xi(\omega)\,d\omega > 0 \text{ for each } \delta > 0\}.$$

Let $(\aleph,\varpi)$ be a b-metric space with parameter $s \geq 1$ and $\mathbb{S}$ be a self-mapping on $\aleph$. For any $u, v \in \aleph$, set

$$\triangle(\mathfrak{u},\mathfrak{v}) = \max\{\varpi(\mathfrak{u},\mathfrak{v}),\varpi(\mathfrak{u},\mathbb{S}\mathfrak{u}),\varpi(\mathfrak{v},\mathbb{S}\mathfrak{v}),\frac{\varpi(\mathfrak{u},\mathbb{S}\mathfrak{v})+\varpi(\mathfrak{v},\mathbb{S}\mathfrak{u})}{2s}\}.$$

3. Main results

In this part, we introduce the new concept of α_{s^p}-admissible mapping and other definitions, which will be used to prove the fixed point theorems of the integral type in b-metric space. Moreover, we also provide two examples to support our results.

Let

$\Theta_1 = \{\theta|\theta : (0,+\infty) \to (1,+\infty)$ satisfies the following conditions (1) and (3)$\}$,

$\Theta_2 = \{\theta|\theta : (0,+\infty) \to (0,1)$ satisfies the following conditions (2) and (3)$\}$,

where

(1) θ is nondecreasing and continuous;

(2) θ is nonincreasing and continuous;

(3) for each sequence $\{\beta_\iota\}_{\iota=1}^{+\infty} \subset (0,+\infty)$, $\lim_{\iota\to+\infty}\theta(\beta_\iota) = 1 \Rightarrow \lim_{\iota\to+\infty}\beta_\iota = 0$.

Definition 3.1. Let $(\aleph,\varpi)$ be a b-metric space with parameter $s \geq 1$ and $p \geq 1$ be an integer. A mapping $\mathbb{S} : \aleph \to \aleph$ is said to be α_{s^p}−admissible if for all $\mathfrak{z},\mathfrak{w} \in \aleph$, one has

$$\alpha(\mathfrak{z},\mathfrak{w}) \geq s^p \Rightarrow \alpha(\mathbb{S}\mathfrak{z},\mathbb{S}\mathfrak{w}) \geq s^p$$

where $\alpha : \aleph\times\aleph \to [0,+\infty)$ is a given function.

Lemma 3.2. Let $\varphi \in \Im$ and $\{\kappa_\iota\}_{\iota\in\mathbb{N}}$ be a nonnegative sequence. If $\limsup_{\iota\to+\infty}\kappa_\iota = \kappa$, then

$$\int_0^\kappa\varphi(\omega)\,d\omega \leq \limsup_{\iota\to+\infty}\int_0^{\kappa_\iota}\varphi(\omega)\,d\omega.$$

If $\liminf_{\iota\to+\infty}\kappa_\iota = \kappa$, then

$$\liminf_{\iota\to+\infty}\int_0^{\kappa_\iota}\varphi(\omega)\,d\omega \leq \int_0^\kappa\varphi(\omega)\,d\omega.$$

Proof. It follows from $\limsup_{\iota\to+\infty}\kappa_\iota = \kappa$ that there exists a subsequence $\{\kappa_{\iota_\varsigma}\}$ of $\{\kappa_\iota\}$ such that

$$\lim_{\varsigma\to+\infty}\kappa_{\iota_\varsigma} = \kappa.$$

In view of Lemma 2.6, we deduce that

$$\int_0^{\kappa} \varphi(\omega)\, d\omega = \lim_{\varsigma\to+\infty} \int_0^{\kappa_{\iota_\varsigma}} \varphi(\omega)\, d\omega \le \limsup_{\iota\to+\infty} \int_0^{\kappa_\iota} \varphi(\omega)\, d\omega.$$

Similarly, one can prove another inequality. □

Theorem 3.3. Let $(\aleph, \varpi)$ be a complete b-metric space with parameter $s \ge 1$ and $\mathbb{S} : \aleph \to \aleph$ be a given self-mapping. Assume that $\alpha : \aleph \times \aleph \to [0, +\infty)$ and $p \ge 3$. If
(i) $\mathbb{S}$ is α_{s^p}-admissible,
(ii) there is $\mathfrak{p}_0 \in \aleph$ satisfying $\alpha(\mathfrak{p}_0, \mathbb{S}\mathfrak{p}_0) \ge s^p$,
(iii) α satisfies transitive property, i.e., for $\xi, \eta, \zeta \in \aleph$ if

$$\alpha(\xi, \eta) \ge s^p \text{ and } \alpha(\eta, \zeta) \ge s^p \Rightarrow \alpha(\xi, \zeta) \ge s^p,$$

(iv) if $\{\mathfrak{p}_\iota\}$ is a sequence in $\aleph$ satisfying $\mathfrak{p}_\iota \to \mathfrak{p}$ as $\iota \to +\infty$, then there exists a subsequence $\{\mathfrak{p}_{\iota(k)}\}_{k=1}^{+\infty}$ of $\{\mathfrak{p}_\iota\}_{\iota=1}^{+\infty}$ with $\alpha(\mathfrak{p}_{\iota(k)}, \mathfrak{p}) \ge s^p$,
(v) $\mathbb{S}$ is a θ-weak contraction, that is, there exists $\ell \in (0, 1)$, $\varphi \in \Im$, $\theta \in \Theta_1$ such that: for any $\mathfrak{u}, \mathfrak{v} \in \aleph$,

$$\alpha(\mathfrak{u}, \mathfrak{v}) \ge s^p, \int_0^{\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega > 0 \Rightarrow \theta\Big(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega\Big) \le \Big[\theta\Big(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega\Big)\Big]^{\ell}, \tag{3.1}$$

then $\mathbb{S}$ has a fixed point in $\aleph$. Furthermore, if
(vi) for $\mathfrak{p}, \mathfrak{q} \in Fix(\mathbb{S})$, one can get the conditions of $\alpha(\mathfrak{p}, \mathfrak{q}) \ge s^p$ and $\alpha(\mathfrak{q}, \mathfrak{p}) \ge s^p$, where $Fix(\mathbb{S})$ represents the collection of all fixed points of $\mathbb{S}$,
then the fixed point is unique.

Proof. Under condition (ii), there is a $\mathfrak{p}_0 \in \aleph$ satisfying $\alpha(\mathfrak{p}_0, \mathbb{S}\mathfrak{p}_0) \ge s^p$. Define sequence $\{\mathfrak{p}_n\}$ in $\aleph$ by $\mathfrak{p}_{n+1} = \mathbb{S}\mathfrak{p}_n$ for $n \in \mathbb{N}$. If $\mathfrak{p}_{n_0} = \mathbb{S}\mathfrak{p}_{n_0}$ for some n_0, then $\mathfrak{p}_{n_0}$ is a fixed point of $\mathbb{S}$. Suppose that $\mathfrak{p}_{n+1} \ne \mathfrak{p}_n$ for $n \in \mathbb{N}$. It follows from condition (i) that

$$\alpha(\mathfrak{p}_0, \mathbb{S}\mathfrak{p}_0) \ge s^p \Rightarrow \alpha(\mathbb{S}\mathfrak{p}_0, \mathbb{S}^2\mathfrak{p}_0) \ge s^p,$$

$$\alpha(\mathfrak{p}_1, \mathfrak{p}_2) \ge s^p \Rightarrow \alpha(\mathbb{S}\mathfrak{p}_1, \mathbb{S}\mathfrak{p}_2) \ge s^p,$$

$$\alpha(\mathfrak{p}_2, \mathfrak{p}_3) \ge s^p \Rightarrow \alpha(\mathbb{S}\mathfrak{p}_2, \mathbb{S}\mathfrak{p}_3) \ge s^p,$$

$$\cdots\cdots\cdots$$

Thus, for all $n \in \mathbb{N}$, we have $\alpha(\mathfrak{p}_{n-1}, \mathfrak{p}_n) \ge s^p$. Using (3.1) by $\mathfrak{u} = \mathfrak{p}_{n-1}$ and $\mathfrak{v} = \mathfrak{p}_n$, one gets

$$\theta\Big(\int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{S}\mathfrak{p}_{n-1},\mathbb{S}\mathfrak{p}_n)} \varphi(\omega)\, d\omega\Big) \le \Big[\theta\Big(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega\Big)\Big]^{\ell} \tag{3.2}$$

where

$$\begin{aligned} \triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_{n-1}, \mathbb{S}\mathfrak{p}_{n-1}), \varpi(\mathfrak{p}_n, \mathbb{S}\mathfrak{p}_n), \frac{\varpi(\mathfrak{p}_{n-1}, \mathbb{S}\mathfrak{p}_n) + \varpi(\mathfrak{p}_n, \mathbb{S}\mathfrak{p}_{n-1})}{2s}\} \\ &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}), \frac{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_{n+1}) + \varpi(\mathfrak{p}_n, \mathfrak{p}_n)}{2s}\} \\ &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1})\}. \end{aligned} \tag{3.3}$$

If $\varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \geq \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n)$ for some $n \in \mathbb{N}$, in view of (3.2) and (3.3), we have $\triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) = \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1})$, so

$$\begin{aligned}\theta(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega) <&\theta(\int_0^{s^p\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega)\\ \leq&\theta(\int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{S}\mathfrak{p}_{n-1},\mathbb{S}\mathfrak{p}_n)} \varphi(\omega)\, d\omega)\\ \leq&[\theta(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)]^{\ell}\\ =&[\theta(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega)]^{\ell}\end{aligned}$$

which is impossible. Hence,

$$\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n) > \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}). \tag{3.4}$$

(3.4) implies that $\triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) = \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n)$ is decreasing. Thus, we have

$$\begin{aligned}\theta(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega) &< [\theta(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)]^{\ell}\\ &< [\theta(\int_0^{\varpi(\mathfrak{p}_{n-2},\mathfrak{p}_{n-1})} \varphi(\omega)\, d\omega)]^{\ell^2}\\ &< \cdots < [\theta(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{p}_1)} \varphi(\omega)\, d\omega)]^{\ell^n}.\end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get

$$1 \leq \lim_{n\to+\infty} \theta(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega) \leq \lim_{n\to+\infty} [\theta(\int_0^{\varpi(\mathfrak{p}_0,\mathfrak{p}_1)} \varphi(\omega)\, d\omega)]^{\ell^n} = 1$$

i.e., $\lim_{n\to+\infty} \theta(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega) = 1$, which by the definition of θ yields that

$$\lim_{n\to+\infty} \int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega = 0$$

which implies

$$\lim_{n\to+\infty} \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}) = 0.$$

Now, we prove $\{\mathfrak{p}_n\}$ is a Cauchy sequence. Suppose $\{\mathfrak{p}_n\}$ is not Cauchy, then there exists $\varepsilon > 0$ for which we can choose sequences $\{\mathfrak{p}_{n(k)}\}$ and $\{\mathfrak{p}_{m(k)}\}$ of $\{\mathfrak{p}_n\}$, such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$\varepsilon \leq \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}), \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) < \varepsilon. \tag{3.5}$$

Under the triangle inequality and (3.5), we get

$$\varepsilon \leq \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) + s\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}) < s\varepsilon + s\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}).$$

Taking the superior limit and inferior limit as $k \to +\infty$, we get

$$\varepsilon \leq \liminf_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq \limsup_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq s\varepsilon. \tag{3.6}$$

Similarly, one can deduce the following inequalities:

$$\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{m(k)-1}) + s^2\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) + s^2\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}), \tag{3.7}$$

$$\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \leq s\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{m(k)}) + s^2\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) + s^2\varpi(\mathfrak{p}_{n(k)}, \mathfrak{p}_{n(k)-1}), \tag{3.8}$$

$$\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{m(k)-1}) + s\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)}), \tag{3.9}$$

$$\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)}) \leq s\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{m(k)}) + s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}), \tag{3.10}$$

$$\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) \leq s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) + s\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}), \tag{3.11}$$

$$\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) \leq s\varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)}) + s\varpi(\mathfrak{p}_{n(k)}, \mathfrak{p}_{n(k)-1}). \tag{3.12}$$

By (3.6)–(3.8), we have

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \leq \limsup_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \leq s^3\varepsilon. \tag{3.13}$$

It follows from (3.6), (3.9), and (3.10) that

$$\frac{\varepsilon}{s} \leq \liminf_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)}) \leq \limsup_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)}) \leq s^2\varepsilon. \tag{3.14}$$

According to (3.6), (3.11), and (3.12), one can obtain

$$\frac{\varepsilon}{s} \leq \liminf_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) \leq \limsup_{k\to+\infty} \varpi(\mathfrak{p}_{m(k)}, \mathfrak{p}_{n(k)-1}) \leq s^2\varepsilon. \tag{3.15}$$

Thus, there exists $N \in \mathbb{N}_0$ such that for $m(k), n(k) \geq N$, $\int_0^{\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1})} \varphi(\omega)\, d\omega > 0$.

In view of the definition of $\triangle(\mathfrak{u}, \mathfrak{v})$, we have

$$\begin{aligned}
\triangle(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) &= \max\{\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}), \varpi(\mathfrak{p}_{m(k)-1}, \mathbb{S}\mathfrak{p}_{m(k)-1}), \varpi(\mathfrak{p}_{n(k)-1}, \mathbb{S}\mathfrak{p}_{n(k)-1}),\\
&\qquad \frac{\varpi(\mathfrak{p}_{m(k)-1}, \mathbb{S}\mathfrak{p}_{n(k)-1}) + \varpi(\mathfrak{p}_{n(k)-1}, \mathbb{S}\mathfrak{p}_{m(k)-1})}{2s}\}\\
&= \max\{\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}), \varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{m(k)}), \varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}),\\
&\qquad \frac{\varpi(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)}) + \varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{m(k)})}{2s}\}.
\end{aligned} \tag{3.16}$$

Letting $k \to +\infty$ in (3.16), we get

$$\liminf_{k\to+\infty} \triangle(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \leq \limsup_{k\to+\infty} \triangle(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \leq \max\{s^3\varepsilon, 0, 0, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\} = s^3\varepsilon. \tag{3.17}$$

The transitivity property of α yields that $\alpha(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \geq s^p$. Choosing $\mathfrak{u} = \mathfrak{p}_{m(k)-1}$ and $\mathfrak{v} = \mathfrak{p}_{n(k)-1}$ in (3.1), by Lemma 3.2, one can deduce

$$\begin{aligned}
\theta(\int_0^{s^3\varepsilon} \varphi(\omega)\,d\omega) &\leq \liminf_{k\to+\infty} \theta(\int_0^{s^p\varpi(\mathfrak{p}_{m(k)},\mathfrak{p}_{n(k)})} \varphi(\omega)\,d\omega) \\
&\leq \liminf_{k\to+\infty} \theta(\int_0^{\alpha(\mathfrak{p}_{m(k)-1},\mathfrak{p}_{n(k)-1})\varpi(\mathbb{S}\mathfrak{p}_{m(k)-1},\mathbb{S}\mathfrak{p}_{n(k)-1})} \varphi(\omega)\,d\omega) \\
&\leq \liminf_{k\to+\infty} [\theta(\int_0^{\triangle(\mathfrak{p}_{m(k)-1},\mathfrak{p}_{n(k)-1})} \varphi(\omega)\,d\omega)]^\ell \\
&\leq [\theta(\int_0^{s^3\varepsilon} \varphi(\omega)\,d\omega)]^\ell
\end{aligned}$$

which is a contradiction. So, $\{\mathfrak{p}_n\}$ is Cauchy. As $\aleph$ is complete, there exists $\mathfrak{p}^* \in \aleph$ such that $\mathfrak{p}_n \to \mathfrak{p}^*$ as $n \to +\infty$.

Next, we prove the point $\mathfrak{p}^*$ to be a fixed point of $\mathbb{S}$. So, we think about a set, say $\mathbb{Q} = \{n \in \mathbb{N}_0 : \mathfrak{p}_n = \mathbb{S}\mathfrak{p}^*\}$, then it has two situations. One, if $\mathbb{Q}$ is an infinite set, then there exists a subsequence $\{\mathfrak{p}_{n(k)}\} \subseteq \{\mathfrak{p}_n\}$, which converges to $\mathbb{S}\mathfrak{p}^*$. By the uniqueness of limit, we have $\mathbb{S}\mathfrak{p}^* = \mathfrak{p}^*$. The other, if $\mathbb{Q}$ is a finite set, then there is $N^* \in \mathbb{N}$ such that $\int_0^{\varpi(\mathfrak{p}_n,\mathbb{S}\mathfrak{p}^*)} \varphi(\omega)\,d\omega > 0$ for any $n \geq N^*$. By (iv), we obtain that there exists a subsequence $\{\mathfrak{p}_{n(k)}\} \subseteq \{\mathfrak{p}_n\}$ such that $\alpha(\mathfrak{p}_{n(k)-1}, \mathfrak{p}^*) \geq s^p$ and $\int_0^{\varpi(\mathfrak{p}_{n(k)},\mathbb{S}\mathfrak{p}^*)} \varphi(\omega)\,d\omega > 0$, $\forall k \geq N^*$. Taking $\mathfrak{u} = \mathfrak{p}_{n(k)-1}$ and $\mathfrak{v} = \mathfrak{p}^*$ in (3.1), we get

$$\theta(\int_0^{\alpha(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)\varpi(\mathbb{S}\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*)} \varphi(t)\,dt) \leq [\theta(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)} \varphi(\omega)\,d\omega)]^\ell \tag{3.18}$$

where

$$\begin{aligned}
\triangle(\mathfrak{p}_{n(k)-1}, \mathfrak{p}^*) &= \max\{\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}^*), \varpi(\mathfrak{p}_{n(k)-1}, \mathbb{S}\mathfrak{p}_{n(k)-1}), \varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}_{n(k)-1}, \mathbb{S}\mathfrak{p}^*) + d\varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}_{n(k)-1})}{2s}\} \\
&= \max\{\varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}^*), \varpi(\mathfrak{p}_{n(k)-1}, \mathfrak{p}_{n(k)}), \varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}_{n(k)-1}, \mathbb{S}\mathfrak{p}^*) + \varpi(\mathfrak{p}^*, \mathfrak{p}_{n(k)})}{2s}\}.
\end{aligned} \tag{3.19}$$

Putting the limit as $k \to +\infty$ in (3.19), we get

$$\lim_{k\to+\infty} \triangle(\mathfrak{p}_{n(k)-1}, \mathfrak{p}^*) = \max\{0, 0, \varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}^*)}{2}\} = \varpi(\mathfrak{p}^*, \mathbb{S}\mathfrak{p}^*).$$

According to (3.18), (3.19), and Lemma 2.5, we get

$$\begin{aligned}\theta(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega) &< \theta(\int_0^{s^3\cdot\frac{1}{s}\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ &\leq \limsup_{n\to+\infty}\theta(\int_0^{s^p\varpi(\mathbb{S}\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ &\leq \limsup_{n\to+\infty}\theta(\int_0^{\alpha(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)\varpi(\mathbb{S}\mathfrak{p}_{n(k)-1}\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ &\leq \limsup_{n\to+\infty}[\theta(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)}\varphi(\omega)\,d\omega)]^\ell\\ &= [\theta(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)]^\ell\end{aligned}$$

which is contradiction. Hence, $\mathbb{S}\mathfrak{p}^* = \mathfrak{p}^*$.

For the uniqueness, let $\mathfrak{q}^*$ be one more fixed point of $\mathbb{S}$, then (vi) yields $\alpha(\mathfrak{p}^*,\mathfrak{q}^*) \geq s^p$. Using (3.1), one can arrive at

$$\theta(\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{S}\mathfrak{p}^*,\mathbb{S}\mathfrak{q}^*)}\varphi(\omega)\,d\omega) \leq [\theta(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)]^\ell$$

where

$$\begin{aligned}\triangle(\mathfrak{p}^*,\mathfrak{q}^*) &= \max\{\varpi(\mathfrak{p}^*,\mathfrak{q}^*),\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*),\varpi(\mathfrak{q}^*,\mathbb{S}\mathfrak{q}^*),\frac{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{q}^*)+\varpi(\mathfrak{q}^*,\mathbb{S}\mathfrak{p}^*)}{2s}\}\\ &= \max\{\varpi(\mathfrak{p}^*,\mathfrak{q}^*),0,0,\frac{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)+\varpi(\mathfrak{q}^*,\mathfrak{p}^*)}{2s},0,0\} = \varpi(\mathfrak{p}^*,\mathfrak{q}^*).\end{aligned}$$

So, we have

$$\begin{aligned}\theta(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega) &< \theta(\int_0^{s^3\cdot\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)\\ &\leq \theta(\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{S}\mathfrak{p}^*,\mathbb{S}\mathfrak{q}^*)}\varphi(\omega)\,d\omega)\\ &\leq [\theta(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)]^\ell\\ &= [\theta(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)]^\ell\end{aligned}$$

a contradiction. Thus, $\mathfrak{p}^* = \mathfrak{q}^*$, which proves the uniqueness of the fixed point. This completes the proof. □

Example 3.4. Let $\aleph = [0,1]$ and $\varpi(\mathfrak{p},\mathfrak{q}) = (\mathfrak{p}-\mathfrak{q})^2$. It is easy to show that $(\aleph,\varpi)$ is a b-metric space with parameter $s = 2$. Define mappings $\mathbb{S} : \aleph \to \aleph$ by

$$\mathbb{S}\mathfrak{p} = \begin{cases} -\dfrac{\mathfrak{p}}{4}+1, & \mathfrak{p}\in[0,1),\\ \dfrac{7}{8}, & \mathfrak{p}=1\end{cases}$$

and $\alpha : \aleph \times \aleph \to [0, +\infty)$ by

$$\alpha(\mathfrak{p}, \mathfrak{q}) = 2^3, \forall \mathfrak{p}, \mathfrak{q} \in \aleph.$$

Define $\theta : [0, +\infty) \to (1, +\infty)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\theta(\omega) = e^{256\omega + \sin\omega} \text{ and } \varphi(\omega) = 2\omega.$$

It is easy to get that $\alpha(\mathfrak{u}, \mathfrak{v}) \geq 2^3$, $\int_0^{\varpi(\mathbb{S}\mathfrak{u}, \mathbb{S}\mathfrak{v})} \varphi(\omega) d\omega > 0 \Leftrightarrow \mathfrak{u}, \mathfrak{v} \in [0, 1]$ and $\mathfrak{u} \neq \mathfrak{v}$. We consider the two following cases:

Case 1. $\mathfrak{u}, \mathfrak{v} \in [0, 1)$. It follows that

$$\begin{aligned}
\theta\Big(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega\Big) &= \theta\Big(\int_0^{2^3(-\frac{\mathfrak{u}}{4}+1+\frac{\mathfrak{v}}{4}-1)^2} 2\omega\, d\omega\Big) \\
&= \theta\Big(\frac{1}{4}(\mathfrak{u} - \mathfrak{v})^4\Big) \\
&= e^{64(\mathfrak{u}-\mathfrak{v})^4 + \sin(\frac{1}{4}(\mathfrak{u}-\mathfrak{v})^4)},
\end{aligned}$$

$$\begin{aligned}
\Big[\theta\Big(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega\Big)\Big]^{\frac{1}{2}} &\geq \Big[\theta\Big(\int_0^{(\mathfrak{u}-\mathfrak{v})^2} 2\omega\, d\omega\Big)\Big]^{\frac{1}{2}} \\
&= [\theta((\mathfrak{u} - \mathfrak{v})^4]^{\frac{1}{2}} \\
&= e^{128(\mathfrak{u}-\mathfrak{v})^4 + \frac{\sin((\mathfrak{u}-\mathfrak{v})^4)}{2}}.
\end{aligned}$$

Case 2. $\mathfrak{u} \in [0, 1), \mathfrak{v} = 1$. One can deduce that

$$\begin{aligned}
\theta\Big(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega\Big) &= \theta\Big(\int_0^{2^3(-\frac{\mathfrak{u}}{4}+1-\frac{7}{8})^2} 2\omega\, d\omega\Big) \\
&= \theta\Big(\frac{1}{4}(\mathfrak{u} - \frac{1}{2})^4\Big) \\
&\leq \theta\Big(\frac{1}{4 \times 16}\Big) \\
&= e^{4 + \sin\frac{1}{64}},
\end{aligned}$$

$$\begin{aligned}
\Big[\theta\Big(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega\Big)\Big]^{\frac{1}{2}} &\geq \Big[\theta\Big(\int_0^{\frac{\varpi(\mathfrak{u},\mathbb{S}\mathfrak{v})+\varpi(\mathfrak{v},\mathbb{S}\mathfrak{u})}{2\cdot 2}} 2\omega\, d\omega\Big)\Big]^{\frac{1}{2}} \\
&= \Big[\theta\Big(\int_0^{\frac{1}{4}[(\mathfrak{u}-\frac{7}{8})^2+\frac{\mathfrak{u}^2}{16}]} 2\omega\, d\omega\Big)\Big]^{\frac{1}{2}} \\
&= \Big[\theta\Big(\int_0^{\frac{1}{4}\cdot\frac{17}{16}[(\mathfrak{u}-\frac{14}{17})^2+(\frac{7}{8})^2-(\frac{14}{17})^2]} 2\omega\, d\omega\Big)\Big]^{\frac{1}{2}} \\
&\geq \Big[\theta\Big(\int_0^{\frac{1}{4}\cdot\frac{17}{16}[(\frac{7}{8})^2-(\frac{14}{17})^2]} 2\omega\, d\omega\Big)\Big]^{\frac{1}{2}} \\
&\geq \Big[\theta\Big(\frac{1}{16}\Big)\Big]^{\frac{1}{2}} \\
&= e^{8 + \frac{\sin(\frac{1}{16})}{2}}.
\end{aligned}$$

Clearly, as $\ell = \frac{1}{2}$, we have

$$\theta(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega) \le [\theta(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega)]^{\ell}.$$

Hence, (3.1) holds. It follows that all conditions of Theorem 3.3 are satisfied with $s = 2$ and $p = 3$. Here, $\frac{4}{5}$ is the fixed point of $\mathbb{S}$.

Remark 3.5. If $(\aleph, \varpi)$ is a metric space and $\alpha(\mathfrak{u}, \mathfrak{v}) = 1$ in Theorem 3.3, then one can obtain Theorem 1.1 immediately.

Theorem 3.6. Let $(\aleph, \varpi)$ be a complete b-metric space with parameter $s \ge 1$ and $\mathbb{S} : \aleph \to \aleph$ be a given self-mapping. Assume that $\alpha : \aleph \times \aleph \to [0, +\infty)$ and $p \ge 3$. If

(i) $\mathbb{S}$ is α_{s^p}-admissible,

(ii) there is $\mathfrak{p}_0 \in \aleph$ satisfying $\alpha(\mathfrak{p}_0, \mathbb{S}\mathfrak{p}_0) \ge s^p$,

(iii) α satisfies transitive property, i.e., for $\xi, \eta, \zeta \in \aleph$ if

$$\alpha(\xi, \eta) \ge s^p \text{ and } \alpha(\eta, \zeta) \ge s^p \Rightarrow \alpha(\xi, \zeta) \ge s^p,$$

(iv) if $\{\mathfrak{p}_\iota\}$ is a sequence in $\aleph$ satisfying $\mathfrak{p}_\iota \to \mathfrak{p}$ as $\iota \to +\infty$, then there is a subsequence $\{\mathfrak{p}_{\iota(k)}\}_{k=1}^{+\infty}$ of $\{\mathfrak{p}_\iota\}_{\iota=1}^{+\infty}$ with $\alpha(\mathfrak{p}_{\iota(k)}, \mathfrak{p}) \ge s^p$,

(v) $\mathbb{S}$ is a θ-ψ-weak contraction, that is, there exists $\varphi \in \Im$, $\theta \in \Theta_2$ such that: for any $\mathfrak{u}, \mathfrak{v} \in \varphi$

$$\begin{aligned} &\alpha(\mathfrak{u}, \mathfrak{v}) \ge s^p, \int_0^{\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega > 0 \\ \Rightarrow \psi(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega) &\le \theta(\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega), \end{aligned} \tag{3.20}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and increasing function with $\psi(\omega) = 0$ if, and only if, $\omega = 0$,

then $\mathbb{S}$ has a fixed point in $\aleph$. Moreover, if

(vi) for $\mathfrak{p}, \mathfrak{q} \in Fix(\mathbb{S})$, one can get the conditions of $\alpha(\mathfrak{p}, \mathfrak{q}) \ge s^p$ and $\alpha(\mathfrak{q}, \mathfrak{p}) \ge s^p$, where $Fix(\mathbb{S})$ represents the collection of all fixed points of $\mathbb{S}$,

then the fixed point of $\mathbb{S}$ is unique.

Proof. As in the proof of Theorem 3.3, we infer $\alpha(\mathfrak{p}_{n-1}, \mathfrak{p}_n) \ge s^p$. Using (3.16) with $\mathfrak{u} = \mathfrak{p}_{n-1}$ and $\mathfrak{v} = \mathfrak{p}_n$, one can deduce that

$$\psi(\int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{S}\mathfrak{p}_{n-1},\mathbb{S}\mathfrak{p}_n)} \varphi(\omega)\, d\omega) \le \theta(\psi(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega) \tag{3.21}$$

where

$$\begin{aligned} \triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_{n-1}, \mathbb{S}\mathfrak{p}_{n-1}), \varpi(\mathfrak{p}_n, \mathbb{S}\mathfrak{p}_n), \frac{\varpi(\mathfrak{p}_{n-1}, \mathbb{S}\mathfrak{p}_n) + \varpi(\mathfrak{p}_n, \mathbb{S}\mathfrak{p}_{n-1})}{2s}\} \\ &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}), \frac{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_{n+1}) + \varpi(\mathfrak{p}_n, \mathfrak{p}_n)}{2s}\} \\ &= \max\{\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n), \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1})\}. \end{aligned} \tag{3.22}$$

If $\varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}) \geq \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n)$ for some $n \in \mathbb{N}$, according to (3.22), one can obtain $\triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) = \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1})$. It follows that

$$\begin{aligned}
&\psi(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega) \\
\leq&\psi(\int_0^{\alpha(\mathfrak{p}_{n-1},\mathfrak{p}_n)\varpi(\mathbb{S}\mathfrak{p}_{n-1},\mathbb{S}\mathfrak{p}_n)} \varphi(\omega)\, d\omega) \\
\leq&\theta(\psi(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega) \\
=&\theta(\psi(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega))\psi(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega)
\end{aligned}$$

which is a contradiction. Thus,

$$\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n) > \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}). \tag{3.23}$$

By (3.23), we get that $\triangle(\mathfrak{p}_{n-1}, \mathfrak{p}_n) = \varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n)$ is a decreasing sequence. Hence, there exists $\rho \geq 0$ such that $\varpi(\mathfrak{p}_{n-1}, \mathfrak{p}_n) = \rho$. If $\rho > 0$, then

$$\frac{\psi(\int_0^{\varpi(\mathfrak{p}_n,\mathfrak{p}_{n+1})} \varphi(\omega)\, d\omega)}{\psi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)} \leq \theta(\psi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)).$$

Taking $n \to +\infty$, we obtain

$$1 \leq \lim_{n\to+\infty} \theta(\psi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)) \leq 1$$

which implies $\lim\limits_{n\to+\infty} \theta(\psi(\int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega)) = 1$. In view of the definition of θ and ψ, one can deduce that

$$\lim_{n\to+\infty} \int_0^{\varpi(\mathfrak{p}_{n-1},\mathfrak{p}_n)} \varphi(\omega)\, d\omega = 0$$

i.e.,

$$\lim_{n\to+\infty} \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}) = 0,$$

which is contradiction. It follows that $\lim\limits_{n\to+\infty} \varpi(\mathfrak{p}_n, \mathfrak{p}_{n+1}) = 0$.

Next, we want to show $\{\mathfrak{p}_n\}$ is a Cauchy sequence. As in the proof of Theorem 3.3, we obtain that (3.13)–(3.17) hold. The transitivity property of α implies that $\alpha(\mathfrak{p}_{m(k)-1}, \mathfrak{p}_{n(k)-1}) \geq s^p$. Putting

$\mathfrak{u} = \mathfrak{p}_{m(k)-1}$ and $\mathfrak{v} = \mathfrak{p}_{n(k)-1}$ into (3.20), we get

$$\begin{aligned}
\psi(\int_0^{s^3\varepsilon} \varphi(\omega)\, d\omega) &\leq \liminf_{k\to+\infty} \psi(\int_0^{s^p \varpi(\mathfrak{p}_{m(k)},\mathfrak{p}_{n(k)})} \varphi(\omega)\, d\omega) \\
&\leq \liminf_{k\to+\infty} \psi(\int_0^{\alpha(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})\varpi(\mathbb{S}\mathfrak{p}_{m_k-1},\mathbb{S}\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega) \\
&\leq \liminf_{k\to+\infty} [\theta(\psi(\int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega)] \\
&\leq \limsup_{k\to+\infty} \theta(\psi(\int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega) \cdot \liminf_{k\to+\infty} \psi(\int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega) \\
&= \theta(\liminf_{k\to+\infty} \psi(\int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega)) \cdot \psi(\liminf_{k\to+\infty} \int_0^{\triangle(\mathfrak{p}_{m_k-1},\mathfrak{p}_{n_k-1})} \varphi(\omega)\, d\omega) \\
&< \psi(\int_0^{s^3\varepsilon} \varphi(\omega)\, d\omega)
\end{aligned}$$

which is a contradiction. Hence, $\{\mathfrak{p}_n\}$ is Cauchy. The completeness of $\aleph$ ensures that there exists $\mathfrak{p}^* \in \aleph$ such that $\{\mathfrak{p}_n\} \to \mathfrak{p}^*$ as $n \to +\infty$.

Next, we prove the point $\mathfrak{p}^*$ to be a fixed point of $\mathbb{S}$. Similar to the discussion related to Theorem 3.4, taking $\mathfrak{u} = \mathfrak{p}_{n(k)-1}$ and $\mathfrak{v} = \mathfrak{p}^*$ in (3.20), we get

$$\begin{aligned}
&\psi(\int_0^{\alpha(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)\varpi(\mathbb{S}\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*)} \varphi(\omega)\, d\omega) \\
\leq &\theta(\psi(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)} \varphi(\omega)\, d\omega)
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*) &= \max\{\varpi(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*), \varpi(\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}_{n(k)-1}), \varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*) + \varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}_{n(k)-1})}{2s}\} \\
&= \max\{\varpi(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*), \varpi(\mathfrak{p}_{n(k)-1},\mathfrak{p}_{n(k)}), \varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*) + \varpi(\mathfrak{p}^*,\mathfrak{p}_{n(k)})}{2s}\}.
\end{aligned} \tag{3.25}$$

Taking the limit as $n \to +\infty$ in (3.25), we get

$$\lim_{n\to+\infty} \triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*) = \max\{0, 0, \varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*), \frac{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}{2}\} = \varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*). \tag{3.26}$$

According to (3.24), (3.26), and Lemma 2.5, we get

$$\begin{aligned}\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(t)\,dt) \leq&\psi(\int_0^{s^3\cdot\frac{1}{s}\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(t)\,dt)\\ \leq&\lim_{n\to+\infty}\psi(\int_0^{\alpha(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)\varpi(\mathbb{S}\mathfrak{p}_{n(k)-1},\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ \leq&\lim_{n\to+\infty}\theta(\psi(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)}\varphi(\omega)\,d\omega))\psi(\int_0^{\triangle(\mathfrak{p}_{n(k)-1},\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ =&\theta(\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega))\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\\ <&\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*)}\varphi(\omega)\,d\omega)\end{aligned}$$

which is impossible. It follows that $\mathbb{S}\mathfrak{p}^* = \mathfrak{p}^*$.

At last, we show the uniqueness of the fixed point of $\mathbb{S}$. Suppose $\mathfrak{q}^*$ is another fixed point of $\mathbb{S}$. It follows from the condition (iv) that $\alpha(\mathfrak{p}^*,\mathfrak{q}^*) \geq s^p$. In light of (3.20), one can get

$$\psi(\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{S}\mathfrak{p},\mathbb{S}\mathfrak{q})}\varphi(\omega)\,d\omega) \leq\theta(\psi(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega))\psi(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega),$$

$$\begin{aligned}\triangle(\mathfrak{p}^*,\mathfrak{q}^*) &= \max\{\varpi(\mathfrak{p}^*,\mathfrak{q}^*),\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{p}^*),\varpi(\mathfrak{q}^*,\mathbb{S}\mathfrak{q}^*),\frac{\varpi(\mathfrak{p}^*,\mathbb{S}\mathfrak{q}^*)+\varpi(\mathfrak{q}^*,\mathbb{S}\mathfrak{p}^*)}{2s}\}\\ &= \max\{\varpi(\mathfrak{p}^*,\mathfrak{q}^*),0,0,\frac{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)+\varpi(\mathfrak{q}^*,\mathfrak{p}^*)}{2s}\} = \varpi(\mathfrak{p}^*,\mathfrak{q}^*).\end{aligned}$$

Then

$$\begin{aligned}\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega) \leq&\psi(\int_0^{\alpha(\mathfrak{p}^*,\mathfrak{q}^*)\varpi(\mathbb{S}\mathfrak{p},\mathbb{S}\mathfrak{q})}\varphi(\omega)\,d\omega)\\ \leq&\theta(\psi(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega))\psi(\int_0^{\triangle(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)\\ <&\psi(\int_0^{\varpi(\mathfrak{p}^*,\mathfrak{q}^*)}\varphi(\omega)\,d\omega)\end{aligned}$$

a contradiction, which implies that $\mathfrak{p}^* = \mathfrak{q}^*$. This completes the proof. □

Example 3.7. Let $\aleph = [0,1]$ and $\varpi(\mathfrak{p},\mathfrak{q}) = (\mathfrak{p}-\mathfrak{q})^2$. Define mappings $\mathbb{S}:\aleph\to\aleph$ by

$$\mathbb{S}\mathfrak{p} = \begin{cases}\dfrac{\mathfrak{p}}{32\sqrt[16]{e}}, & \mathfrak{p}\in[0,\frac{1}{2}],\\ \dfrac{1}{32\sqrt[16]{e}}, & \mathfrak{p}\in(\frac{1}{2},1]\end{cases}$$

and $\alpha:\aleph\times\aleph\to[0,+\infty)$ by

$$\alpha(\mathfrak{p},\mathfrak{q}) = 2^4,\ \mathfrak{p},\mathfrak{q}\in[0,1].$$

Define $\theta : [0, +\infty) \to (0, 1)$ and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by

$$\theta(\omega) = e^{-4\omega},\ \psi(\omega) = \omega \ \text{ and } \ \varphi(\omega) = 2\omega.$$

One can deduce that $\alpha(\mathfrak{u}, \mathfrak{v}) \geq 2^4$, $\int_0^{\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)d\omega > 0 \Leftrightarrow \mathfrak{u}, \mathfrak{v} \in [0, 1]$ with $\mathfrak{u} \neq \mathfrak{v}$. It follows that we also consider two cases:
Case 1. $\mathfrak{u}, \mathfrak{v} \in [0, \frac{1}{2}]$, then

$$\begin{aligned}\psi(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega) &= \int_0^{2^4(\frac{\mathfrak{u}}{32\sqrt[16]{e}} - \frac{\mathfrak{v}}{32\sqrt[16]{e}})^2} 2\omega\, d\omega \\ &= \frac{1}{64^2 \times \sqrt[4]{e}}(\mathfrak{u} - \mathfrak{v})^4,\end{aligned}$$

$$\begin{aligned}\theta(\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega) &= e^{-4\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} 2\omega\, d\omega} \cdot \int_0^{\triangle(\mathfrak{u},\mathfrak{v})} 2\omega\, d\omega \\ &\geq \frac{1}{\sqrt[4]{e}}(\mathfrak{u} - \mathfrak{v})^4.\end{aligned}$$

Case 2. $\mathfrak{u} \in [0, \frac{1}{2}], \mathfrak{v} \in (\frac{1}{2}, 1]$. It is easy to obtain that

$$\begin{aligned}\psi(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega) &= \int_0^{2^4(\frac{\mathfrak{u}}{32\sqrt[16]{e}} - \frac{1}{32\sqrt[16]{e}})^2} 2\omega\, d\omega \\ &= \frac{1}{64^2 \times \sqrt[4]{e}}(\mathfrak{u} - 1)^4 \\ &\leq \frac{1}{64^2 \times \sqrt[4]{e}},\end{aligned}$$

$$\begin{aligned}\theta(\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega) &= e^{-4\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} 2\omega\, d\omega} \cdot \int_0^{\triangle(\mathfrak{u},\mathfrak{v})} 2\omega\, d\omega \\ &\geq \frac{1}{e^4} \times \frac{1}{16} \times (1 - \frac{1}{32\sqrt[16]{e}})^4 \\ &\geq \frac{1}{64^2 \times \sqrt[4]{e}}.\end{aligned}$$

That is,

$$\psi(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\varpi(\mathbb{S}\mathfrak{u},\mathbb{S}\mathfrak{v})} \varphi(\omega)\, d\omega) \leq \theta(\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega))\psi(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega).$$

It follows that all conditions of Theorem 3.6 are satisfied with $s = 2$ and $p = 4$. It is easy to get that 0 is the unique fixed point of $\mathbb{S}$.

4. An application

In this section, by using the fixed point theorems obtained in Section 3, we study the existence of solutions of the following functional Eq (4.2).

Let O and P be two Banach spaces and $S \subseteq O$ and $D \subseteq P$ be the state and decision spaces. $B(S)$ denotes the Banach space of all bounded real-valued functions on S with norm

$$\| m \| = \sup\{|m(\xi)| : \xi \in S\} \text{ for any } m \in B(S). \tag{4.1}$$

Bellman [20] was the first to investigate the existence and uniqueness of solutions for the following functional equations arising in dynamic programming:

$$f(x) = \inf_{y \in D} \max\{r(x, y), s(x, y), f(b(x, y))\},$$

$$f(x) = \inf_{y \in D} \max\{r(x, y), f(b(x, y))\}$$

in a complete metric space $BB(S)$. As suggested in Bellman and Lee [21], the basic form of the functional equations in dynamic programming is as follows:

$$f(x) = opt_{y \in D}\{H(x, y, f(T(x, y)))\}, \forall x \in S$$

where the *opt* represents sup or inf. Bhakta and Mitra [22] obtained the existence and uniqueness of solutions for the functional equations

$$f(x) = \sup_{y \in D}\{p(x, y) + A(x, y, f(a(x, y)))\}$$

in a Banach space $B(S)$ and

$$f(x) = \sup_{y \in D}\{p(x, y) + f(a(x, y))\}$$

in $BB(S)$, respectively. After that, many authors established the existence and uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by means of various fixed and common fixed point theorems (see [23–25]).

It is easy to get that $(B(S), \varpi)$ is a complete b-metric space with

$$\varpi(\xi, \eta) = \| \xi - \eta \|^2, \forall \xi, \eta \in B(S).$$

Consider the functional equations arising in dynamic programming:

$$\mathfrak{f}(x) = \inf_{y \in D}\{u(x, y) + H(x, y, \mathfrak{f}(\mathrm{T}(x, y)))\}, \forall x \in S \tag{4.2}$$

where $u : S \times D \to \mathbb{R}$, $\mathrm{T} : S \times D \to S$ and $H : S \times D \times \mathbb{R} \to \mathbb{R}$ are mappings. Let

$$\mathbb{S}\mathfrak{f}(x) = \inf_{y \in D}\{u(x, y) + H(x, y, \mathfrak{f}(\mathrm{T}(x, y)))\}, \forall (x, \mathfrak{f}) \in S \times B(S). \tag{4.3}$$

Theorem 4.1. Let $u : S \times D \to \mathbb{R}$, $\mathrm{T} : S \times D \to S$, $H : S \times D \times \mathbb{R} \to \mathbb{R}$, $\mathbb{S} : B(S) \to B(S)$, $\alpha : B(S) \times B(S) \to \mathbb{R}$. If

(i) u and H are bounded,
(ii) $\mathbb{S}$ is α_{s^p}-admissible,
(iii) there is $\mathfrak{p}_0 \in B(S)$ satisfying $\alpha(\mathfrak{p}_0, \mathbb{S}\mathfrak{p}_0) \geq s^p$,
(iv) α satisfies transitive property, i.e., for $\xi, \eta, \zeta \in B(S)$ such that

$$\alpha(\xi, \eta) \geq s^p \text{ and } \alpha(\eta, \zeta) \geq s^p \Rightarrow \alpha(\xi, \zeta) \geq s^p,$$

(v) if $\{\mathfrak{p}_n\}$ is a sequence in $B(S)$ satisfying $\mathfrak{p}_n \to \mathfrak{p}$ as $n \to +\infty$, then there is a subsequence $\{\mathfrak{p}_{n(k)}\}$ of $\{\mathfrak{p}_n\}$ with $\alpha(\mathfrak{p}_{n(k)}, \mathfrak{p}) \geq s^p$,
(vi) for $\mathfrak{p}, \mathfrak{q} \in Fix(\mathbb{S})$, one can get the condition of $\alpha(\mathfrak{p}, \mathfrak{q}) \geq s^p$ and $\alpha(\mathfrak{q}, \mathfrak{p}) \geq s^p$, where $Fix(\mathbb{S})$ represents the collection of all fixed points of $\mathbb{S}$,
(vii) if there exists $\ell \in (0, 1)$, $\varphi \in \mathfrak{I}$ such that

$$\begin{aligned} &\alpha(\mathfrak{u}, \mathfrak{v}) \geq s^p, \int_0^{\|\mathbb{S}\mathfrak{u} - \mathbb{S}\mathfrak{v}\|^2} \varphi(\omega)\, d\omega > 0 \\ \Rightarrow& \\ &\exp\Big(\int_0^{2\alpha(\mathfrak{u},\mathfrak{v})|H(\mathfrak{u},\mathfrak{v},\mathfrak{g}(\mathrm{T}(\mathfrak{u},\mathfrak{v}))) - H(\mathfrak{u},\mathfrak{v},\mathfrak{h}(\mathrm{T}(\mathfrak{u},\mathfrak{v})))|^2} \varphi(\omega)\, d\omega\Big) \leq \Big[\exp\Big(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega\Big)\Big]^{\ell}, \end{aligned} \tag{4.4}$$

then the functional Eq (4.2) has a unique solution $\mathfrak{p}^* \in B(S)$.

Proof. It follows from (i) that there exists $M > 0$ satisfying

$$\sup\{|u(x, y)|, |H(x, y, t)| : (x, y, t) \in S \times D \times \mathbb{R}\} \leq M.$$

It is easy to see that $\mathbb{S}$ is a self-mapping in $B(S)$. Define $\alpha : B(S) \times B(S) \to [0, \infty)$ by

$$\alpha(\mathfrak{u}, \mathfrak{v}) = \begin{cases} s^p, & \varpi(\mathbb{S}\mathfrak{u}, \mathbb{S}\mathfrak{v}) > 0, \\ 0, & \text{otherwise}. \end{cases}$$

By (i) and $\varphi \in \mathfrak{I}$, we have for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_C \varphi(t)dt < \varepsilon, \forall C \subset [0, 2M] \text{ with } m(C) \leq \delta, \tag{4.5}$$

where $m(C)$ denotes the Lebesgue measure of C.

Let $\mathfrak{u} \in S, \mathfrak{h}, \mathfrak{g} \in B(S)$. By (4.3), there exists $\mathfrak{v}, \mathfrak{w} \in D$ satisfying

$$\mathbb{S}\mathfrak{g}(\mathfrak{u}) > u(\mathfrak{u}, \mathfrak{v}) + H(\mathfrak{u}, \mathfrak{v}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))) - \frac{\sqrt{2\delta}}{2},$$

$$\mathbb{S}\mathfrak{h}(\mathfrak{u}) > u(\mathfrak{u}, \mathfrak{w}) + H(\mathfrak{u}, \mathfrak{w}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))) - \frac{\sqrt{2\delta}}{2},$$

$$\mathbb{S}\mathfrak{g}(\mathfrak{u}) \leq u(\mathfrak{u}, \mathfrak{w}) + H(\mathfrak{u}, \mathfrak{w}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))),$$

$$\mathbb{S}\mathfrak{h}(\mathfrak{u}) \leq u(\mathfrak{u}, \mathfrak{v}) + H(\mathfrak{u}, \mathfrak{v}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))).$$

Thus,

$$\begin{aligned}\mathbb{S}\mathfrak{g}(\mathfrak{u}) - \mathbb{S}\mathfrak{h}(\mathfrak{u}) <& H(\mathfrak{u}, \mathfrak{w}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))) - H(\mathfrak{u}, \mathfrak{w}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))) + \frac{\sqrt{2\delta}}{2}\\ \leq& |H(\mathfrak{u}, \mathfrak{w}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))) - H(\mathfrak{u}, \mathfrak{w}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{w})))| + \frac{\sqrt{2\delta}}{2},\end{aligned}$$

$$\begin{aligned}\mathbb{S}\mathfrak{h}(\mathfrak{u}) - \mathbb{S}\mathfrak{g}(\mathfrak{u}) <& H(\mathfrak{u}, \mathfrak{v}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))) - H(\mathfrak{u}, \mathfrak{v}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))) + \frac{\sqrt{2\delta}}{2}\\ \leq& |H(\mathfrak{u}, \mathfrak{v}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))) - H(\mathfrak{u}, \mathfrak{v}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{v})))| + \frac{\sqrt{2\delta}}{2}.\end{aligned}$$

It follows that

$$\|\mathbb{S}\mathfrak{g} - \mathbb{S}\mathfrak{h}\| = \sup_{\mathfrak{u}\in S} |\mathbb{S}\mathfrak{g}(\mathfrak{u}) - \mathbb{S}\mathfrak{h}(\mathfrak{u})| \leq \max\{\mathrm{T}_1, \mathrm{T}_2\} + \frac{\sqrt{2\delta}}{2}, \tag{4.6}$$

where

$$\mathrm{T}_1 = |H(\mathfrak{u}, \mathfrak{w}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{w}))) - H(\mathfrak{u}, \mathfrak{w}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{w})))|,$$

$$\mathrm{T}_2 = |H(\mathfrak{u}, \mathfrak{v}, \mathfrak{h}(\mathrm{T}(\mathfrak{u}, \mathfrak{v}))) - H(\mathfrak{u}, \mathfrak{v}, \mathfrak{g}(\mathrm{T}(\mathfrak{u}, \mathfrak{v})))|.$$

It is easy to get that $\|\mathbb{S}\mathfrak{g} - \mathbb{S}\mathfrak{h}\|^2 \leq \max\{2{\mathrm{T}_1}^2, 2{\mathrm{T}_2}^2\} + \delta$. Under (4.4) and (4.6), we have

$$\begin{aligned}&\exp(\int_0^{s^p\|\mathbb{S}\mathfrak{g}(\mathfrak{u})-\mathbb{S}\mathfrak{h}(\mathfrak{u})\|^2} \varphi(\omega)\, d\omega)\\ \leq& \exp(\int_0^{s^p \max\{2\mathrm{T}_1^2, 2\mathrm{T}_2^2\}+\delta} \varphi(\omega)\, d\omega)\\ =& \max\{\exp(\int_0^{2s^p\mathrm{T}_1^2+\delta} \varphi(\omega)\, d\omega), \exp(\int_0^{2s^p\mathrm{T}_2^2+\delta} \varphi(\omega)\, d\omega)\}\\ =& \max\{\exp(\int_0^{2s^p\mathrm{T}_1^2} \varphi(\omega)\, d\omega) \cdot \exp(\int_{2s^p\mathrm{T}_1^2}^{2s^p\mathrm{T}_1^2+\delta} \varphi(\omega)\, d\omega),\\ &\exp(\int_0^{2s^p\mathrm{T}_2^2} \varphi(\omega)\, d\omega) \cdot \exp(\int_{2s^p\mathrm{T}_2^2}^{2s^p\mathrm{T}_2^2+\delta} \varphi(\omega)\, d\omega)\}\\ \leq& \max\{\exp(\int_0^{2s^p\mathrm{T}_1^2} \varphi(\omega)\, d\omega), \exp(\int_0^{2s^p\mathrm{T}_2^2} \varphi(\omega)\, d\omega)\}\\ &\cdot \max\{\exp(\int_{2s^p\mathrm{T}_1^2}^{2s^p\mathrm{T}_1^2+\delta} \varphi(\omega)\, d\omega), \exp(\int_{2s^p\mathrm{T}_2^2}^{2s^p\mathrm{T}_2^2+\delta} \varphi(\omega)\, d\omega)\}\\ \leq& [\exp(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\, d\omega)]^\ell \cdot \exp(\varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0^+$ in the above inequality, we get

$$\exp(\int_0^{\alpha(\mathfrak{u},\mathfrak{v})\|\mathbb{S}\mathfrak{g}-\mathbb{S}\mathfrak{h}\|^2} \varphi(\omega)\,d\omega) \leq [\exp(\int_0^{\triangle(\mathfrak{u},\mathfrak{v})} \varphi(\omega)\,d\omega)]^{\ell}.$$

Thus, the conditions of Theorem 3.3 are satisfied by taking $\theta(\omega) = \exp(\omega)$, so the functional Eq (4.2) has a unique fixed sloution $\mathfrak{p}^* \in B(S)$. This completes the proof. □

5. Conclusions

In this manuscript, we first defined two new types of weak contractions named θ-weak contraction and θ-ψ-weak contraction. Second, we presented the conditions of existence and uniqueness of fixed points for them in b-metric spaces. After that, two examples were given to demonstrate the practicability of our theorems. As an application, the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming were discussed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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