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## Trees with paired-domination number twice their domination number

by

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# Trees with paired-domination number twice their domination number

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**Abstract**

In this paper, we continue the study of paired-domination in graphs introduced by Haynes and Slater (Networks 32 (1998) 199–206). A paired-dominating set of a graph $G$ with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of $G$ is the minimum cardinality of a paired-dominating set of $G$. For $k \geq 2$, a $k$-packing in $G$ is a set $S$ of vertices of $G$ that are pairwise at distance greater than $k$ apart. The $k$-packing number of $G$ is the maximum cardinality of a $k$-packing in $G$. Haynes and Slater observed that the paired-domination number is bounded above by twice the domination number. We give a constructive characterization of the trees attaining this bound that uses labelings of the vertices. The key to our characterization is the observation that the trees with paired-domination number twice their domination number are precisely the trees with 2-packing number equal to their 3-packing number.

**Keywords:** domination, packing number, paired-domination
**AMS subject classification: 05C69**

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# 1 Introduction

In this paper, we continue the study of domination and paired-domination in graphs. For a graph $G = (V, E)$, a set $S$ is a *dominating set*, denoted DS, if every vertex in $V \setminus S$ has a neighbor in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$*-set*. Domination and its many variations have been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [8, 9]. We are interested in a variation of domination called paired-domination.

A *matching* in a graph $G$ is a set of independent edges in $G$. A *perfect matching* $M$ in $G$ is a matching in $G$ such that every vertex of $G$ is incident to a vertex of $M$. A *paired-dominating set*, denoted PDS, of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to some vertex in $S$ and the subgraph $G[S]$ induced by $S$ contains a perfect matching $M$ (not necessarily induced). Two vertices joined by an edge of $M$ are said to be *paired* in $S$. Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The *paired-domination number* of $G$, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a PDS. A PDS of cardinality $\gamma_{\rm pr}(G)$ we call a $\gamma_{\rm pr}(G)$-set. Paired-domination was introduced by Haynes and Slater in [10, 11] as a model for assigning backups to guards for security purposes, and is studied, for example, in [2, 3, 4, 6, 7, 13, 15, 16] and elsewhere.

In general we follow the notation and graph theory terminology in [8]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For any vertex $v \in V$, the *open neighborhood of* $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. If $v \in S$, then a vertex $w \in V \setminus S$ is an *external private neighbor of* $v$ (with respect to $S$) if $N(w) \cap S = \{v\}$; and the *external private neighbor set of* $v$ *with respect to* $S$, denoted $\text{epn}(v, S)$, is the set of all external private neighbors of $v$. We denote the degree of $v$ in $G$ by $d_G(v)$. We denote a path on $n$ vertices by $P_n$.

For ease of presentation, we mostly consider *rooted trees*. For a vertex $v$ in a (rooted) tree $T$, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. A vertex of degree one is called a *leaf* and its neighbor is called a *support* vertex. A *strong support vertex* is adjacent to at least two leaves.

For $k \geq 2$, Meir and Moon [14] defined a *k-packing* in a graph $G$ as a set $S$ of vertices of $G$ that are pairwise at distance greater than $k$ apart, i.e., if $u, v \in S$, then $d_G(u, v) > k$. A 1-packing is an independent set. The *k-packing number* of $G$, denote $\rho_k(G)$, is the maximum cardinality of a $k$-packing in $G$. We call a $k$-packing of cardinality $\rho_k(G)$ a $\rho_k(G)$*-set*. By definition, we have the following observation.

**Observation 1** *For $k \geq 1$ and for any graph $G$, $\rho_k(G) \geq \rho_{k+1}(G)$.*

A 2-packing is also called a *packing* and the 2-packing number is also called the *packing number* denoted by $\rho(G)$. Thus a $\rho(G)$-set is a packing of maximum cardinality $\rho(G)$. By Observation 1, $\rho(G) \geq \rho_3(G)$.

**Definition 1** *We define a graph $G$ to be a $(\rho = \rho_3)$-graph if $\rho(G) = \rho_3(G)$.*

The following bound was first observed by Haynes and Slater [10]. The proof follows from the fact that every graph without isolated vertices has a minimum DS in which each vertex has an external private neighbor.

**Observation 2** ([10]) *For every graph $G$ without isolated vertices, $\gamma_{\rm pr}(G) \leq 2\gamma(G)$.*

We call the graphs obtaining the upper bound in Observation 2, $(\gamma_{\rm pr} = 2\gamma)$-graphs. It is our goal in this paper to characterize $(\gamma_{\rm pr} = 2\gamma)$-trees. Recently, a characterization of $(\gamma_{\rm pr} = 2\gamma)$-trees was given in [12]. The characterization we present here is a constructive characterization using labelings that is simpler than that presented in [12]. The key to our characterization is the observation that the family of $(\gamma_{\rm pr} = 2\gamma)$-trees is precisely the family of $(\rho = \rho_3)$-trees.

**Observation 3** *Let $T$ be a tree of order at least 2. Then, $T$ is a $(\gamma_{\rm pr} = 2\gamma)$-tree if and only if $T$ is a $(\rho = \rho_3)$-tree.*

Observation 3 is an immediate consequence of the following two results.[1]

**Theorem 1** (Moon and Meir [14]) *For every tree $T$, $\gamma(T) = \rho(T)$.*

**Theorem 2** (Bresar et. al. [1]) *For every tree $T$ of order at least 2, $\gamma_{\rm pr}(T) = 2\rho_3(T)$.*

Hence, by Observation 3, to characterize the $(\gamma_{\rm pr} = 2\gamma)$-trees it suffices to characterize the $(\rho = \rho_3)$-trees.

## 2 $(\rho = \rho_3)$-Trees

Our aim in this section is to give a constructive characterization of $(\rho = \rho_3)$-trees (and hence, by Observation 3, of $(\gamma_{\rm pr} = 2\gamma)$-trees). The main idea to our constructive characterization is to find a labeling of the vertices that indicates the roles each vertex plays in the sets associated with both parameters. This idea of labeling the vertices is introduced in [5], where trees with equal domination and independent domination numbers as well as trees with equal domination and total domination numbers are characterized.

[1] Since the result of Bresar et. al. [1] may not be readily available to the referees, we provide a proof–for refereeing purposes only–of Theorem 2 in the appendix.

## 2.1 Main Result

By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. We define a *labeling* of a tree $T$ as a weak partition $S = (S_A, S_B, S_C, S_D)$ of $V(T)$. The *label* or *status* of a vertex $v$, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. Our aim it to describe a procedure to build $(\rho = \rho_3)$-trees in terms of labelings.

We begin by defining three labeled trees $T_1$, $T_2$ and $T_3$ as follows. Let $T_1$ be a $P_2$ with one vertex labeled $A$ and the other vertex $C$. Let $T_2$ be a $P_2$ with one vertex labeled $B$ and the other vertex $D$. Let $T_3$ be a $P_3$ with the central vertex labeled $B$ and with one leaf labeled $A$ and the other leaf $C$.

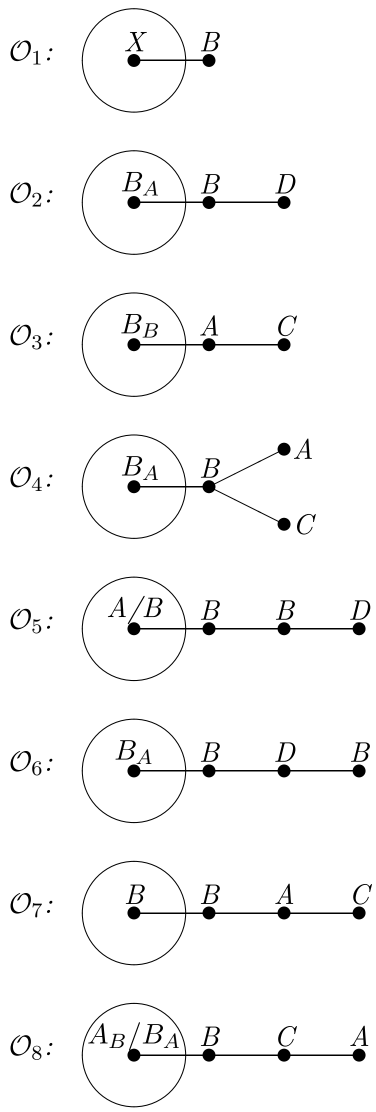
**Definition 2** *Let $\mathcal{T}$ be the family of labeled trees that:* (i) *contains the labeled trees $T_1$, $T_2$ and $T_3$; and* (ii) *is closed under the eleven operations $\mathcal{O}_j$ $(j = 1, 2, \cdots, 11)$ listed below, which extend the tree $T$ by attaching a tree to the vertex $v \in V(T)$.*

- **Operation $\mathcal{O}_1$.** Assume $v$ is a support vertex of $T$. Add a new vertex $u$ of status $B$ and the edge $uv$. (Note that $v$ can have any status.)
- **Operation $\mathcal{O}_2$.** Assume $v$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $A$. Add a path $u_1, u_2$ and the edge $vu_1$. Let $\text{sta}(u_1) = B$ and $\text{sta}(u_2) = D$.
- **Operation $\mathcal{O}_3$.** Assume $v$ and all its neighbors have status $B$. Add a path $u_1, u_2$ and the edge $vu_1$. Let $\text{sta}(u_1) = A$ and $\text{sta}(u_2) = C$.
- **Operation $\mathcal{O}_4$.** Assume $v$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $A$. Add a path $u_1, u_2, u_3$ and the edge $vu_2$. Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = B$, and $\text{sta}(u_3) = C$.
- **Operation $\mathcal{O}_5$.** Assume $v$ has status $A$ or $B$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\text{sta}(u_1) = \text{sta}(u_2) = B$ and $\text{sta}(u_3) = D$.
- **Operation $\mathcal{O}_6$.** Assume $v$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $A$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = D$ and $\text{sta}(u_3) = B$.
- **Operation $\mathcal{O}_7$.** Assume $v$ has status $B$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = A$ and $\text{sta}(u_3) = C$.
- **Operation $\mathcal{O}_8$.** Assume $v$ has status $A$ and all its neighbors have status $B$ or assume $v$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $A$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = C$ and $\text{sta}(u_3) = A$.
- **Operation $\mathcal{O}_9$.** Assume $v$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $C$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = B$ and $\text{sta}(u_3) = C$.

- **Operation $\mathcal{O}_{10}$.** Assume $v$ and all its neighbors have status $B$. Add a path $u_1, u_2, u_3$ and the edge $vu_1$. Let $\mathrm{sta}(u_1) = A$, $\mathrm{sta}(u_2) = C$ and $\mathrm{sta}(u_3) = B$.

- **Operation $\mathcal{O}_{11}$.** Assume $v$ has status $A$ or $B$. Add a star $K_{1,3}$ with center $u$ and leaves $u_1$, $u_2$ and $u_3$, and add the edge $vu_1$. Let $\mathrm{sta}(u) = \mathrm{sta}(u_1) = B$, $\mathrm{sta}(u_2) = A$, $\mathrm{sta}(u_3) = C$.

These eleven operations are illustrated in Figure 1 where by status $X$ we mean a support vertex of any status (namely, $A$, $B$, $C$ or $D$). For $Y \in \{A, B\}$, by status $Y_B$ we mean a vertex of status $Y$ all of whose neighbors have status $B$. For $Z \in \{A, C\}$, by status $B_Z$ we mean a vertex of status $B$ all of whose neighbors have status $B$, except possibly for one neighbor which has status $Z$.



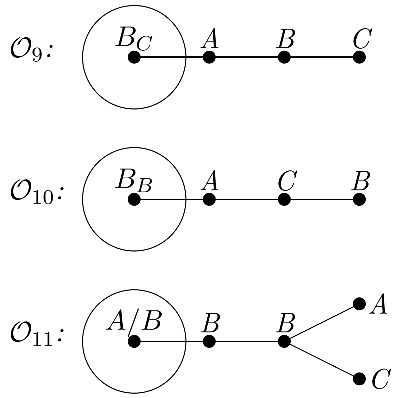



**Figure 1.** The eleven operations.

Our main result is the following constructive characterization of $(\rho = \rho_3)$-trees that uses labelings.

**Theorem 3** *The $(\rho = \rho_3)$-trees are precisely those trees $T$ such that $(T, S) \in \mathcal{T}$ for some labeling $S$.*

Theorem 3 is an immediate consequence of Theorem 4 presented in Section 2.2 and Theorem 5 in Section 2.3.

### 2.2 $(\rho, \rho_3)$-labelings

**Definition 3** *We define a $(\rho, \rho_3)$-**labeling** of a tree $T = (V, E)$ as a weak partition $S = (S_A, S_B, S_C, S_D)$ of $V$ such that*

- $S_A \cup S_D$ *is a* $\rho(T)$*-set,*
- $S_C \cup S_D$ *is a* 3*-packing, and*
- $|S_A| = |S_C|$.

We are now in a position to characterize $(\rho = \rho_3)$-trees in terms of $(\rho, \rho_3)$-labelings.

**Theorem 4** *A tree is a $(\rho = \rho_3)$-tree if and only if has a $(\rho, \rho_3)$-labeling.*

**Proof.** Suppose $T$ is a tree that has a $(\rho, \rho_3)$-labeling $(S_A, S_B, S_C, S_D)$. Then, $\rho_3(T) \le \rho(T) = |S_A \cup S_D| = |S_C \cup S_D| \le \rho_3(T)$. Hence we must have equality throughout this inequality chain. In particular, $\rho_3(T) = \rho(T)$; that is, $T$ is a $(\rho = \rho_3)$-tree.

Conversely, suppose that $T$ is a $(\rho = \rho_3)$-tree. Let $P$ be a $\rho(T)$-set and let $L$ be a $\rho_3(T)$-set. Let $S_A = P \setminus L$, $S_B = V \setminus (P \cup L)$, $S_C = L \setminus P$, and $S_D = L \cap P$. Then, $P = S_A \cup S_D$ is a $\rho(T)$-set and $L = S_C \cup S_D$ is a 3-packing. Further since $|P| = \rho(T) = \rho_3(T) = |L|$, we have $|S_A| = |S_C|$. Hence, $(S_A, S_B, S_C, S_D)$ is a $(\rho, \rho_3)$-labeling of $T$. $\square$

### 2.3 $(\rho, \rho_3)$-labelled Trees

Given a $(\rho, \rho_3)$-labeling $S = (S_A, S_B, S_C, S_D)$ of a tree $T$, we will refer to the pair $(T, S)$ as a $(\rho, \rho_3)$-labelled **tree**. In this subsection, we characterize $(\rho, \rho_3)$-labelled trees.

**Theorem 5** *A labeled tree is a $(\rho, \rho_3)$-labelled tree if and only if it is in $\mathcal{T}$.*

**Proof.** It is clear that the eleven operations $\mathcal{O}_i$, $1 \leq i \leq 11$, preserve a $(\rho, \rho_3)$-labeling, whence every element of $\mathcal{T}$ is a $(\rho, \rho_3)$-labelled tree.

Conversely, the proof that every $(\rho, \rho_3)$-labelled tree $(T, S)$ belongs to $\mathcal{T}$ is by induction on the order of $T$. The smallest $(\rho, \rho_3)$-labelled trees are the labeled trees $T_1$ and $T_2$ defined earlier, both of which are in $\mathcal{T}$. So fix a $(\rho, \rho_3)$-labelled tree $(T, S)$ of order at least 3, and assume that any smaller $(\rho, \rho_3)$-labelled tree is in $\mathcal{T}$. By Theorem 4, $T$ is a $(\rho = \rho_3)$-tree.

To complete the proof, we need to identify a set $P$ of vertices that can be pruned to leave a $(\rho, \rho_3)$-labelled tree (by induction, this pruned tree is in $\mathcal{T}$) and an operation $\mathcal{R}$ or a sequence of operations that restores the pruned vertices.

If $T = P_3$, then either $T = T_3$ or $T$ can be obtained from either $T_1$ or $T_2$ by operation $\mathcal{O}_1$, implying that $T \in \mathcal{T}$.

Suppose there is a leaf $u$ in $S_B$ adjacent to a strong support vertex. Let $T' = T - u$ and let $S'$ be the restriction of $S$ to $T'$. Then, $(T', S')$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = \{u\}$ and $\mathcal{R} = \mathcal{O}_1$. So we may assume that no leaf of status $B$ is adjacent to a strong support vertex. Hence we may assume that $\text{diam}(T) \geq 3$.

Among all longest paths in $T$, let $v_0, v_1, v_2, \ldots, v_{\text{diam}(T)}$ be chosen so that $\deg v_1$ is as large as possible. Let $T$ be rooted at the vertex $v_{\text{diam}(T)}$.

**Case 1.** $d_T(v_1) \geq 3$, i.e., $v_1$ is a strong support vertex. By assumption, no leaf of $v_1$ has status $B$. Since $S_A \cup S_D$ is a packing and $S_C \cup S_D$ is a 3-packing, $v_1$ therefore has exactly two children, say $u_0$ and $v_0$, one of status $A$ and the other of status $C$. We may assume $\text{sta}(u_0) = A$ and $\text{sta}(v_0) = C$. Further, $\text{sta}(v_1) = \text{sta}(v_2) = B$ and $\text{sta}(w) = B$ for each $w \in N(v_2)$, except possibly for one neighbor of $v_2$ which has status $A$.

Let $T' = T - D[v_1]$. Then, $(S_A \cup S_D) \setminus \{u_0\}$ is a packing in $T'$, and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then $(T', S')$ is a $(\rho, \rho_3)$-labelled tree where $S'$ is the restriction of $S$ to $T'$, and we can take $P = D[v_1]$ and $\mathcal{R} = \mathcal{O}_4$. Hence we may assume that $\rho(T') = \rho(T)$. Thus, by Theorem 1, $\gamma(T') = \gamma(T)$. Let $D$ be a $\gamma(T)$-set. Then, $v_1 \in D$. If $v_2 \notin \text{epn}(v, D)$, then $D \setminus \{v_1\}$ is a DS of $T'$ of cardinality $\gamma(T) - 1$, contrary to assumption. Hence, for every $\gamma(T)$-set $D$ of $T$ we must have that $v_2 \in \text{epn}(v, D)$. This implies that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_A \cup S_B$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_{11}$.

**Case 2.** $d_T(v_1) = 2$. We consider four possibilities.

**Case 2.1.** $v_0 \in S_D$. Then, $\{v_1, v_2\} \subseteq S_B$ and every neighbor of $v_2$ is in $S_B$, except possibly for one neighbor which is in $S_A$. Let $T' = T - \{v_0, v_1\}$. Then, $(S_A \cup S_D) \setminus \{v_0\}$ is a packing in $T'$, and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then $(T', S')$ is a $(\rho, \rho_3)$-labelled tree where $S'$ is the restriction of $S$ to $T'$, and we can take $P = \{v_0, v_1\}$ and $\mathcal{R} = \mathcal{O}_2$. Hence we may assume that $\rho(T') = \rho(T)$. This implies, as argued in Case 1, that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_A \cup S_B$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_5$.

**Case 2.2.** $v_0 \in S_C$. Then either $v_1 \in S_A$ or $v_1 \in S_B$.

Suppose first that $v_1 \in S_A$. Then, $v_2 \in S_B$ and every neighbor of $v_2$ except for $v_1$ is in $S_B$. Let $T' = T - \{v_0, v_1\}$. Then, $(S_A \cup S_D) \setminus \{v_0\}$ is a packing in $T'$, and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then $(T', S')$ is a $(\rho, \rho_3)$-labelled tree where $S'$ is the restriction of $S$ to $T'$, and we can take $P = \{v_0, v_1\}$ and $\mathcal{R} = \mathcal{O}_3$. Hence we may assume that $\rho(T') = \rho(T)$. This implies that $\gamma(T') = \gamma(T)$ and hence that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_B$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_7$.

Suppose secondly that $v_1 \in S_B$. If $v_2 \in S_B$, then $S_A \cup S_D \cup \{v_0\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, which is impossible. Hence, $v_2 \in S_A$. Thus, $N(v_2) \subseteq S_B$. Suppose $d_T(v_2) \geq 3$. Since $S_A \cup S_D$ is a packing, no descendant of $v_2$ is in $S_A \cup S_D$. Hence if $u_1$ is a child of $v_2$ different from $v_1$, then $(S_A \cup S_D \cup \{u_1, v_0\}) \setminus \{v_2\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, which is impossible. Therefore, $d_T(v_2) = 2$. As observed earlier, $v_3 \in S_B$. If $T = P_4$, then $\rho(T) = 2$, contradicting the fact that $S_A \cup S_D = \{v_2\}$ is a $\rho(T)$-set. Hence, $n \geq 5$. Every neighbor of $v_3$ different from $v_2$ has status $B$, except possibly for one neighbor of $v_3$ which has status $C$. We now consider the tree $T' = T - D[v_2]$. Let $S'$ be the restriction of $S$ to $T'$. Then, $(T', S')$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_9$.

**Case 2.3.** $v_0 \in S_A$. Then either $v_1 \in S_B$ or $v_1 \in S_C$.

Suppose that $v_1 \in S_B$. If $v_2 \in S_B$, then the set $(S_C \cup S_D) \cup \{v_0\}$ is a packing in $T$ of cardinality $|S_C| + |S_D| + 1 = |S_A| + |S_D| + 1 = \rho(T) + 1$, which is impossible. Hence, $v_2 \in S_C$. Since $S_C \cup S_D$ is a 3-packing, the vertex $v_3$ is therefore at distance at least 3 from every vertex in $(S_C \cup S_D) \setminus \{v_2\}$. But this implies that $(S_C \cup S_D \cup \{v_0, v_3\}) \setminus \{v_2\}$ is a packing in $T$ of cardinality $|S_C| + |S_D| + 1 = \rho(T) + 1$, which is impossible. Hence, $v_1 \notin S_B$, implying that $v_1 \in S_C$ and $v_2 \in S_B$.

Let $T' = T - \{v_0, v_1\}$. Then, $(S_C \cup S_D \cup \{v_2\}) \setminus \{v_1\}$ is a packing in $T'$, and so by Theorem 1, $\gamma(T') = \rho(T') \geq |S_C \cup S_D| = |S_A \cup S_D| = \rho(T) = \gamma(T)$. Consequently, $\gamma(T') = \gamma(T)$. This implies that $d_T(v_2) = 2$. Since $S_C \cup S_D$ is a 3-packing, we observe that either $v_3$ has status $A$ and all its neighbors have status $B$ or $v_3$ and all its neighbors have status $B$, except possibly for one neighbor of $v$ which has status $A$. We now consider the tree $T^* = T - D[v_2]$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_8$.

**Case 2.4.** $v_0 \in S_B$.

Suppose that $v_1 \in S_A \cup S_B \cup S_C$. Let $T' = T - \{v_0, v_1\}$. If $v_1 \in S_A \cup S_B$, then $S_C \cup S_D$ is a packing in $T'$, while if $v_1 \in S_B \cup S_C$, then $S_A \cup S_D$ is a packing in $T'$. It follows that $\rho(T') = \rho(T)$ and therefore that $d_T(v_2) = 2$. If $v_2 \in S_D$, then every vertex at distance 1 and 2 from $v_2$ is in $S_B$, while every vertex at distance 3 from $v_2$ is in $S_A \cup S_B$. But then $(S_C \cup S_D \cup \{v_0, v_3\}) \setminus \{v_2\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, which is impossible. Hence, $\{v_1, v_2\} \cap S_D = \emptyset$. If $\{v_1, v_2\} \cap S_A = \emptyset$, then $S_A \cup S_D \cup \{v_0\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, which is impossible. If $\{v_1, v_2\} \cap S_C = \emptyset$, then $S_C \cup S_D \cup \{v_0\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, which is impossible. Hence, either $v_1 \in S_A$ and $v_2 \in S_C$ or $v_1 \in S_C$ and $v_2 \in S_A$. The former case cannot occur because $v_1 \in S_A$ implies that $v_3 \in S_B$ and that every neighbor of $v_3$ different from $v_2$ is in $S_B$ except possibly one $v_3$-neighbor which may belong to $S_A$. But now $(S_C \cup S_D) \setminus \{v_2\} \cup \{v_0, v_3\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, a contradiction. Thus the latter case $v_1 \in S_C$, $v_2 \in S_A$ occurs and every vertex at distance 1 and 2 from $v_2$ except for $v_1$ is labelled B. Recall $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_{10}$.

Hence we may assume that $v_1 \in S_D$. With our earlier assumptions, we may therefore assume that every leaf of a path of length $\operatorname{diam}(T)$ has status $B$ and its neighbor (of degree-2) has status $D$. Suppose that $d_T(v_2) \geq 3$. Let $u_1$ be a child of $v_2$ different from $v_1$. If $u_1$ is not a leaf, then, by our earlier assumptions, $u_1$ has status $D$, and so we have two vertices of status $D$ at distance 2 apart, contradicting the fact that $S_D$ is a packing. Hence, $u_1$ is a leaf and $u_1 \in S_B$. But then $(S_A \cup S_D \cup \{u_1, v_0\}) \setminus \{v_1\}$ is a packing in $T$ of cardinality $\rho(T) + 1$, a contradiction. Hence, $d_T(v_2) = 2$. Since $v_1 \in S_D$, $\{v_2, v_3\} \subset S_B$ and every neighbor of $v_3$ have status $B$, except possibly for one neighbor of $v_3$ which has status $A$. We now consider the tree $T^* = T - D[v_2]$. Let $S^*$ be the restriction of $S$ to $T^*$. Then, $(T^*, S^*)$ is a $(\rho, \rho_3)$-labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_6$. $\square$

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## Appendix: For refereeing purposes only

In this appendix we present a proof of the result due to Bresar et. al. [1] that the paired domination number of a tree is equal to twice its 3-packing number. We begin with two observations from [1]. Recall that the *independence number* $\beta(G)$ of a graph $G$ is the maximum cardinality of a set of independent vertices in $G$.

**Observation 4** (Bresar et. al. [1]) *If $D$ is a $\gamma_{\rm pr}(G)$-set in a graph $G$ without isolated vertices, then $|D| \ge 2\beta(G[D])$.*

**Proof.** Let $D' \subset D$ be a maximum independent set in $G[D]$. Since each vertex of $D'$ has a partner, and these partners are distinct, $\gamma_{\rm pr}(G) = |D| \ge 2|D'| = 2\beta(G[D])$. $\Box$

**Observation 5** (Bresar et. al. [1]) *For any graph $G$ without isolated vertices, $\gamma_{\rm pr}(G) \ge 2\rho_3(G)$.*

**Proof.** Let $D$ be a $\gamma_{\rm pr}(G)$-set and let $S$ be a $\rho_3(G)$-set. For each vertex $v \in S$, let $v'$ be a vertex of $D$ that dominates $v$ and let $D' = \cup_{v\in S}\{v'\}$. Since the vertices in $S$ are pairwise at distance at least 4 apart, the vertices $v'$, where $v \in S$, are distinct and the set $D'$ is an independent set in $G[D]$. Hence, by Observation 4, $\gamma_{\rm pr}(G) = |D| \ge 2|D'| = 2\rho_3(G)$. $\Box$

Recall Theorem 2:

**Theorem 2** (Bresar et. al. [1]) *For every tree $T$ of order at least 2, $\gamma_{\rm pr}(T) = 2\rho_3(T)$.*

**Proof.** We proceed by induction on the order $n \ge 2$ of a tree $T$. If $n = 2$, then $T = K_2$ and $\gamma_{\rm pr}(T) = 2 = 2\rho_3(T)$. This establishes the base case. Assume then that $n \ge 3$ and that all nontrivial trees $T'$ of order less than $n$ satisfy $\gamma_{\rm pr}(T') = 2\rho_3(T')$. Let $T$ be a tree of order $n$. If $T$ is star or a double star, then $\gamma_{\rm pr}(T) = 2 = 2\rho_3(T)$. Hence we may assume that $\mathrm{diam}(T) \ge 4$.

In the proof we shall frequently prune the tree $T$ to a tree $T'$ and then establish that $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T')+2k$ and $\rho_3(T) \ge \rho_3(T')+k$ for some integer $k \ge 0$. Since $\gamma_{\rm pr}(T) \ge 2\rho_3(T)$ and $\gamma_{\rm pr}(T') = 2\rho_3(T')$, it then follows that $\gamma_{\rm pr}(T')+2k \ge \gamma_{\rm pr}(T) \ge 2\rho_3(T) \ge 2(\rho_3(T')+k) = \gamma_{\rm pr}(T') + 2k$, whence $\gamma_{\rm pr}(T) = 2\rho_3(T)$, as desired.

Suppose $T$ has a strong support vertex $v$. Let $u$ be a leaf neighbor of $v$, and let $T' = T - u$. Any $\gamma_{\rm pr}(T')$-set contains the support vertex $v$ and is therefore a PDS of $T$, and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T')$. Any $\rho_3(T')$-set is also a 3-packing in $T$, and so $\rho_3(T) \ge \rho_3(T')$. Thus, $\gamma_{\rm pr}(T) = 2\rho_3(T)$. Hence we may assume that $T$ has no strong support vertex.

Let $T$ be rooted at a leaf $r$ of a longest path $P$. Let $P$ be a $r$–$u$ path, and let $v$ be the neighbor of $u$. Then, $u$ is a leaf of $T$ and, since $T$ has no strong support vertex, $\deg_T v = 2$. Let $w$ denote the parent of $v$ on this path and $x$ the parent of $w$.

Suppose $\deg_T w \ge 3$ and $w$ is a support vertex. Let $v'$ be the leaf-neighbor of $w$, and let $T' = T - v'$. Then there exists a $\gamma_{\rm pr}(T')$-set that contains $w$ (if $w$ is not in some $\gamma_{\rm pr}(T')$-set, then $u$ and $v$ are paired in such a $\gamma_{\rm pr}(T')$-set and we can simply replace $u$ with the vertex $w$ thereby pairing $v$ and $w$ in the new $\gamma_{\rm pr}(T')$-set). Such a PDS is also a PDS of $T$, and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T')$. Clearly, $\rho_3(T) \ge \rho_3(T')$. Thus, $\gamma_{\rm pr}(T) = 2\rho_3(T)$.

Suppose $\deg_T w \ge 3$ and $w$ is not a support vertex. Then, each child of $w$ is a support vertex of degree 2. Let $v'$ be a child of $w$ different from $v$, and let $u'$ be the leaf-neighbor of $v'$. Let $T' = T - u' - v'$. Any $\gamma_{\rm pr}(T')$-set can be extended to a PDS of $T$ by adding to it the vertices $u'$ and $v'$ (with $u'$ and $v'$ paired), and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. Let $S'$ be a $\rho_3(T')$-set that contains as many leaves as possible. Then, $S' \cap N[w] = \emptyset$ (for example, if $S'$ contains a vertex from the set $\{v, w, x\}$, then we can simply replace such a vertex with the vertex $u$). Hence, $S'$ can be extended to a 3-packing of $T$ by adding to it the leaf $u'$, and so $\rho_3(T) \ge |S'| + 1 = \rho_3(T') + 1$. Thus, $\gamma_{\rm pr}(T) = 2\rho_3(T)$. Hence we may assume that $\deg_T w = 2$ for otherwise $\gamma_{\rm pr}(T) = 2\rho_3(T)$, as desired.

Suppose $\deg_T x = 2$. Let $T' = T - \{u, v, w, x\}$. Any ${\rm pr}(T')$-set can be extended to a PDS of $T$ by adding to it the vertices $v$ and $w$ (with $v$ and $w$ paired), and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. Any $\rho_3(T')$-set can be extended to a 3-packing of $T$ by adding to it the vertex $u$, and so $\rho_3(T) \ge \rho_3(T') + 1$. Thus, $\gamma_{\rm pr}(T) = 2\rho_3(T)$. Hence we may assume $\deg_T x \ge 3$.

Let $T' = T - \{u, v, w\}$. Every $\gamma_{\rm pr}(T')$-set can be extended to a PDS of $T$ by adding to it the vertices $v$ and $w$ (with $v$ and $w$ paired), and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. Every $\rho_3(T')$-set that does not contain $x$ (if $x$ belongs to a some $\rho_3(T')$-set, then we can simply replace $x$ with a child of $x$ in $T'$) can be extended to a 3-packing of $T$ by adding to it the vertex $u$, and so $\rho_3(T) \ge \rho_3(T') + 1$. Thus, $\gamma_{\rm pr}(T) = 2\rho_3(T)$. $\Box$