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by

Lasse Borup and Morten Nielsen

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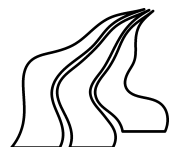
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FRAME DECOMPOSITION OF DECOMPOSITION SPACES

LASSE BORUP AND MORTEN NIELSEN[†]

ABSTRACT. A new construction of tight frames for $L_2(\mathbb{R}^d)$ with flexible time-frequency localization is considered. The frames can be adapted to form atomic decompositions for a large family of smoothness spaces on \mathbb{R}^d , a class of so-called decomposition spaces. The decomposition space norm can be completely characterized by a sparseness condition on the frame coefficients. As examples of the general construction, new tight frames yielding decompositions of Besov space, anisotropic Besov spaces, α -modulation spaces, and anisotropic α -modulation spaces are considered. Finally, curvelet-type tight frames are constructed on \mathbb{R}^d , $d \geq 2$.

1. INTRODUCTION

In applicable harmonic analysis, smoothness spaces are often designed following the principle that smoothness should be characterized by (or at least imply) some decay or sparseness of an associated discrete expansion. An elementary example is provided by the fact that a function in $C^1(\mathbb{T})$ has Fourier coefficients with decay $O(1/N)$. For applications, a particular important example is given by orthonormal wavelets. It is known, see [38], that suitable sparseness of a wavelet expansion is equivalent to smoothness measure in a Besov space. The fact that smoothness leads to sparse expansions has many important practical implications. It is possible to use a sparse representation of a function to compress that function simply by thresholding the expansion coefficients. Again wavelets provide an important example that has been successfully applied to compress sound signals and images with smoothness in some Besov space, see e.g. [13, 12]. Other interesting examples include sparse expansions in modulation spaces [33, 3], and sparse curvelet expansions [43, 7].

In this paper we consider a general construction of smoothness spaces, a subclass of so-called decomposition spaces, defined on \mathbb{R}^d for which it is possible to find adapted tight frames for $L_2(\mathbb{R}^d)$. Each frame forms an atomic decomposition of the smoothness space, and the space can be completely characterized by a sparseness condition on the frame coefficients. It is therefore possible to compress the elements of such smoothness spaces using the frame. The smoothness spaces considered here are special cases of a very general construction of decomposition spaces introduced by Feichtinger and Gröbner [18] and Feichtinger [15]. The family of spaces are based on structured decompositions of the frequency space \mathbb{R}^d , so strictly speaking they are decomposition spaces on the Fourier side. This is a fairly standard approach to define smoothness spaces. For example, Besov spaces correspond to a dyadic decomposition while the family of modulation spaces introduced by Feichtinger [17] correspond to a uniform decomposition.

Recently the search for more efficient methods to compress natural images has shown that new (often redundant) decomposition systems can produce sparser representations

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of certain natural images than e.g. wavelets. One such new decomposition system is the curvelet frame [43]. Bandlets and brushlets are other prominent examples, see [35] and [37]. Curvelets correspond to a decomposition of the frequency plane that is significantly different from both dyadic and uniform decompositions, and as a consequence, sparseness for curvelet expansions cannot be described in terms of classical smoothness spaces. The class of smoothness spaces considered here can be adapted to decompositions of curvelet-type. The connection between nonlinear approximation properties and sparse expansions has been studied from an abstract point of view in a number of papers, see for example [33, 14, 25, 28, 29].

Several authors have considered function spaces built using ideas related to decomposition spaces. Gröbner [30] used the decomposition space methods in [18] to define the α -modulation spaces as a family of intermediate spaces between modulation and Besov spaces. Banach frames for α -modulation spaces have been considered by Fornasier [22] in the univariate case and by the authors [1] in the multivariate setting. Group theoretical constructions of function spaces, including smoothness spaces, have been studied by Feichtinger and Gröchenig [16, 19, 20, 21, 31]. Frazier and Jawerth constructed frames (their so-called φ -transform) for Besov and Triebel-Lizorkin spaces in [24, 23]. Their results were generalized recently by Bownik and Ho to the anisotropic case [5, 4]. Another important application of discrete decompositions of spaces is to simplify the analysis of operators acting on it. Pseudo-differential and Fourier integral operators on Besov and modulation spaces have been studied extensively, see [2, 6, 8, 10, 11, 32, 39, 41, 44, 45] and references herein.

The outline of this paper is as follows. In Section 2 we define the family of decomposition spaces based on suitable coverings of the frequency space. In Section 3 we restrict our attention to a smaller family of decomposition spaces based on what we call a structured splitting of the frequency space \mathbb{R}^d . Structured splittings are obtained by applying a countable family of invertible affine transformations to some fixed neighborhood of the origin. The key step to obtain tight frames for such spaces is to construct a nice resolution of the identity adapted to the structured decomposition. Tight frames for structured coverings are introduced in Section 3. Characterizations of the norm in the decomposition spaces in terms of frame coefficients are given in Section 4. The application of the characterization to nonlinear approximation is considered in Section 4.2. Best n -term approximation with error measured in L_2 and more general decomposition spaces is studied, and Jackson type estimates are derived. The characterization given in Section 4 also implies that the tight frame from Section 3 gives an atomic decomposition of the associated decomposition space. In Section 5 we consider the problem of constructing interesting structured coverings of \mathbb{R}^d . A general method to construct structured coverings made up of open balls in a space of homogeneous type over \mathbb{R}^d is introduced. The covering balls are chosen to have diameters given by a fixed function of their centers. Section 6 contains applications of the methods of Section 5 to obtain examples related to classical smoothness spaces. Besov spaces, modulation spaces, and anisotropic Besov spaces are shown to correspond to special cases of the construction. In Section 6 we also define a new class of anisotropic α -modulation spaces. Finally, in Section 7 we demonstrate that the construction yields new smoothness spaces adapted to curvelet-type decompositions of the frequency space \mathbb{R}^d , $d \geq 2$. The tight frame corresponding to $d = 2$ is a variation of the so-called second generation curvelets considered by Candés and Donoho in [7]. Embedding results for curvelet-type decomposition spaces relative to Besov spaces are also obtained.

Let us summarize some of the notation used throughout this paper. We let $\mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx$, $f \in L_1(\mathbb{R}^d)$, denote the Fourier transform. By $F \asymp G$ we mean that there exist two constants $0 < C_1 \leq C_2 < \infty$ such that $C_1 F \leq G \leq C_2 F$.

2. DECOMPOSITION SPACES

Let us introduce the family of smoothness spaces that will be considered throughout this paper. The spaces are special cases of the family of decomposition spaces introduced by Feichtinger and Gröbner [18] and Feichtinger [15]. First we define admissible coverings of \mathbb{R}^d , which we will consider as the frequency domain.

Definition 2.1. A set $\mathcal{Q} := \{Q_i\}_{i \in I}$ of measurable subsets $Q_i \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{i \in I} Q_i$ and there exists $n_0 < \infty$ such that $\#\{j \in I : Q_i \cap Q_j \neq \emptyset\} \leq n_0$ for all $i \in I$.

Remark 2.2. By considering coverings of all of \mathbb{R}^d we restricted our attention to spaces of inhomogeneous type. However, the results in this paper can easily be modified to a statement on homogeneous type spaces by considering admissible coverings of $\mathbb{R}^d \setminus \{0\}$.

Notice that an admissible covering has finite height, i.e.,

$$\sum_{i \in I} \chi_{Q_i}(\xi) \leq n_0 \quad \text{for all } \xi \in \mathbb{R}^d.$$

The converse is not true as it is easy to find a covering of finite height which is not admissible. We also need partitions of unity compatible with the covers from Definition 2.1. The Fourier transform condition in Definition 2.3 is a consequence of the fact that we consider decompositions of the frequency domain, and we wish to use the partition of unity to induce a uniformly bounded family of multiplier operators on $L_p(\mathbb{R}^d)$.

Definition 2.3. Given an admissible covering $\{Q_i\}_{i \in I}$ of \mathbb{R}^d . A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\Psi = \{\psi_i\}_{i \in I}$ satisfying

- $\text{supp}(\psi_i) \subset Q_i$, $i \in I$,
- $\sum_{i \in I} \psi_i(\xi) = 1$, $\forall \xi \in \mathbb{R}^d$,
- $\sup_{i \in I} |Q_i|^{1/p-1} \|\mathcal{F}^{-1} \psi_i\|_{L_p} < \infty$, $\forall p \in (0, 1]$.

Given $\psi_i \in \Psi$, we define the multiplier $\psi_i(D)f := \mathcal{F}^{-1}(\psi_i \mathcal{F} f)$, $f \in L_2(\mathbb{R}^d)$. By a standard result on band-limited multipliers [46, Proposition 1.5.1], the conditions in Definition 2.3 ensure that $\psi_i(D)$ extends to a bounded operator on $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, uniformly in $i \in I$.

We can now give the definition of a decomposition space on the Fourier side. For particular choices of coverings, the decomposition spaces yield classical spaces such as Besov and modulation spaces, see [18]. Recall that a (quasi-)Banach sequence space Y on I is called solid if $|a_i| \leq |b_i|$ for all i implies that $\|\{a_i\}\|_Y \leq \|\{b_i\}\|_Y$.

Definition 2.4. Let $\mathcal{Q} = \{Q_i\}_{i \in I}$ be an admissible covering of \mathbb{R}^d for which there exists a BAPU Ψ . Let Y be a solid (quasi-)Banach sequence space on I satisfying that $\ell_0(I)$ is dense in Y . Then for $p \in (0, \infty]$, we define the decomposition space $D(\mathcal{Q}, L_p, Y)$ as the set of functions $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying

$$(2.1) \quad \|f\|_{D(\mathcal{Q}, L_p, Y)} := \left\| \left\{ \|\psi_i(D)f\|_{L_p} \right\}_{i \in I} \right\|_Y < \infty,$$

Remark 2.5. Using the assumption that $\ell_0(I)$ is dense in Y , it can be verified that $\mathcal{S}(\mathbb{R}^d)$ is dense in $D(\mathcal{Q}, L_p, Y)$, see e.g. [18].

Remark 2.6. We use a slightly different notation than [18]. Since we measure the “local component” of a function in \mathcal{FL}_p , $D(\mathcal{Q}, L_p, Y)$ is written $D(\mathcal{Q}, \mathcal{FL}_p, Y)$ in the notation of [18].

Given a decomposition space $D(\mathcal{Q}, L_p, Y)$, it is important to know how stable the spaces is relative to the choice of \mathcal{Q} and BAPU, i.e., if we modify \mathcal{Q} or the BAPU slightly, will $D(\mathcal{Q}, L_p, Y)$ change too? Usually one would prefer $D(\mathcal{Q}, L_p, Y)$ to be independent of the particular choice of BAPU and to remain unchanged under small geometric modifications of \mathcal{Q} . This problem has been studied in [18], and we list some of the results in the following section.

2.1. Some basic properties of decomposition spaces. Let us introduce some notation needed to study properties of admissible coverings. Given an admissible covering $\{Q_i\}_{i \in I}$ of \mathbb{R}^d and a subset $J \subset I$, we define

$$\tilde{J} := \{i \in I : \exists j \in J \text{ s.t. } Q_i \cap Q_j \neq \emptyset\}.$$

Furthermore, let $\tilde{J}^{(0)} := J$, and define inductively $\tilde{J}^{(k+1)} := \widetilde{\tilde{J}^{(k)}}$, $k \geq 0$. We write $\tilde{i}^{(k)} := \widetilde{\{i\}^{(k)}}$ and $\tilde{i} := \widetilde{\{i\}}$ for a singleton set. Notice that $\tilde{i} := \{j \in I : Q_i \cap Q_j \neq \emptyset\}$. Finally we define

$$\widetilde{Q}_i^{(k)} := \bigcup_{j \in \tilde{i}^{(k)}} Q_j \quad \text{and} \quad \tilde{\psi}_i := \sum_{j \in \tilde{i}} \psi_j,$$

for $\{\psi_i\}_{i \in I}$ an associated BAPU.

Next we consider the notion of a \mathcal{Q} -regular sequence space. Such sequence spaces induce decomposition spaces which are independent of the particular choice of BAPU.

Definition 2.7. A solid (quasi-)Banach sequence space Y on I is called \mathcal{Q} -regular if $h \in Y$ implies that $h^+ \in Y$, where $h^+(i) = \sum_{j \in \tilde{i}} h(j)$, $i \in I$.

It can be verified that the decomposition space $D(\mathcal{Q}, L_p, Y)$ is independent of the particular BAPU provided that Y is \mathcal{Q} -regular, see [18, Thm. 2.3].

To study the stability of $D(\mathcal{Q}, L_p, Y)$ under a modification of \mathcal{Q} , we need the notion of equivalent coverings.

Definition 2.8. Let $\mathcal{Q} = \{Q_i\}_{i \in I}$ and $\mathcal{P} = \{P_j\}_{j \in J}$ be two coverings of \mathbb{R}^d . \mathcal{Q} is called subordinate to \mathcal{P} if for every $i \in I$ there exists $j \in J$ such that $Q_i \subseteq P_j$. \mathcal{Q} is called almost subordinate to \mathcal{P} (written $\mathcal{Q} \leq \mathcal{P}$) if there exists $k \in \mathbb{N}$ such that \mathcal{Q} is subordinate to $\{\tilde{P}_j^{(k)}\}_{j \in J}$. If $\mathcal{Q} \leq \mathcal{P}$ and $\mathcal{P} \leq \mathcal{Q}$ the two coverings are called *equivalent* and we write $\mathcal{Q} \sim \mathcal{P}$.

Furthermore, to ensure stability of $D(\mathcal{Q}, L_p, Y)$ under a modification of \mathcal{Q} , we need some stability criteria for sequence spaces.

Definition 2.9. Let $\mathcal{Q} = \{Q_i\}_{i \in I}$ be an admissible covering. A strictly positive function w on \mathbb{R}^d is called \mathcal{Q} -moderate if there exists $C > 0$ such that $w(x) \leq Cw(y)$ for all $x, y \in Q_i$ and all $i \in I$. A strictly positive \mathcal{Q} -moderate weight on I (derived from w) is a sequence $v_i = w(x_i)$, $i \in I$, with $x_i \in Q_i$ and w a \mathcal{Q} -moderate function.

For Y a solid (quasi-)Banach sequence space on I , we define the weighted space $Y_v := \{d_i\}_{i \in I} : \{d_i v_i\}_{i \in I} \in Y\}$.

Remark 2.10. Consider two equivalent admissible coverings $\mathcal{Q} = \{Q_i\}_{i \in I}$ and $\mathcal{P} = \{U_j\}_{j \in J}$ and assume that w is a \mathcal{Q} -moderate function. Let $\{v_i\}_{i \in I}$ and $\{u_j\}_{j \in J}$ be weights on I and J , respectively, derived from w . Then one can easily check that (uniformly) $v_i \asymp v_j$ whenever $Q_i \cap U_j \neq \emptyset$.

A solid (quasi-)Banach sequence space on I is called symmetric if it is invariant under permutation $\rho: I \rightarrow I$. Notice that for a moderate weight v and a symmetric solid (quasi-)Banach sequence space Y we have

$$\|f\|_{D(\mathcal{Q}, L_p, Y_v)} \asymp \left\| \left\{ v_i \|\tilde{\psi}_i(D)f\|_{L_p} \right\}_{i \in I} \right\|_Y.$$

The following important result on the stability of decomposition spaces was proved in [18, p. 117]. The result shows that decomposition spaces are independent of the choice of BAPU and they remain unchanged under certain geometric modifications of the admissible cover \mathcal{Q} . We say that a solid (quasi-)Banach sequence space Y on a countable index set I corresponds to a solid (quasi-)Banach sequence space \bar{Y} on \mathbb{N} if there exists a bijection $\gamma: I \rightarrow \mathbb{N}$ such that $Y = \gamma_*(\bar{Y})$.

Theorem 2.11. *Let \mathcal{Q} and \mathcal{P} be two equivalent countable admissible coverings of \mathbb{R}^d , and let $\Psi = \{\psi_i\}_{i \in I}$ and $\Phi = \{\varphi_j\}_{j \in J}$ be corresponding BAPUs. Assume that Y and W are symmetric (quasi-)Banach sequence space derived from the same (quasi-)Banach sequence space \bar{Y} on \mathbb{N} , and that $\{v_i\}_{i \in I}$ and $\{u_j\}_{j \in J}$ are weights derived from the same moderate function w . Then*

$$D(\mathcal{Q}, L_p, Y_v) = D(\mathcal{P}, L_p, W_u),$$

with equivalent norms.

One elementary example of a symmetric (quasi-)Banach sequence space that will be used in Section 4.2 is $\ell_\tau(I)$, $0 < \tau < \infty$.

Remark 2.12. To be precise, Theorem 2.11 is only considered for Banach spaces, $B = L_p(\mathbb{R}^d)$ and Y , in [18]. However, all the arguments in the proof given in [18] hold true for the quasi-Banach spaces in Theorem 2.11 using the fact that the multipliers $\psi_i(D)$, $i \in I$, and $\varphi_j(D)$, $j \in J$, are uniformly bounded on $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$.

3. STRUCTURED COVERINGS AND ASSOCIATED FRAMES

In this section we consider a restricted class of admissible coverings having some additional structure that will allow us to construct associated tight frames for $L_2(\mathbb{R}^d)$ that will also give atomic decompositions of the associated decomposition spaces. The coverings are simple in the sense that they are obtained by applying a countable family of invertible affine maps to a fixed subset $Q \subset \mathbb{R}^d$.

For an invertible matrix $A \in GL(d, \mathbb{R})$ and a constant $c \in \mathbb{R}^d$, we define the affine transformation

$$T\xi := A\xi + c, \quad \xi \in \mathbb{R}^d.$$

For a subset $Q \subset \mathbb{R}^d$ we let $Q_T := T(Q)$, and for notational convenience we define $|T| := |A| := |\det A|$. For admissible coverings generated by a nice family of affine transformations it is possible to construct a corresponding BAPU. The following result is based on the elementary fact that the Fourier transform is well-behaved under an affine change of variables.

Proposition 3.1. *Given a countable family \mathcal{T} of invertible affine transformations on \mathbb{R}^d . Suppose there exist two bounded open sets $P \subset Q \subset \mathbb{R}^d$, with P compactly contained in Q , satisfying*

$$(3.1) \quad \{P_T\}_{T \in \mathcal{T}} \quad \text{and} \quad \{Q_T\}_{T \in \mathcal{T}} \quad \text{are admissible coverings.}$$

Then there exist

- (i) A BAPU $\{\psi_T\}_{T \in \mathcal{T}} \subset \mathcal{S}(\mathbb{R}^d)$ corresponding to $\{Q_T\}_{T \in \mathcal{T}}$.
- (ii) A system $\{\varphi_T\}_{T \in \mathcal{T}} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying
 - $\text{supp}(\varphi_T) \subset Q_T, \forall T \in \mathcal{T}$,
 - $\sum_{T \in \mathcal{T}} \varphi_T^2(\xi) = 1, \forall \xi \in \mathbb{R}^d$,
 - $\sup_{T \in \mathcal{T}} |T|^{\frac{1}{p}-1} \|\mathcal{F}^{-1} \varphi_T\|_{L_p} < \infty, \forall p \in (0, 1]$.
- (iii) A pair of systems $\{\widetilde{g}_T\}_{T \in \mathcal{T}}, \{g_T\}_{T \in \mathcal{T}} \subset \mathcal{S}(\mathbb{R}^d)$, where $g_T(\xi) = \Phi(T^{-1}\xi)$ for a fixed function $\Phi \in \mathcal{S}(\mathbb{R}^d)$. The functions satisfy
 - $\text{supp}(\widetilde{g}_T), \text{supp}(g_T) \subset Q_T, \forall T \in \mathcal{T}$,
 - $\sum_{T \in \mathcal{T}} \widetilde{g}_T(\xi) g_T(\xi) = 1, \forall \xi \in \mathbb{R}^d$,
 - $\sup_{T \in \mathcal{T}} |T|^{\frac{1}{p}-1} \|\mathcal{F}^{-1} \widetilde{g}_T\|_{L_p} < \infty, \forall p \in (0, 1]$
 - $\sup_{T \in \mathcal{T}} |T|^{\frac{1}{p}-1} \|\mathcal{F}^{-1} g_T\|_{L_p} < \infty, \forall p \in (0, 1]$.

Proof. Notice that $|P_T| \asymp |T| \asymp |Q_T|$ uniformly in $T \in \mathcal{T}$. We begin by proving (i). Pick a non-negative function $\Phi \in C^\infty(\mathbb{R}^d)$ with $\Phi(\xi) = 1$ for $\xi \in P$ and $\text{supp}(\Phi) \subset Q$. For $T \in \mathcal{T}$, we let $g_T(\xi) := \Phi(T^{-1}\xi)$. Clearly, $g_T \in C^\infty(\mathbb{R}^d)$ with $P_T \subset \text{supp}(g_T) \subset Q_T$. Since we want a BAPU, we consider the (locally finite) sum $g(\xi) := \sum_{T \in \mathcal{T}} g_T(\xi)$. Notice that there exists a constant N such that $1 \leq g(\xi) \leq N$ since $\{Q_T\}_{T \in \mathcal{T}}$ has finite height and $\{P_T\}_{T \in \mathcal{T}}$ covers \mathbb{R}^d . Thus, we can define a smooth resolution of the identity by $\psi_T(\xi) := g_T(\xi)/g(\xi)$.

In order to conclude, we need to verify that $\sup_T |T|^{1/p-1} \|\mathcal{F}^{-1} \psi_T\|_{L_p} < \infty$, for all $p \in (0, 1]$. Let

$$h_T(\xi) := \psi_T(T\xi) = \frac{\Phi(\xi)}{g(T\xi)}.$$

According to Lemma 3.2 below we have

$$\|\mathcal{F}^{-1} \psi_T\|_{L_p} = |T|^{1-1/p} \|\mathcal{F}^{-1} h_T\|_{L_p}.$$

It is easy to verify that for every $\beta \in \mathbb{N}_0^d$ there exists a constant C_β independent of $T \in \mathcal{T}$ such that

$$(3.2) \quad |\partial_\xi^\beta h_T(\xi)| \leq C_\beta \chi_Q(\xi).$$

Thus, using integration by parts, we have

$$\begin{aligned} \|\mathcal{F}^{-1} h_T\|_{L_p}^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} h_T(\xi) e^{ix \cdot \xi} d\xi \right|^p dx \\ &\leq C_d \left(\sum_{|\beta| \leq \lceil (d+1)/p \rceil} \|\partial^\beta h_T\|_{L_1} \right)^p \int_{\mathbb{R}^d} (1 + |x|)^{-d-1} dx < \infty. \end{aligned}$$

We conclude that $\{\psi_T\}_{T \in \mathcal{T}}$ is a BAPU corresponding to the admissible covering $\{Q_T\}_{T \in \mathcal{T}}$. This proves (i). To prove (ii), we repeat the above argument using the functions $\varphi_T(\xi) = g_T(\xi)/\sqrt{\sum_{T' \in \mathcal{T}} g_{T'}^2(\xi)}$, $T \in \mathcal{T}$. Finally, for (iii) we use the functions g_T from above with “dual” system defined by $\widetilde{g}_T = g_T/(\sum_{T' \in \mathcal{T}} g_{T'}^2(\xi))$. It is then easy to verify that all conditions of (iii) are satisfied. \square

The following elementary Lemma is used in the proof of Proposition 3.1.

Lemma 3.2. *Given $p \in (0, \infty]$, and an invertible affine transformation T . For a function $f \in L_p(\mathbb{R}^d)$ let $\hat{f}_T(\xi) := \hat{f}(T^{-1}\xi)$. Then*

$$\|f_T\|_{L_p} = |T|^{1-1/p} \|f\|_{L_p}, \quad 0 < p \leq \infty.$$

Proof. Notice that if $T = A \cdot + c$, then $f_T(x) = |T|e^{ix \cdot c} f(A^\top x)$. The result now follows by a simple substitution. \square

3.1. Frames and structured admissible coverings. The purpose of this section is to construct frames and tight frames for $L_2(\mathbb{R}^d)$ with frequency localization compatible with any given structured covering of \mathbb{R}^d satisfying the regularity conditions given in Proposition 3.1.

Definition 3.3. Given a family \mathcal{T} of invertible affine transformations on \mathbb{R}^d . Suppose there exist two bounded open sets $P \subset Q \subset \mathbb{R}^d$, with P compactly contained in Q , such that

$$(3.3) \quad \{P_T\}_{T \in \mathcal{T}} \quad \text{and} \quad \{Q_T\}_{T \in \mathcal{T}} \quad \text{are admissible coverings.}$$

Then we call $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ a *structured admissible covering* and \mathcal{T} a structured family of affine transformations.

In the subsequent sections, given a structured admissible covering, the functions φ_T and ψ_T will always be those given in Proposition 3.1 if nothing else is mentioned.

3.1.1. Tight frames. Let us now consider a construction of a tight frame adapted to a given admissible covering based on the partition of unity given by Proposition 3.1(ii). The idea we employ has been used to construct time-frequency frames before. A similar approach was used by Candés and Donoho in their second generation curvelet construction [43] and by the authors in [1]. Curvelets correspond to one particular decomposition of the frequency space, and generalized curvelet-type tight frames will be considered in Section 7.

Consider a structured admissible covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$. Suppose K_a is a cube in \mathbb{R}^d (aligned with the coordinate axes) with side-length $2a$ satisfying $Q \subseteq K_a$. Then we define

$$e_{n,T}(\xi) := (2a)^{-\frac{d}{2}} |T|^{-\frac{1}{2}} \chi_{K_a}(T^{-1}\xi) e^{i\frac{\pi}{a}n \cdot T^{-1}\xi}, \quad n \in \mathbb{Z}, T \in \mathcal{T}.$$

and

$$(3.4) \quad \hat{\eta}_{n,T} := \varphi_T e_{n,T} \quad n \in \mathbb{Z}^d, T \in \mathcal{T},$$

with φ_T given in Proposition 3.1. We can also obtain an explicit representation of $\eta_{n,T}$ in direct space. Suppose $T = A \cdot + c$, and $\hat{\mu}_T(\xi) := \varphi_T(T\xi)$. Then

$$(3.5) \quad \eta_{n,T}(x) = (2a)^{-\frac{d}{2}} |T|^{1/2} \mu_T\left(\frac{\pi}{a}n + A^\top x\right) e^{ix \cdot c}.$$

It can easily be verified that $\hat{\mu}_T$ satisfies $|\partial_\xi^\beta \hat{\mu}_T(\xi)| \leq C_\beta \chi_Q(\xi)$ for every $\beta \in \mathbb{N}^d$. Thus, we have for any $N \in \mathbb{N}$,

$$(3.6) \quad |\mu_T(x)| \leq C(1 + |x|)^{-N} \left| \sum_{|\beta| \leq N} x^\beta \mu_T(x) \right| \leq C(1 + |x|)^{-N} \sum_{|\beta| \leq N} \|\partial_\xi^\beta \hat{\mu}_T\|_{L_1} \leq C_N(1 + |x|)^{-N},$$

independent of $T \in \mathcal{T}$. We notice that $\eta_{n,T}$ is obtained by translating, “dilating”, and modulating a unit-scale element μ_T . In some sense, $\eta_{n,T}$ is a mix between a Gabor and a wavelet system. We discuss this matter in more detail in Section 3.2.

Let us study some basic properties of the system $\{\eta_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}}$. Our starting point is the simple observation that $\{e_{n,T}\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis for $L_2(T(K_a))$. We can actually use this fact to conclude that $F := F(\mathcal{T}) := \{\eta_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}}$ is a tight frame for $L_2(\mathbb{R}^d)$.

Proposition 3.4. *Let $F := F(\mathcal{T}) := \{\eta_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}}$ be the system defined in (3.4) for a structured admissible covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$. For all $f \in L_2(\mathbb{R}^d)$ we have*

$$\|f\|_{L_2} = \left(\sum_{T \in \mathcal{T}} \sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T} \rangle|^2 \right)^{1/2},$$

and equivalently,

$$(3.7) \quad f = \sum_{T \in \mathcal{T}} \sum_{n \in \mathbb{Z}^d} \langle f, \eta_{n,T} \rangle \eta_{n,T}.$$

Proof. Since $\{e_{n,T}\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis for $L_2(T(K_a))$ and $\text{supp}(\varphi_T) \subset T(K_a)$ we have

$$\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T} \rangle|^2 = \sum_{n \in \mathbb{Z}^d} |\langle \varphi_T \hat{f}, e_{n,T} \rangle|^2 = \|\varphi_T \hat{f}\|_{L_2}^2.$$

But $\{\varphi_T^2\}_{T \in \mathcal{T}}$ is a partition of unity, so

$$\sum_{T \in \mathcal{T}} \|\varphi_T \hat{f}\|_{L_2}^2 = \int_{\mathbb{R}^d} \sum_{T \in \mathcal{T}} \varphi^2(\xi) |\hat{f}(\xi)|^2 d\xi = \|f\|_{L_2}^2,$$

proving that $F(\mathcal{T})$ is a tight frame. Let us derive (3.7) directly. Notice that

$$\varphi_T \hat{f} = \sum_{n \in \mathbb{Z}^d} \langle \varphi_T \hat{f}, e_{n,T} \rangle e_{n,T} = \sum_{n \in \mathbb{Z}^d} \langle \hat{f}, \hat{\eta}_{n,T} \rangle e_{n,T}$$

and hence,

$$(3.8) \quad \varphi_T^2 \hat{f} = \sum_{n \in \mathbb{Z}^d} \langle \hat{f}, \hat{\eta}_{n,T} \rangle \varphi_T e_{n,T} = \sum_{n \in \mathbb{Z}^d} \langle \hat{f}, \hat{\eta}_{n,T} \rangle \hat{\eta}_{n,T}.$$

Again, since $\{\varphi_T^2\}_T$ is a partition of unity,

$$\hat{f} = \sum_{T \in \mathcal{T}} \varphi_T^2 \hat{f} = \sum_{T \in \mathcal{T}} \sum_{n \in \mathbb{Z}^d} \langle \hat{f}, \hat{\eta}_{n,T} \rangle \hat{\eta}_{n,T}$$

which gives the tight frame expansion (3.7). \square

Remark 3.5. Proposition 3.4 shows that we obtain a tight frame for $L_2(\mathbb{R}^d)$ regardless of the choice of cube K_a as long as $Q \subseteq K_a$. However, for practical purposes one should choose a as small as possible to avoid unnecessary “oversampling” which shows up in the formula for $e_{n,T}$ as the constant $(2a)^{-d/2}$.

3.1.2. An alternative construction. The tight frame in (3.5) is obtained by translating and “dilating” a unit-scale element μ_T . We notice that the unit-scale generator μ_T depends on T unlike for usual wavelet and Gabor bases where there is only a finite number of generators. In some cases it may be advantageous to have a fixed generator of the frame, and by sacrificing tightness of the frame we can achieve one generator by using the partition of unity given by Proposition 3.1.(iii). With the same notation and setup as in Section 3.1.1, we define two systems $\tilde{\Gamma} = \{\widetilde{\gamma}_{n,T}\}_{n,T}$ and $\Gamma = \{\gamma_{n,T}\}_{n,T}$ by

$$(3.9) \quad \widetilde{\gamma}_{n,T} := \widetilde{g}_T e_{n,T} \quad \text{and} \quad \hat{\gamma}_{n,T} := g_T e_{n,T}, \quad n \in \mathbb{Z}^d, T \in \mathcal{T},$$

with \widetilde{g}_T and g_T given in Proposition 3.1.(iii). The representation of $\widetilde{\gamma}_{n,T}$ in direct space is similar to (3.5) [use $\hat{\mu}_T(\xi) := \widetilde{g}_T(T\xi)$], but for $\gamma_{n,T}$ we have a simplified formula. Suppose $T = A \cdot + c$, and $\hat{\Psi}(\xi) := g_T(T\xi) = \Phi(\xi)$. Then

$$(3.10) \quad \gamma_{n,T}(x) = (2a)^{-d/2} |T|^{1/2} \Psi\left(\frac{\pi}{a}n + A^\top x\right) e^{ix \cdot c},$$

with generator $\Psi \in \mathcal{S}(\mathbb{R}^d)$ that does not depend on T . We can use a slight modification of the proof of Proposition 3.4 to obtain the frame expansion

$$(3.11) \quad f = \sum_{n,T} \langle f, \gamma_{n,T} \rangle \widetilde{\gamma}_{n,T} = \sum_{n,T} \langle f, \widetilde{\gamma}_{n,T} \rangle \gamma_{n,T}, \quad \forall f \in L_2(\mathbb{R}^d).$$

The expansion (3.11) gives us the freedom to analyze a function with the simple system Γ and reconstruct with the more “complicated” dual system $\tilde{\Gamma}$, or the other way around. In what follows, we will only state our results for the tight frame from Section 3.1.1, but the reader can verify that each result can be modified in a straightforward way to a statement on the dual frames Γ and $\tilde{\Gamma}$.

3.2. Some remarks on the structure of the frame. The frames defined in (3.5) and (3.10) both have a structure that combines some important features of wavelet and Gabor systems. Consider the frame given by (3.10). The frame is obtained operating on a fixed unit-scale element Ψ by “dilation” by A , translation, and modulation. The dilation and translation structure is a feature also found in a generic wavelet system $\mathcal{W} = \{|A|^{1/2} \psi(A^j x - k)\}_{j,k}$ associated with a dilation matrix A . However, in (3.10) we do not require A to be expansive or even to preserve \mathbb{Z}^d , any invertible matrix will work as long as a structured admissible covering is created. Modulation is an integral part of generic Gabor systems, $\mathcal{G} = \{g(x - na) e^{ibm \cdot x}\}_{m,n}$, but in (3.10) modulation is combined with dilation to form a type of “mixed” atoms.

There are also differences between the frame in (3.10) and the systems \mathcal{W} and \mathcal{G} . The frame in (3.10) is not necessarily associated with a group or grid structure like \mathcal{W} and \mathcal{G} , and this gives us much freedom to create adaptable coverings of \mathbb{R}^d . The price for this flexibility is that the frame is redundant and cannot be made into an orthonormal basis unlike \mathcal{W} and \mathcal{G} (by choosing generators and grid points in a suitable way). From this point of view, the frame expansion (3.11) should perhaps be considered an adaptable variant of the φ -transform of Frazier and Jawerth [24, 23], see also the generalization in [5, 4].

A special type of structured admissible coverings generate frames that are of more “pure” wavelet type. Consider a structured admissible covering associated with a collection of linear transformations (i.e., the affine transformations all have translation factor 0). For such coverings, the oscillating factor disappears in (3.10) and we obtain a system more similar to a wavelet system. In fact, the fundamental sets in (3.3) cannot contain the origin when \mathcal{T} is a family of invertible linear maps, and consequently all moments of Ψ vanish similar to e.g. the Meyer wavelet. The other extreme case is more restricted. There we keep A fixed and equal to the identity map, and the family of affine transformations reduces to a family of translation operators. In this case we basically obtain a Gabor system with (possibly) irregular grid points.

4. CHARACTERIZATION OF DECOMPOSITION SPACES

Here we consider the tight frame $F(\mathcal{T})$ defined in Section 3.1 as a tool to study and characterize decomposition spaces. Since our interest lies beyond frame decompositions

of $L_2(\mathbb{R}^d)$, we will need the following lemma providing some basic facts about stability of the system $F(\mathcal{T})$ defined in (3.4) for a structured admissible covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$. The lemma below will be used later in Proposition 4.3 to obtain an explicit characterization of decomposition space by the canonical frame coefficients. We remind the reader that the functions φ_T and ψ_T are defined in Proposition 3.1.

Lemma 4.1. *Given $f \in \mathcal{S}(\mathbb{R}^d)$. For $0 < p \leq \infty$, we have*

$$\left(\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T} \rangle|^p \right)^{1/p} \leq C |T|^{\frac{1}{p} - \frac{1}{2}} \|\tilde{\psi}_T(D)f\|_{L_p}, \quad \text{and}$$

$$\|\psi_T(D)f\|_{L_p} \leq C' |T|^{\frac{1}{2} - \frac{1}{p}} \sum_{T' \in \tilde{\mathcal{T}}} \left(\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T'} \rangle|^p \right)^{1/p}.$$

with equivalence independent of $T \in \mathcal{T}$. When $p = \infty$ the sum over $n \in \mathbb{Z}^d$ is changed to sup.

Proof. Notice that the representation (3.5) together with the estimate (3.6) yields

$$(4.1) \quad \sup_{x \in \mathbb{R}^d} \|\{\eta_{n,T}(x)\}_{n \in \mathbb{Z}^d}\|_{\ell_p} \leq C_p |T|^{\frac{1}{2}} \quad \text{and} \quad \sup_{n \in \mathbb{Z}^d} \|\eta_{n,T}\|_{L_p} \leq C'_p |T|^{\frac{1}{2} - \frac{1}{p}}.$$

Suppose $p \leq 1$. It can easily be verified that $\text{supp}(\tilde{\psi}_T * \hat{\eta}_{n,T}(T \cdot)) \subset K_{2a}$ a cube with side-length $4a$. Hence, using the technical Lemma 4.2 below we obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T} \rangle|^p &= \sum_{n \in \mathbb{N}_0^d} |\langle \tilde{\psi}_T(D)f, \eta_{n,T} \rangle|^p \leq \sum_{n \in \mathbb{Z}^d} \|(\tilde{\psi}_T(D)f)\eta_{n,T}\|_{L_1}^p \\ &= C |T|^{1-p} \sum_{n \in \mathbb{Z}^d} \|(\tilde{\psi}_T(D)f)\eta_{n,T}\|_{L_p}^p \leq C' |T|^{1-\frac{p}{2}} \|\tilde{\psi}_T(D)f\|_{L_p}^p, \end{aligned}$$

giving the first inequality in the lemma. For the other estimate, notice that

$$\|\psi_T(D)f\|_{L_p} \leq C \sum_{T' \in \tilde{\mathcal{T}}} \|\varphi_{T'}^2(D)f\|_{L_p},$$

and by (3.8)

$$\|\varphi_{T'}^2(D)f\|_{L_p}^p \leq \sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T'} \rangle|^p \|\eta_{n,T'}\|_{L_p}^p \leq C |T|^{\frac{p}{2}-1} \sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T'} \rangle|^p.$$

For $1 < p < \infty$ the estimates follow using (4.1) for $p = 1$, together with Hölder's inequality (see e.g. [38, Sec. 2.5]). The case $p = \infty$ is left for the reader. \square

The following technical lemma was used to prove Lemma 4.1.

Lemma 4.2. *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$ satisfies $\text{supp}(f) \subset B(0, r)$ for some constant $r > 0$. Given an invertible affine transformation T , let $\hat{f}_T(\xi) := \hat{f}(T^{-1}\xi)$. Then for $0 < p \leq q \leq \infty$*

$$\|f_T\|_{L_q} \leq C |T|^{1/p-1/q} \|f\|_{L_p},$$

for a constant C independent of T .

Proof. Using Lemma 3.2 and a Nikolskii-Plancherel-Polya type inequality on f (see e.g. [46, p. 18]), we obtain

$$\|f_T\|_{L_q} = |T|^{1-1/q} \|f\|_{L_q} \leq |T|^{1-1/q} C \|f\|_{L_p}.$$

\square

We now turn to a characterization of $D(\mathcal{Q}, L_p, Y_v)$ in terms of the canonical frame coefficients. To simplify the notation, we let

$$\eta_{n,T}^p := |T|^{1/2-1/p} \eta_{n,T}$$

denote the function $\eta_{n,T}$ “normalized” in $L_p(\mathbb{R}^d)$, $p \in (0, \infty]$. We use Lemma 4.1, and the fact that $\mathcal{S}(\mathbb{R}^d)$ is dense in $D(\mathcal{Q}, L_p, Y_v)$, in Definition 2.4 to obtain the following result.

Proposition 4.3. *Let $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ be a structured admissible covering. Let Y be a symmetric (quasi-)Banach sequence space on \mathcal{T} , and let v be a \mathcal{Q} -moderate weight. Then, for $0 < p \leq \infty$, we have the characterization*

$$\|f\|_{D(\mathcal{Q}, L_p, Y_v)} \asymp \left\| \left\{ \left(\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T}^p \rangle|^p \right)^{1/p} \right\}_{T \in \mathcal{T}} \right\|_{Y_v},$$

with the usual modification for $p = \infty$.

Inspired by the characterization given by Proposition 4.3 we define the following coefficient space.

Definition 4.4. Given a structured admissible covering $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$, a symmetric (quasi-)Banach sequence space Y on \mathcal{T} , and a \mathcal{Q} -moderate weight v . Then for $0 < p \leq \infty$ we define the space $d(\mathcal{Q}, \ell_p, Y_v)$ as the set of coefficients $c = \{c_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}} \subset \mathbb{C}$ satisfying

$$\|c\|_{d(\mathcal{Q}, \ell_p, Y_v)} := \left\| \left\{ \left(\sum_{n \in \mathbb{Z}^d} |c_{n,T}|^p \right)^{1/p} \right\}_{T \in \mathcal{T}} \right\|_{Y_v} < \infty.$$

4.1. Atomic decomposition of decomposition spaces. The characterization given by Proposition 4.3 is an important step towards proving that the tight frame $F(\mathcal{T})$ from Section 3.1 gives an atomic decomposition of $D(\mathcal{Q}, L_p, Y_v)$. What remains is to study the stability of the canonical reconstruction operator from $d(\mathcal{Q}, \ell_p, Y_v)$ into $D(\mathcal{Q}, L_p, Y_v)$. We study the reconstruction operator later in this section, but let us first recall the formal definition of an atomic decomposition of a Banach space (see [31]).

Definition 4.5. Let X be a Banach space and let X_d be an associated Banach sequence space on \mathbb{N} . An atomic decomposition of X with respect to X_d is a sequence $\{(y_n, x_n) : n \in \mathbb{N}\} \subset X' \times X$, with X' the dual space to X , such that the following properties hold.

- (1) The coefficient operator $C_X : f \rightarrow \{\langle f, y_n \rangle\}_{n \in \mathbb{N}}$ is bounded from X into X_d .
- (2) Norm equivalence:

$$\|f\|_X \asymp \|\{\langle f, y_n \rangle\}_{n \in \mathbb{N}}\|_{X_d}, \quad f \in X.$$

- (3) We have

$$f = \sum_{n \in \mathbb{N}} \langle f, y_n \rangle x_n, \quad \forall f \in X.$$

When $X = L_2(\mathbb{R}^d)$ and $X_d = \ell_2(\mathbb{Z}^d)$, Definition 4.5 coincides with the usual definition of a frame for $L_2(\mathbb{R}^d)$.

Notice that the first two requirements of Definition 4.5 are satisfied by the sequence $\{\eta_{n,T}^p\}_{n,T}$ according to Proposition 4.3 for the space $D(\mathcal{Q}, \ell_p, Y_v)$ with associated sequence space $d(\mathcal{Q}, \ell_p, Y_v)$.

Let us now consider the canonical reconstruction operator $R : d(\mathcal{Q}, \ell_p, Y_v) \rightarrow D(\mathcal{Q}, L_p, Y_v)$. We need the following lemma to show that the reconstruction operator can be defined as a bounded linear operator.

Lemma 4.6. *Let $\mathcal{Q} = \{Q_T\}_{T \in \mathcal{T}}$ be a structured admissible covering. Let Y be a symmetric (quasi-)Banach sequence space on \mathcal{T} , and let v be a \mathcal{Q} -moderate weight. Then, for $0 < p \leq \infty$, and for any finite sequence $\{c_{n,T}\}_{n \in \mathbb{N}_0^d, T \in \mathcal{T}}$ of complex numbers we have*

$$\left\| \sum_{n \in \mathbb{Z}^d, T \in \mathcal{T}} c_{n,T} |T|^{1/p-1/2} \eta_{n,T} \right\|_{D(\mathcal{Q}, L_p, Y_v)} \leq C \|\{c_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}}\|_{d(\mathcal{Q}, \ell_p, Y_v)}.$$

Proof. Let $f = \sum_{n \in \mathbb{Z}^d, T' \in \mathcal{T}} c_{n,T'} \eta_{n,T'}$. For $T \in \mathcal{T}$ we have

$$\|\psi_T(D)f\|_{L_p} = \left\| \psi_T(D) \left(\sum_{T' \in \tilde{\mathcal{T}}} \sum_{n \in \mathbb{Z}^d} c_{n,T'} \tilde{\eta}_{n,T'} \right) \right\|_{L_p} \leq C \sum_{T' \in \tilde{\mathcal{T}}} \left\| \sum_{n \in \mathbb{Z}^d} c_{n,T'} \eta_{n,T'} \right\|_{L_p}$$

since $\psi_T(D)$ is a bounded operator on $L_p(\mathbb{R}^d)$ uniformly in T . Now, the same technique as in the proof of Lemma 4.1 yields

$$\left\| \sum_{n \in \mathbb{N}_0^d} c_{n,T'} \eta_{n,T'} \right\|_{L_p} \leq C |T'|^{\frac{1}{2} - \frac{1}{p}} \|\{c_{n,T'}\}_{n \in \mathbb{Z}^d}\|_{\ell_p}, \quad 0 < p \leq \infty,$$

uniformly in $T \in \mathcal{T}$. The lemma follows since Y is symmetric and v is \mathcal{Q} -moderate. \square

We can now finally verify that $F(\mathcal{T})$ forms a Banach frame for the decomposition spaces. Using Lemma 4.6, we define the coefficient operator $C : D(\mathcal{Q}, L_p, Y_v) \rightarrow d(\mathcal{Q}, \ell_p, Y_v)$ by $Cf = \{\langle f, \eta_{n,T}^p \rangle\}_{n,T}$ and the reconstruction operator $R : d(\mathcal{Q}, \ell_p, Y_v) \rightarrow D(\mathcal{Q}, L_p, Y_v)$, by $\{c_{n,T}\}_{n,T} \rightarrow \sum_{n,T} c_{n,T} |T|^{1/p-1/2} \eta_{n,T}$.

Theorem 4.7. *Given $0 < p \leq \infty$. Then the coefficient operator C and reconstruction operator R are both bounded and makes $D(\mathcal{Q}, L_p, Y_v)$ a retract of $d(\mathcal{Q}, \ell_p, Y_v)$, i.e., $RC = \text{Id}_{D(\mathcal{Q}, L_p, Y_v)}$. In particular, for $p \geq 1$, $F(\mathcal{T})$ is an atomic decomposition of the spaces $D(\mathcal{Q}, L_p, Y_v)$.*

Proof. According to Proposition 4.3, the coefficient operator C is a bounded linear operator, and using Lemma 4.6 it is easy to verify that the reconstruction operator R is a bounded linear operator. Thus, (3.7) in Proposition 3.4 extends to a bounded splitting $RC = \text{Id}_{D(\mathcal{Q}, L_p, Y_v)}$ as illustrated in the following commuting diagram.

$$\begin{array}{ccc} D(\mathcal{Q}, L_p, Y_v) & \xrightarrow{\text{Id}_{D(\mathcal{Q}, L_p, Y_v)}} & D(\mathcal{Q}, L_p, Y_v) \\ & \searrow C & \nearrow R \\ & & d(\mathcal{Q}, \ell_p, Y_v) \end{array}$$

\square

As an interesting corollary to Proposition 4.7 and the characterization given by Proposition 4.3, we can introduce the following equivalent (quasi-)norm on $D(\mathcal{Q}, L_p, Y_v)$.

Corollary 4.8. *We have the following (quasi-)norm equivalence on $D(\mathcal{Q}, L_p, Y_v)$,*

$$(4.2) \quad \|f\|_{D(\mathcal{Q}, L_p, Y_v)} \asymp \inf \left\{ \|\{c_{n,T}\}_{n,T}\|_{d(\mathcal{Q}, \ell_p, Y_v)} : f = \sum_{n,T} c_{n,T} |T|^{1/p-1/2} \eta_{n,T} \right\}.$$

The corollary shows that when sparseness of expansion coefficients is measured by $d(\mathcal{Q}, \ell_p, Y_v)$, then the sparsest expansions (up to a constant) is actually the canonical frame expansion.

4.2. Application to nonlinear approximation. One of the important applications of sparse discrete expansions is to approximation problems. One can use efficient approximations of a given function to obtain a (lossy) compression of the function. Here we discuss nonlinear n -term approximation using the tight frames constructed in Section 3.1.1.

We begin by introducing a family of sparseness/smoothness spaces $S_{p,q}^\beta$ associated with a certain type of admissible covering and a special class of weights. The spaces $S_{p,q}^\beta$ have very simple characterizations in terms of frame coefficients.

Definition 4.9. Let \mathcal{T} be a structured countable family of invertible affine transformations with associated admissible covering \mathcal{Q} . Given $\beta \in \mathbb{R}$ and a \mathcal{Q} -moderate function w , define $v_{w,\beta} := \{(w(b_T))^\beta\}_{A_T \cdot + b_T \in \mathcal{T}}$. We let $S_{p,q}^\beta(\mathcal{T}, w)$ denote the decomposition space $D(\mathcal{Q}, L_p, (\ell_q)_{v_{w,\beta}})$ for $\beta \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q < \infty$.

Section 5 contains several examples to which Definition 4.9 applies, see also Section 5.1.1.

Let $\{\eta_{n,T}\}$ be the tight frame from Section 3.1 associated with \mathcal{T} . Notice that,

$$|\langle f, \eta_{n,T}^\tau \rangle| = |T|^{1/p-1/\tau} |\langle f, \eta_{n,T}^p \rangle|, \quad \text{for } 0 < \tau, p \leq \infty.$$

Suppose there exists a constant $\delta > 0$ such that $w(b_T) \asymp w_\delta(b_T) := |T|^{1/\delta}$ for $T \in \mathcal{T}$. Then, according to Proposition 4.3 we have the characterization

$$\begin{aligned} \|f\|_{S_{p,q}^\beta} &\asymp \left(\sum_T |T|^{\beta q/\delta} \left(\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T}^p \rangle|^p \right)^{q/p} \right)^{1/q} \\ (4.3) \quad &= \left(\sum_{T \in \mathcal{T}} \left(\sum_{n \in \mathbb{Z}^d} |\langle f, \eta_{n,T}^r \rangle|^p \right)^{q/p} \right)^{1/q}, \quad \text{when } \frac{\beta}{\delta} = \frac{1}{p} - \frac{1}{r}. \end{aligned}$$

In Section 5 we will see that several function spaces satisfy the above criterion.

We can use this simple characterization in conjunction with Corollary 4.8 to study the effect of thresholding the frame coefficients for functions in $\mathcal{S}_{\tau,\tau}^\gamma(\mathcal{T}, w_\delta)$. Suppose $f \in \mathcal{S}_{\tau,\tau}^\gamma(\mathcal{T}, w_\delta)$, and consider its frame expansion

$$f = \sum_{n,T} \langle f, \eta_{n,T} \rangle \eta_{n,T}.$$

Let $\{\theta_m\}_{m \in \mathbb{N}}$ be a decreasing rearrangement of the frame coefficients $\{|\langle f, \eta_{n,T}^r \rangle|\}_{n,T}$, where r is given by $\gamma/\delta = 1/\tau - 1/r$, and let f_n^F be the n -term approximation of f obtained by extracting from the frame expansion of f the terms corresponding to the n largest coefficients $\{\theta_m\}_{m=1}^n$. Assume $\beta \in \mathbb{R}$, and $p > 0$ satisfy $(\gamma - \beta)/\delta = 1/\tau - 1/p > 0$. Then using Corollary 4.8 and the fact that $\beta/\delta = 1/p - 1/r$, the approximation error in $S_{p,p}^\beta$ obeys

$$\begin{aligned} \|f - f_n^F\|_{S_{p,p}^\beta} &\leq C \left(\sum_{m>n} |\theta_m|^p \right)^{1/p} \\ (4.4) \quad &\leq C \|\theta\|_{w_{\ell_\tau}} \cdot n^{-(\gamma-\beta)/\delta} \leq C' \|f\|_{\mathcal{S}_{\tau,\tau}^\gamma} \cdot n^{-(\gamma-\beta)/\delta}, \end{aligned}$$

which follows from standard arguments, see e.g. [27]. An important special case of (4.4) is for $p = 2$ and $\beta = 0$, where $S_{2,2}^0(\mathcal{T}, w_\delta) = L_2(\mathbb{R}^d)$, and we obtain

$$\|f - f_n^F\|_{L_2} \leq C' \|f\|_{\mathcal{S}_{\tau,\tau}^\gamma} \cdot n^{-\gamma/\delta},$$

for $\gamma/\delta = 1/\tau - 1/2$.

The estimate (4.4) can also be used to obtain a Jackson inequality for nonlinear n -term approximation with $\mathcal{F}(\mathcal{T})$. Consider the nonlinear set of all possible n -term expansions using $\mathcal{F}(\mathcal{T})$,

$$\Sigma_n := \left\{ S = \sum_{(n,T) \in \Lambda} c_{n,T} \eta_{n,T} : \#\Lambda \leq n \right\},$$

and define the error of best n -term approximation as

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{S_{p,p}^\beta} := \inf_{g \in \Sigma_n} \|f - g\|_{S_{p,p}^\beta}.$$

Notice that $f_n^F \in \Sigma_m$ so (4.4) clearly implies that

$$(4.5) \quad \sigma_n(f, \mathcal{F}(\mathcal{T}))_{S_{p,p}^\beta} \leq C \|f\|_{S_{\tau,\tau}^\gamma} \cdot n^{-(\gamma-\beta)/\delta}, \quad n \geq 0,$$

for $1/\tau - 1/p = (\gamma - \beta)/\delta$.

5. CONSTRUCTION OF DECOMPOSITION SPACES AND ASSOCIATED FRAMES

The remaining part of the paper is devoted to constructing decomposition spaces with a suitable structure such that the results of Section 3 can be applied. Most importantly, given a reasonable partition of the frequency space, we would like to construct an adapted admissible covering along with an associated BAPU in order to define families of decomposition spaces based on various weights. In addition, an equivalent structured covering is needed to construct the tight frames of Section 3.1. In Section 5.1 below we consider a fairly general method to construct structured coverings made up of open balls in a space of homogeneous type over \mathbb{R}^d . The covering balls are chosen to have diameters given by a fixed function of their centers. The advantage of dealing with spaces of homogeneous type is that the decomposition spaces and associated frames can be adapted to any given anisotropy on \mathbb{R}^d . In Section 6 we apply the construction to obtain tight frames for Besov and anisotropic Besov spaces and we introduce a family of anisotropic α -modulation spaces which generalizes anisotropic Besov spaces.

5.1. A general construction of structured coverings. We now consider a general construction of structured coverings and associated decomposition spaces. The construction is inspired by Feichtinger's construction in [15] and uses the notion of a space of homogeneous type over \mathbb{R}^d along with an associated regulation function to create a structured covering based on balls with certain nice geometric properties.

Let us first recall the definition of a space of homogeneous type on \mathbb{R}^d . Spaces of homogeneous type over any nonempty set were introduced by Coifman and Weiss, see [9]. A function $d : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ is called a quasi-distance on \mathbb{R}^d if the following conditions are satisfied:

- a) $\forall x, y \in \mathbb{R}^d, d(x, y) = 0$ if and only if $x = y$,
- b) $\forall x, y \in \mathbb{R}^d, d(x, y) = d(y, x)$,
- c) there exists a constant $K \geq 1$ such that

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for every x, y and z in \mathbb{R}^d .

The d -ball $B_d(x, r)$ is defined by $B_d(x, r) = \{y \in \mathbb{R}^d : d(x, y) < r\}$. We equip \mathbb{R}^d with a topology by saying that $O \subseteq \mathbb{R}^d$ is open if for any $x \in O$ there exists $r > 0$ such that $B_d(x, r) \subseteq O$. In general, $B_d(x, r)$ may not be an open subset in \mathbb{R}^d but there is a result of

Macías and Segovia [36] which says that it is possible to replace d by a continuous quasi-distance d' that is equivalent to d , i.e., $d(x, y) \asymp d'(x, y)$ for every $x, y \in \mathbb{R}^d$. In particular, every d' -ball is open in \mathbb{R}^d . Below we will assume that d is a continuous quasi-distance on \mathbb{R}^d .

Let μ be a positive Borel measure on \mathbb{R}^d such that every d -ball $B_d(x, r)$ has finite μ -measure. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$(5.1) \quad 0 < \mu(B_d(x, 2r)) \leq A\mu(B_d(x, r)) < \infty.$$

We call the structure (\mathbb{R}^d, d, μ) a space of homogeneous type on \mathbb{R}^d .

Example 5.1. An example that will be used later is the quasi-distance $d(x, y)_{\mathbf{a}} := |x - y|_{\mathbf{a}} := \sum_{j=1}^d |x_j - y_j|^{1/a_j}$, where each $a_j > 0$. Then $(\mathbb{R}^d, d_{\mathbf{a}}, dx)$ is a spaces of homogeneous type on \mathbb{R}^d , see [9, Chap. 3]. We also notice that the topology induced by $d(x, y)_{\mathbf{a}}$ on \mathbb{R}^d is equivalent to the standard Euclidean topology.

We now turn to the construction of admissible coverings of \mathbb{R}^d . To keep the construction geometrically simple, we use a suitable collection of d -balls to cover \mathbb{R}^d . Another simplification is that we choose the radius of a given d -ball in the cover as a suitable function of its center. The following class of regulation functions will be useful for that purpose.

Definition 5.2. Let (\mathbb{R}^d, d, μ) be a space of homogeneous type. A function $h : \mathbb{R}^d \rightarrow (0, \infty)$ is called d -moderate if there exist constants $R, \delta_0 > 0$ such that $d(x, y) \leq \delta_0 h(x)$ implies $R^{-1} \leq h(y)/h(x) \leq R$.

Several examples of d -moderate functions will be considered in Section 6. We can use a d -moderate function to define a continuous covering of \mathbb{R}^d and then prune it to a “nice” admissible covering.

Lemma 5.3. Suppose (\mathbb{R}^d, d, μ) is a space of homogeneous type, and let $h : \mathbb{R}^d \rightarrow (0, \infty)$ be d -moderate with constants δ_0 and R , and pick $0 < \delta < \delta_0$. Then

- a) there exist an admissible cover $\{B_d(x_j, \delta h(x_j))\}_{j \in J}$ of \mathbb{R}^d and a constant $0 < \delta' < \delta$ such that $\{B_d(x_j, \delta' h(x_j))\}_{j \in J}$ are pairwise disjoint.
- b) Any two admissible coverings

$$\{B_d(x_i, \delta_1 h(x_i))\}_{i \in I} := \{\mathcal{A}_i\}_{i \in I} \quad \text{and} \quad \{B_d(y_j, \delta_2 h(y_j))\}_{j \in J} := \{\mathcal{B}_j\}_{j \in J}$$

of the type considered in a) satisfy $\sup_{i \in I} \#J(i) < \infty$ and $\sup_{j \in J} \#I(j) < \infty$, where

$$J(i) := \{j \mid j \in J, \mathcal{A}_i \cap \mathcal{B}_j \neq \emptyset\}, \quad I(j) := \{i \mid i \in I, \mathcal{B}_j \cap \mathcal{A}_i \neq \emptyset\}.$$

Proof. We begin by proving a). Fix $0 < \delta < \delta_0$. Let $U_x := B_d(x, \delta h(x))$. Define $\nu(x) = \{y \in \mathbb{R}^d \mid U_x \cap U_y \neq \emptyset\}$, and let $U'_x = \cup_{y \in \nu(x)} U_y$. Suppose $z \in U'_x$. We would like to estimate $d(x, z)$. Notice that $z \in U_y$ for some U_y with $U_x \cap U_y \neq \emptyset$. Pick a point $z_0 \in U_x \cap U_y$. Then

$$(5.2) \quad \begin{aligned} d(x, z) &\leq K(d(x, z_0) + K(d(z_0, y) + d(y, z))) \\ &< \delta K(h(x) + 2Kh(y)) \\ &\leq \delta Kh(x)(1 + 2KR^2), \end{aligned}$$

where K is the constant in the quasi-triangle inequality for d . Thus, $U'_x \subset B_d(x, R_1 \delta h(x))$ for $R_1 := K(1 + 2KR^2)$. Using Zorn’s Lemma, choose a maximal set $\{x_j\}_{j \in J} \subset \mathbb{R}^d$ such

that $\mathcal{C}_1 := \{V_{x_j}\}_{j \in J}$ are pairwise disjoint, with $V_x := B_d(x, \frac{\delta}{R_1}h(x))$. The maximality of the set ensures that $\{V'_{x_j}\}_{j \in J}$ covers \mathbb{R}^d , and consequently $\mathcal{Q} = \{B_d(x_j, \delta h(x_j))\}_{j \in J}$ also covers \mathbb{R}^d . Thus we only need to show that for \mathcal{Q} , $\#\tilde{i}$ is uniformly bounded for all $i \in J$. Let m be a constant satisfying $2^m \geq R_1$. Fix an $i \in J$. Then (5.1) yields

$$\mu\{B_d(x_j, \frac{\delta}{R_1}h(x_j))\} \geq A^{-2m} \mu\{B_d(x_j, R_1\delta h(x_j))\} \geq A^{-2m} \mu\{U_{x_i}\} \quad \text{for all } j \in \tilde{i}.$$

Now, since \mathcal{C}_1 are pairwise disjoint, we have

$$(5.3) \quad \#\tilde{i} \leq \sup_{j \in \tilde{i}} \frac{\mu\{B_d(x_i, R_1\delta h(x_i))\}}{\mu\{B_d(x_j, \frac{\delta}{R_1}h(x_j))\}} \leq A^{2m} \frac{\mu\{B_d(x_i, R_1\delta h(x_i))\}}{\mu\{U_{x_i}\}} \leq A^{3m}.$$

We now turn to the proof of b). Fix $i \in I$, and pick $j \in J(i)$. An estimate similar to the one in (5.2) shows that there exists a constant $C = C(K, R, \delta_1, \delta_2)$ such that for any $z \in B(y_j, \delta_2 h(y_j))$, $d(x_i, z) \leq Ch(x_i)$. Pick $\delta'_2 < \delta_2$ such that $\{B(y_j, \delta'_2 h(y_j))\}_{j \in J}$ are pairwise disjoint. Using the d -moderation of h we have $h(y_j) \geq R^{-2}h(x_i)$ for $j \in J(i)$, so there exists a constant C' such that

$$\#J(i) \leq \frac{\mu\{B_d(x_i, Ch(x_i))\}}{\mu\{B_d(y_j, \delta'_2 R^{-2}h(x_i))\}} \leq C'.$$

The estimate for $I(j)$ is similar. □

It is an open problem whether the coverings given by Lemma 5.3 are structured for a general quasi-distance d . However, if we impose one additional assumption on the quasi-distance d , it is possible to show that the coverings are indeed structured. We assume that the quasi-distance d is translation invariant and induced by a positive function $\rho : \mathbb{R}^d \rightarrow [0, \infty)$, i.e., $d(x, y) = \rho(x - y)$. Recall that an anisotropy on \mathbb{R}^d is a vector $\mathbf{a} = (a_1, a_2, \dots, a_d)$ of positive numbers, which we assume is normalized such that $\sum_{j=1}^d a_j = d$. For $t \geq 0$ the anisotropic dilation matrix $D_{\mathbf{a}}(t)$ is defined by $D_{\mathbf{a}}(t) = \text{diag}(t^{a_1}, \dots, t^{a_d})$. The function ρ is called homogeneous with respect to the anisotropy \mathbf{a} if $\rho(D_{\mathbf{a}}(t)\xi) = t\rho(\xi)$ for $t > 0$ and $\xi \in \mathbb{R}^d$. We have the following result on structured coverings.

Theorem 5.4. *Let \mathbf{a} be an anisotropy and suppose that the quasi-distance d is induced by ρ , which we assume is homogeneous with respect to \mathbf{a} . Suppose $h : \mathbb{R}^d \rightarrow (0, \infty)$ is d -moderate. Then*

- a) *the family $\mathcal{Q} = \{B_d(x_j, \delta h(x_j))\}_{j \in J}$ given in Lemma 5.3 is a structured admissible covering of \mathbb{R}^d . Moreover, the covering is countable if the topology induced by d is finer than the Euclidean topology on \mathbb{R}^d .*
- b) *Any two such families of structured admissible coverings are equivalent in the sense of Definition 2.8, i.e., h determines exactly one equivalence class of structured admissible coverings.*
- c) *Let \mathcal{P} and \mathcal{Q} be two equivalent countable coverings of the type from b), and Assume that Y and W are symmetric (quasi-)Banach sequence spaces derived from the same (quasi-)Banach sequence space \bar{Y} on \mathbb{N} , and that $\{v_i\}_{i \in I}$ and $\{u_j\}_{j \in J}$ are weights derived from the same moderate function w . Then*

$$D(\mathcal{Q}, L_p, Y_v) = D(\mathcal{P}, L_p, W_u),$$

with equivalent norms, i.e., h, \bar{Y} , and w determine exactly one decomposition space up to equivalent norms.

Proof. Let $\delta < \delta_0$, and $\{x_j\}_{j \in J}$ be the points given in Lemma 5.3. Define $P := B_d(0, 1)$, $Q = B_d(0, 2)$, and

$$T_j \xi = A_j \xi + x_j, \quad \text{where } A_j = D_{\mathbf{a}}(\delta h(x_j)).$$

If ρ is homogeneous with respect to \mathbf{a} , we have

$$\rho(T_j^{-1} \xi) = \rho\left(D_{\mathbf{a}}\left(\frac{1}{\delta h(x_j)}\right)(\xi - x_j)\right) = \frac{1}{\delta h(x_j)} \rho(\xi - x_j).$$

Thus,

$$P_{T_j} = \{\xi : \rho(T_j^{-1} \xi) < 1\} = \{\xi : \rho(\xi - x_j) < \delta h(x_j)\} = B_d(x_j, \delta h(x_j)).$$

Now, using Lemma 5.3 it is straight forward to verify that $\{P_{T_j}\}_{j \in J}$ and $\{Q_{T_j}\}_{j \in J}$ are admissible coverings. To prove countability of \mathcal{Q} when the topology induced by d is finer than the standard topology, we simply associate an Euclidean ball with rational radius and center to each set in \mathcal{Q} in such a way that these rational balls are pairwise disjoint.

Let us consider the proof of b). Recall that any d -ball $B_d(x, R)$ is open, and we claim that it is also arcwise connected. We can connect x to any $y \in B_d(x, R)$ by the continuous path $\gamma : [0, 1] \rightarrow B_d(x, R)$ given by $\gamma(t) = x + D_{\mathbf{a}}(t)(y - x)$. Consider any two admissible structured coverings \mathcal{Q} and \mathcal{P} of the type from a). It follows that each set in either \mathcal{Q} or \mathcal{P} is open and arcwise connected, and by b) in Lemma 5.3 that they have ‘‘finite overlap’’. We can therefore apply [20, Proposition 3.6] to conclude that $\mathcal{Q} \sim \mathcal{P}$ in the sense of Definition 2.8. According to [20, Proposition 3.5], \sim is an equivalence relation on admissible coverings, so h determines exactly one equivalence class of structured admissible coverings.

Part c) is a direct consequence of Theorem 2.11. \square

Theorem 5.4 ensures that we can construct a BAPU corresponding to the admissible covering given in Lemma 5.3, provided that h is d -moderate. The following result proved in [15] gives an abundance of d -moderate functions.

Lemma 5.5. *Suppose the quasi-distance d is induced by ρ , and let $h : \mathbb{R}^d \rightarrow (0, \infty)$ be a weakly subadditive function, i.e., there exist a constant C_1 such that $h(x + y) \leq C_1(h(x) + h(y))$, for all $x, y \in \mathbb{R}^d$. Assume furthermore that h satisfies the growth condition*

$$(5.4) \quad h(x) \leq C(1 + \rho(x)), \quad \text{for all } x \in \mathbb{R}^d.$$

Then h is d -moderate. In particular, suppose $s : [0, \infty) \rightarrow [1, \infty)$ is a non-decreasing function satisfying $s(2t) \leq K_s s(t)$ and $s(t) \leq C_s(1 + t)$ for all $t \geq 0$, then $h(x) = s(\rho(x))$ is weakly subadditive (and thus d -moderate).

5.1.1. *Generic admissible weights.* Here we consider two families of admissible weights needed to define the decomposition spaces. Suppose the assumptions of Theorem 5.4 are satisfied, and let

$$\mathcal{Q} = \{B_d(x_j, \delta h(x_j))\}_{j \in J}$$

be an associated countable structured covering generated by the affine transformations $\mathcal{T} = \{T_j\}_{j \in J}$. Notice that for any $x \in B_d(x_j, \delta h(x_j))$ we have (uniformly in j),

$$\rho(x) \leq K(d(x, x_j) + \rho(x_j)) \leq C(h(x_j) + \rho(x_j)),$$

and

$$\rho(x_j) \leq K(d(x, x_j) + \rho(x)) \leq C(h(x_j) + \rho(x)) \leq C'(h(x) + \rho(x)).$$

Now, if h satisfies the growth condition (5.4), then $1 + \rho(x)$ is a \mathcal{Q} -moderate function on \mathbb{R}^d , and thus $\{(1 + \rho(b_T))^\beta\}_{A_T + b_T \in \mathcal{T}, \beta \in \mathbb{R}}$ is a \mathcal{Q} -moderate weight.

In particular, Definition 4.9 applies and Theorem 5.4 shows that the space $S_{p,q}^\beta(\mathcal{T}, 1 + \rho)$ is well-defined and depends only on h and ρ . Henceforth, we will use the notation $S_{p,q}^\beta(h, \rho)$ to denote $S_{p,q}^\beta(\mathcal{T}, 1 + \rho)$.

Another natural weight is induced by the regulation function h . Observe that

$$|B_d(x_j, \delta h(x_j))| \asymp |T_j| \asymp \det \text{diag}[(\delta h(x_j))^{a_1}, \dots, (\delta h(x_j))^{a_d}] \asymp h(x_j)^d,$$

since $d = \sum_{i=1}^d a_i$. Hence, $|T_j|^{\beta/d} \asymp h(x_j)^\beta$, and clearly h^β is a \mathcal{Q} -moderate function on \mathbb{R}^d since h is d -moderate, so $\{|T_j|^{\beta/d}\}_{j \in J}$ is a \mathcal{Q} -moderate weight.

6. TIGHT FRAMES FOR SOME NEW (AND OLD) SMOOTHNESS SPACES

In this section we consider some specific applications of Theorem 5.4. The first two examples are based on the same choice of regulation function h but with different metrics on \mathbb{R}^d . For isotropic spaces we use the Euclidean metric in Theorem 5.4 to obtain new tight frames for Besov and α -modulation spaces. For a given anisotropy on \mathbb{R}^d , we use the quasi-metric induced by $|\cdot|_{\mathbf{a}}$ of Example 5.1 to generate tight frames for anisotropic Besov spaces and for anisotropic α -modulation spaces. Anisotropic α -modulation is a new family of spaces that “interpolate” between anisotropic Besov and classical modulation spaces. The third class of examples in Section 7 are not derived from regulation functions, but are inspired by the curvelet tight frames introduced by Candés and Donoho.

6.1. Isotropic spaces. Let us first consider \mathbb{R}^d with the Euclidean norm $|\cdot|$ and a simple family of regulation functions to create covers corresponding to classical isotropic spaces. We let $h_\alpha(x) = (1 + |x|)^\alpha$, where $\alpha \in [0, 1]$ is fixed. According to Lemma 5.5, h_α is moderate w.r.t. Euclidean distance since $(1 + 2t)^\alpha \leq 2^\alpha(1 + t)^\alpha$ for $t \geq 0$. Obviously the Euclidean distance is homogeneous w.r.t. the (an)isotropy $\mathbf{a} = (1, 1, \dots, 1)$, so we can use Theorem 5.4 to conclude that h_α determines one decomposition space (up to equivalent norms) for each choice of a suitable weight Y_v . In particular, $S_{p,q}^\beta(h_\alpha, |\cdot|)$ is well-defined. Observe that any covering ball B in \mathbb{R}^d associated with h_α satisfies the following simple geometric rule: $x \in B \Rightarrow (1 + |x|)^{\alpha d} \asymp |B|$. For $\alpha = 1$ this corresponds to a “dyadic” cover while $\alpha = 0$ corresponds to a uniform cover. Let us consider a more explicit representation of the respective spaces.

6.1.1. The case $\alpha = 1$. First we consider the case $\alpha = 1$, which corresponds to Besov spaces. Let $E_2 = \{\pm 1, \pm 2\}$, $E_1 := \{\pm 1\}$, and $E := E_2^d \setminus E_1^d$. For each $k \in E$, and $j \in \mathbb{N}$ define $b_{j,k} := 2^j(v(k_1), \dots, v(k_d))$, where

$$v(k) = \text{sgn}(k) \cdot \begin{cases} 1/2 & \text{for } k = 1 \\ 3/2 & \text{for } k = 2. \end{cases}$$

Suppose $\mathcal{T} = \{4I, T_{j,k}\}_{j \in \mathbb{N}, k \in E}$ is given by $T_{j,k}\xi = 2^j\xi + b_{j,k}$ and let $Q \subset \mathbb{R}^d$ be an open cube with center 0 and side length $r > 1/2$. Then $\{T_{j,k}Q\}_{j \in \mathbb{N}, k \in E}$ is a structured admissible covering of \mathbb{R}^d . Figure 1 illustrate this fact for $d = 2$. Using \mathcal{T} it can be verified that $S_{p,q}^\beta(h_1, |\cdot|)$ is a Besov space. In fact $S_{p,q}^\beta(h_1, |\cdot|) = B_{p,q}^\beta(\mathbb{R}^d)$, see e.g. [46, p. 85] for details. The tight frame $\mathcal{F}(\mathcal{T})$ yields an atomic decomposition of the Besov space $B_{p,q}^\beta(\mathbb{R}^d)$, and (4.3) gives a Lizorkin-type characterization of the norm on $B_{p,q}^\beta(\mathbb{R}^d)$. Since $|b_{j,k}| \asymp |T_{j,k}|^{1/d}$, Eq. (4.5) provides the Jackson estimate

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{B_{p,p}^\beta(\mathbb{R}^d)} \leq C \|f\|_{B_{\tau,\tau}^\gamma(\mathbb{R}^d)} \cdot n^{-(\gamma-\beta)/d}, \quad n \geq 0,$$

for $1/\tau - 1/p = (\gamma - \beta)/d$.

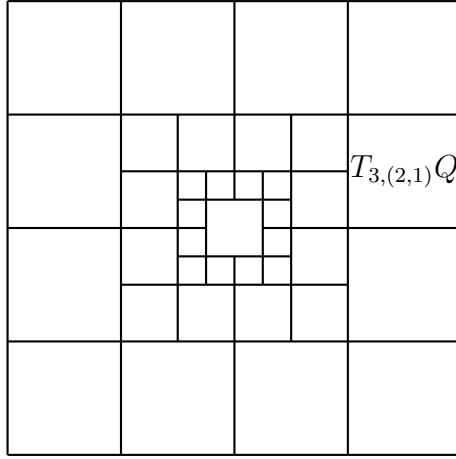


FIGURE 1. A decomposition of the frequency plane using the structured family \mathcal{T} from Section 6.1.1 on a fixed cube Q .

6.1.2. *The case $0 \leq \alpha < 1$.* Next we suppose $0 \leq \alpha < 1$, which corresponds to α -modulation spaces. Define $b_k = |k|k|^{\alpha/(1-\alpha)}$, $k \in \mathbb{Z}^d \setminus \{0\}$, and let $\mathcal{T} = \{T_k\}_{k \in \mathbb{Z}^d \setminus \{0\}}$ be given by $T_k \xi = |k|^{\alpha/(1-\alpha)} \xi + b_k$. This type of “polynomial” covering was first considered by Päiväranta and Somersalo in [40] to study pseudodifferential operators, and it was proven by the authors in [1] that (4.3) gives a characterization of the α -modulation space $M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$ introduced by Gröbner [1, 30]. We have $S_{p,q}^\beta(h_\alpha, |\cdot|) = M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$, so the tight frame $\mathcal{F}(\mathcal{T})$ gives an atomic decomposition of $M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$, and (4.3) gives a characterization of the norm on $M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$. In particular, for $\alpha = 0$, we obtain the classical modulation spaces introduced by Feichtinger [17]. In [1], the authors constructed a nontight frame for $M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$, and as far as we know, the frame given by (3.4) is the first example of a tight frame for $M_{p,q}^{\beta,\alpha}(\mathbb{R}^d)$. We have $|b_k| \asymp |T_k|^{\frac{1}{d\alpha}}$. Thus, if $0 < \alpha < 1$, Eq. (4.5) provides the Jackson estimate

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{M_{p,p}^{\beta,\alpha}(\mathbb{R}^d)} \leq C \|f\|_{M_{\tau,\tau}^{\gamma,\alpha}(\mathbb{R}^d)} \cdot n^{-(\gamma-\beta)/(d\alpha)}, \quad n \geq 0,$$

for $1/\tau - 1/p = (\gamma - \beta)/(d\alpha)$.

6.2. **Anisotropic spaces.** Next we consider the same construction as above but adapted to a given anisotropy \mathbf{a} on \mathbb{R}^d . Let d be induced by $|\cdot|_{\mathbf{a}}$ defined in Example 5.1, and define $h_{\alpha}^{\mathbf{a}}(x) = (1 + |x|_{\mathbf{a}})^{\alpha}$, which is d -moderate according to Lemma 5.5. It is easy to check that $|\cdot|_{\mathbf{a}}$ is homogeneous w.r.t. \mathbf{a} , so we can use Theorem 5.4 to conclude that $h_{\alpha}^{\mathbf{a}}$ determines one decomposition space (up to equivalent norms) for each choice of a suitable weight Y_{ν} . In particular, the space $S_{p,q}^{\beta}(h_{\alpha}^{\mathbf{a}}, |\cdot|_{\mathbf{a}})$ is well-defined. Also notice that for $\mathbf{a} = (1, 1, \dots, 1)$, $h_{\alpha}^{\mathbf{a}}$ is the regulation function considered in the isotropic case. In this particular case, we obtain the same isotropic spaces as in Section 6.1. In general, we obtain anisotropic version of the spaces considered in Section 6.1.

6.2.1. *The case $\alpha = 1$.* First suppose $\alpha = 1$, which will correspond to anisotropic Besov spaces. To see this, let us describe the anisotropic Lizorkin decomposition. Given the anisotropy $\mathbf{a} = (a_1, \dots, a_d)$, define the cubes

$$Q_j = \{x : |x_i| \leq 2^{ja_i}, i = 1, \dots, d\} \quad \text{for } j = 0, 1, \dots,$$

and the corridors $K_j = Q_j \setminus Q_{j-1}$ for $j \in \mathbb{N}$. Recall the set E from the isotropic Besov spaces. For each $j \in \mathbb{N}$ and $k \in E$ define

$$P_{j,k} = \{x \in \mathbb{R}^d : \text{sgn}(x_i) = \text{sgn}(k_i), \text{ and } (|k_i| - 1)2^{j-1} \leq |x_i|^{1/a_i} \leq |k_i|2^{j-1}\}.$$

Clearly, $K_j \subset \cup_{k \in E} P_{j,k}$. The family $\{Q_0, P_{j,k}\}$ gives an anisotropic Lizorkin decomposition of \mathbb{R}^d . The cubes $P_{j,k}$ can be generated by a family of affine transformations \mathcal{T} . In fact, let $c_{j,k}$ be the center of the cube $P_{j,k}$, and define the function $B: E \rightarrow \mathbb{R}^{d \times d}$ by

$$B(k) = \text{diag}(h(k)_1, \dots, h(k)_d), \quad \text{where } h(k)_i = \begin{cases} 2^{-(a_i+1)} & \text{if } |k_i| = 1, \\ (1 - 2^{-a_i})/2 & \text{if } |k_i| = 2. \end{cases}$$

Then the family $\mathcal{T} = \{D_{\mathbf{a}}(1), T_{j,k}\}_{j \in \mathbb{N}, k \in E}$ given by $T_{j,k}\xi = D_{\mathbf{a}}(2^j)B(k)\xi + c_{j,k}$ generates the sets $P_{j,k}$. For this choice it can be verified that $S_{p,q}^\beta(h_{\mathbf{1}}^{\mathbf{a}}, |\cdot|_{\mathbf{a}})$ is an anisotropic Besov space, see e.g. [15, p. 223]. In fact, $S_{p,q}^\beta(h_{\mathbf{1}}^{\mathbf{a}}, |\cdot|_{\mathbf{a}}) = B_{p,q}^{\mathbf{a},\beta}(\mathbb{R}^d)$, so the tight frame $\mathcal{F}(\mathcal{T})$ gives an atomic decomposition of $B_{p,q}^{\mathbf{a},\beta}(\mathbb{R}^d)$, and (4.3) gives a Lizorkin-type characterization of the corresponding norm. Notice that $|c_{j,k}|_{\mathbf{a}} \asymp 2^j \asymp |T_{j,k}|^{1/d}$, since $d = \sum_{i=1}^d a_i$. Thus Eq. (4.5) provides the Jackson estimate

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{B_{p,p}^{\mathbf{a},\beta}(\mathbb{R}^d)} \leq C \|f\|_{B_{\tau,\tau}^{\mathbf{a},\gamma}(\mathbb{R}^d)} \cdot n^{-(\gamma-\beta)/d}, \quad n \geq 0,$$

for $1/\tau - 1/p = (\gamma - \beta)/d$. Wavelet characterizations of anisotropic Besov spaces are considered in e.g. [26, 34].

6.2.2. *The case $0 \leq \alpha < 1$.* Next we consider the case $0 \leq \alpha < 1$, which corresponds to a family of spaces which we will call anisotropic α -modulation spaces. According to Theorem 5.4, $S_{p,q}^\beta(h_{\alpha}^{\mathbf{a}}, |\cdot|_{\mathbf{a}})$ is well-defined and we define the anisotropic α -modulation space $M_{p,q}^{\mathbf{a},\beta,\alpha}(\mathbb{R}^d)$ by

$$M_{p,q}^{\mathbf{a},\beta,\alpha}(\mathbb{R}^d) := S_{p,q}^\beta(h_{\alpha}^{\mathbf{a}}, |\cdot|_{\mathbf{a}}), \quad 0 < p \leq \infty, \quad 0 < q < \infty, \quad \beta \in \mathbb{R}.$$

One can check that for $\alpha = 0$, which corresponds to uniform coverings of the frequency space, $M_{p,q}^{\mathbf{a},\beta,0}(\mathbb{R}^d)$ reduces to the classical modulation space $M_{p,q}^{\beta,0}(\mathbb{R}^d)$. This follows from the fact that the unit ball in $(\mathbb{R}^d, |\cdot|_{\mathbf{a}})$ contains an Euclidean ball and is itself contained in some Euclidean ball.

Theorem 5.4 tells us that there exists a structured covering associated with $h_{\alpha}^{\mathbf{a}}$, but unlike the isotropic case, we do not (in general) know of any explicitly given structured covering. However, suppose $\mathcal{T} = \{A_k \cdot + b_k\}_{k \in \mathbb{Z}^d \setminus \{0\}}$ is one of the equivalent structured covering given by Theorem 5.4. The tight frame $\mathcal{F}(\mathcal{T})$ yields an atomic decomposition of $M_{p,q}^{\mathbf{a},\beta,\alpha}(\mathbb{R}^d)$, and (4.3) gives a characterization of the norm on $M_{p,q}^{\mathbf{a},\beta,\alpha}(\mathbb{R}^d)$. We have $|b_k|_{\mathbf{a}} \asymp |A_k|_{\mathbf{a}}^{1/d}$ since $\sum_{j=1}^d a_j = d$. Thus, if $0 < \alpha < 1$, Eq. (4.5) provides the Jackson estimate

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{M_{p,p}^{\mathbf{a},\beta,\alpha}(\mathbb{R}^d)} \leq C \|f\|_{M_{\tau,\tau}^{\mathbf{a},\gamma,\alpha}(\mathbb{R}^d)} \cdot n^{-(\gamma-\beta)/(d\alpha)}, \quad n \geq 0,$$

for $1/\tau - 1/p = (\gamma - \beta)/(d\alpha)$.

7. CURVELET-TYPE DECOMPOSITION SPACES

This final section is concerned with a study of decomposition spaces associated with certain splittings of the frequency space inspired by the decompositions considered by Candés and Donoho in their construction of curvelet tight frames for $L_2(\mathbb{R}^2)$. In [7] they considered so-called second generation curvelet tight frames obtained by a subdivision of

the dyadic coronas in \mathbb{R}^2 into small wedges satisfying the rule “length = height²”. A similar type of atomic decomposition was considered earlier by Smith in [42] for the purpose of studying Fourier integral operators. Below we adapt our general construction from Section 3 to generate structured coverings of \mathbb{R}^d that can be used to obtain curvelet-type tight frames for $L_2(\mathbb{R}^d)$, $d \geq 2$. We obtain a natural family of sparseness (smoothness) spaces associated to curvelet-type frames, and in Section 7.3 we show that the sparseness spaces for the second generation curvelets from [7] and the sparseness space for the curvelet-type frames from Section 7 are the same.

7.1. Curvelet-type decompositions. Our goal is to define curvelet-type tight frames for $L_2(\mathbb{R}^d)$, $d \geq 2$ by introducing structured decompositions of \mathbb{R}^d compatible with the time-frequency properties of curvelets. The frequency decomposition associated with curvelets is obtained by splitting each dyadic band $\{x \in \mathbb{R}^d : 2^j \leq |x| < 2^{j+1}\}$ into approximately $2^{j/2}$ equally sized “wedges”. For $d = 2$, it is straightforward to obtain such wedges by simply subdividing \mathbb{S}^1 into $2^{\lfloor j/2 \rfloor}$ equal pieces. For $d > 2$, the process is slightly more complicated and we need the following geometric lemma to ensure that we can find a certain number of approximately equidistant points on the unit sphere $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1} : |x| = 1\}$.

Lemma 7.1. *Let (\mathbb{S}^m, ρ) , $m \geq 1$, be the metric space on the unit sphere \mathbb{S}^m with the geodesic metric ρ . Let $B_{\mathbb{S}^m}(x, r) := \{y \in \mathbb{S}^m : \rho(x, y) < r\}$ be an open ball of geodesic radius $r > 0$ around $x \in \mathbb{S}^m$. For $0 < r$ there exists a set $\{x_k\}_{k=1}^L \subset \mathbb{S}^m$ of points satisfying $\{B_{\mathbb{S}^m}(x_k, r)\}_k$ are pairwise disjoint, $\mathbb{S}^m = \cup_k B_{\mathbb{S}^m}(x_k, 3r)$, and (uniformly in r) $L = \#\{x_k\} \asymp r^{-m}$ for $r \rightarrow 0$. Furthermore, there exists a constant $0 < A < \infty$ depending only on m such that $\#\tilde{k} \leq A$ for $k = 1, \dots, L$.*

Proof. Fix $r > 0$. Let $U_x = B_{\mathbb{S}^m}(x, r)$, $\nu(x) = \{y \in \mathbb{S}^m : U_x \cap U_y \neq \emptyset\}$, and $U'_x = \cup_{y \in \nu(x)} U_y$. By Zorn’s lemma there exists a maximal set $\{x_k\}_k \subset \mathbb{S}^m$ satisfying $\{U_{x_k}\}_k$ are pairwise disjoint. The maximality implies that $\{U'_{x_k}\}_k$ covers \mathbb{S}^m . Clearly, $U'_x \subseteq B_{\mathbb{S}^m}(x, 3r)$. Hence $\mathbb{S}^m = \cup_k B_{\mathbb{S}^m}(x_k, 3r)$. Let $d\sigma$ be the surface measure on \mathbb{S}^m . Since $\sigma(U_x) \asymp r^m$ and $\{U_{x_k}\}_k$ are pairwise disjoint, there exists a constant $0 < C_m < \infty$ depending only on m such that $\#\{x_k\} \leq C_m r^{-m}$. Likewise, $\sigma(B_{\mathbb{S}^m}(x, 3r)) \asymp (3r)^m$, which gives a lower bound $\#\{x_k\} \geq c_m 3^{-m} r^{-m}$. Finally, notice that

$$\sigma(B_{\mathbb{S}^m}(x_i, r)) \geq C_m 9^{-m} \sigma(B_{\mathbb{S}^m}(x_k, 9r))$$

for all i, k . Thus

$$\#\tilde{k} \leq \sup_i \frac{\sigma(B_{\mathbb{S}^m}(x_k, 9r))}{\sigma(B_{\mathbb{S}^m}(x_i, r))} \leq 9^m.$$

□

We can now define curvelet-type structured coverings for $d \geq 2$. For each $j \in \mathbb{N}$, let $\{x_{j,\ell}\}_{\ell \in L_j} \subset \mathbb{S}^{d-1}$ be a set of points obtained using Lemma 7.1 with $r = 2^{-j/2}$. It is easy to check that we have the uniform bound

$$\#L_{j+1}/\#L_j \leq C_d$$

for all $j \in \mathbb{N}$. Thus the wedges

$$W_{j,\ell}^1 = [2^j, 2^{j+1}] \times B_{\mathbb{S}^{d-1}}(x_{j,\ell}, 3 \cdot 2^{-j/2}), \quad j \in \mathbb{N}, \ell \in L_j$$

and

$$W_{j,\ell}^2 = [2^{j-1}, 2^{j+2}] \times B_{\mathbb{S}^{d-1}}(x_{j,\ell}, 7 \cdot 2^{-j/2}), \quad j \in \mathbb{N}, \ell \in L_j$$

are both admissible coverings of $\mathbb{R}^d \setminus B(0, 2)$.

Let $R_{j,\ell}$ be an orthogonal transformation which maps $(1, 0, \dots, 0) \in \mathbb{R}^d$ to the point $e_{j,\ell} \in \mathbb{R}^d$ given in polar coordinates by $(1, x_{j,\ell}) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$. Using the hyperspherical coordinates $\varphi_1, \dots, \varphi_{d-1}$, given by the relations

$$\begin{aligned} x_1 &= |x| \cos \varphi_1 \\ x_2 &= |x| \sin \varphi_1 \cos \varphi_2 \\ &\vdots \\ x_{d-1} &= |x| \sin \varphi_1 \cdots \sin \varphi_{d-2} \cos \varphi_{d-1} \\ x_d &= |x| \sin \varphi_1 \cdots \sin \varphi_{d-2} \sin \varphi_{d-1}, \end{aligned}$$

we have that $R_{j,\ell}^{-1}W_{j,\ell}^1$ can be described by $0 \leq \varphi_1 < 3 \cdot 2^{-j/2}$, $0 \leq \varphi_i < \pi$, $i = 2, \dots, d-2$ and $0 \leq \varphi_{d-1} < 2\pi$. A similar description holds for $R_{j,\ell}^{-1}W_{j,\ell}^2$.

Consider the parabolic dilation matrix

$$D_j = \text{diag}(2^j, 2^{j/2}, \dots, 2^{j/2}),$$

and define a family \mathcal{T} of invertible affine transformations on \mathbb{R}^d by

$$T_J \xi = R_J D_J \xi, \quad J \in \mathcal{J},$$

where $\mathcal{J} = \{(j, \ell) : j \in \mathbb{N}, \ell \in L_j\}$. Then by simple estimates on sine and cosine it can be verified that there exists a cube $Q \subset \mathbb{R}^d$ satisfying

$$T_{j,\ell}^{-1}W_{j,\ell}^1 \subset Q \subset T_{j,\ell}^{-1}W_{j,\ell}^2 \quad \forall (j, \ell) \in \mathcal{J}.$$

Thus $\mathcal{T} = \{T_0, T_J\}_{J \in \mathcal{J}}$ is a structured family of affine transformations for some T_0 satisfying $B(0, 2) \subset T_0 Q$.

We now define the associated curvelet-type system $F(\mathcal{T}) = \{\eta_{n,T}\}_{n \in \mathbb{Z}^d, T \in \mathcal{T}}$ by (3.4). According to Proposition 3.4, $F(\mathcal{T})$ is a tight frame for $L_2(\mathbb{R}^d)$.

For $d = 2$ the wedges $W_{j,\ell}^1$ are similar to those considered by Donoho and Candés in [7] (see also Section 7.3.1 below). Figure 2 illustrates a tiling of the frequency plane by such wedges.

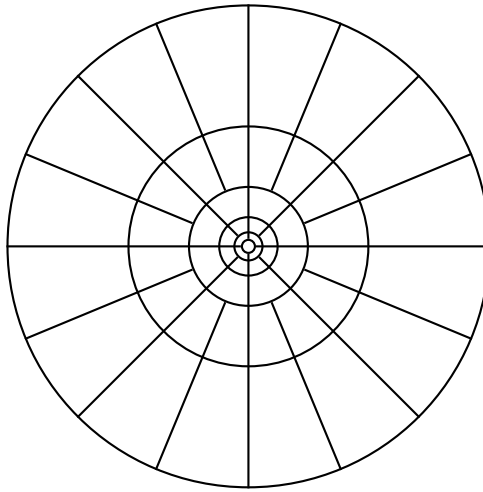


FIGURE 2. Sketch of a tiling of the frequency plane using the polar wedges in Section 7.1.

Remark 7.2. It is possible to use the parabolic dilation matrix

$$D_j = \text{diag}(a^j, b^j, \dots, b^j),$$

in the construction of the structured admissible family \mathcal{T} above provided $1 < b < a < \infty$. In fact, all the above arguments and the following results hold true with obvious modifications.

7.2. Curvelet spaces. Let $G_{p,q}^\beta$ denote the decomposition space corresponding to the curvelet-type decomposition, using the weights $|2^j e_{j,k}|^\beta = 2^{j\beta}$.

Since $2^j = |T_{j,k}|^{2/(d+1)}$, Eq. (4.5) provides the Jackson estimate

$$\sigma_n(f, \mathcal{F}(\mathcal{T}))_{G_{p,p}^\beta(\mathbb{R}^d)} \leq C \|f\|_{G_{\tau,\tau}^\gamma(\mathbb{R}^d)} \cdot n^{-2(\gamma-\beta)/(d+1)}, \quad n \geq 0,$$

for $1/\tau - 1/p = 2(\gamma - \beta)/(d + 1)$.

7.2.1. Stability of curvelet spaces. A number of choices were made in the construction of the curvelet-type frames (in particular, multiple points on \mathbb{S}^{d-1} are chosen), so it is a serious concern whether the associated decomposition spaces depend on the particular choice. The following lemma shows that this is not the case; any two choices will lead to decomposition spaces that agree up to equivalence of norms.

Lemma 7.3. *Let $G_{p,q}^\beta$ and $\widetilde{G}_{p,q}^\beta$ be any pair of curvelet spaces associated with the construction of Section 7.1. Then $G_{p,q}^\beta = \widetilde{G}_{p,q}^\beta$ with equivalent norms.*

Proof. Let $\{\varphi_{j,k}\}$ and $\{\widetilde{\varphi}_{j,k}\}$ be BAPUs generating the two curvelet spaces in question. It follows easily from the construction of the curvelet structured decomposition that $\text{supp}(\varphi_{j,k}) \cap \text{supp}(\widetilde{\varphi}_{j',k'}) \neq \emptyset$ implies $|j - j'| \leq 1$. Moreover, for $|j - j'| \leq 1$, we claim that

$$\sup_{k'} \#\{k : \text{supp}(\varphi_{j,k}) \cap \text{supp}(\widetilde{\varphi}_{j',k'}) \neq \emptyset\} < \infty,$$

and

$$\sup_k \#\{k' : \text{supp}(\varphi_{j,k}) \cap \text{supp}(\widetilde{\varphi}_{j',k'}) \neq \emptyset\} < \infty.$$

The claim follows from an argument similar to the one used to prove part b) of Lemma 5.3, where we use the properties of the points obtained from Lemma 7.1. The lemma now follows from Theorem 2.11 and this finite overlap property. \square

7.2.2. Curvelet and Besov spaces. It is well-known that Besov spaces can be defined using a partition of unity adapted to the dyadic frequency bands $\{x \in \mathbb{R}^d : 2^j \leq |x| < 2^{j+1}\}$, and the curvelet splitting of the frequency space can therefore be considered a refinement of the Besov space one. We use this observation below to obtain embedding results between curvelet spaces and Besov spaces.

Lemma 7.4. *Let $K = \frac{d-1}{2}$. For $0 < p \leq \infty$, $0 < q < \infty$, and $\beta \in \mathbb{R}$ we have*

$$B_{p,q}^{\beta+s}(\mathbb{R}^d) \hookrightarrow G_{p,q}^\beta(\mathbb{R}^d),$$

where $s = K \frac{1}{q}$. Likewise,

$$G_{p,q}^\beta(\mathbb{R}^d) \hookrightarrow B_{p,q}^{\beta-s'}(\mathbb{R}^d),$$

where $s' = K(\max(1, 1/p) - \min(1, 1/q))$

Proof. Let $\{\varphi_{j,k}\}$ be a BAPU corresponding to a curvelet-type decomposition. Let $\{\psi_j\}_{j \in \mathbb{N}_0}$ be a partition of unity with essential support on the dyadic frequency bands $\{x \in \mathbb{R}^d : 2^j \leq |x| < 2^{j+1}\}$ and satisfying

$$\|f\|_{B_{p,q}^\beta} \asymp \left(\sum_{j \in \mathbb{N}_0} (2^{\beta j} \|\psi_j(D)f\|_p)^q \right)^{1/q},$$

$$\text{supp}(\varphi_{j,k}) \subset \text{supp}(\tilde{\psi}_j), \quad k \in L_j, \quad \text{and} \quad \text{supp}(\psi_j) \subset \cup_{k \in L_j} \text{supp}(\tilde{\varphi}_{j,k})$$

for all $j \in \mathbb{N}_0$. Such a partition clearly exist, due to the construction of the curvelet-type decomposition.

For the first embedding we notice that

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} \sum_{k \in L_j} 2^{\beta q j} \|\varphi_{j,k}(D)f\|_p^q &= \sum_{j \in \mathbb{N}_0} \sum_{k \in L_j} 2^{\beta q j} \|\varphi_{j,k}(D)\tilde{\psi}_j(D)f\|_p^q \leq C \sum_{j \in \mathbb{N}_0} \sum_{k \in L_j} 2^{\beta q j} \|\tilde{\psi}_j(D)f\|_p^q \\ &\leq C \sum_{j \in \mathbb{N}_0} 2^{jK} 2^{\beta q j} \|\tilde{\psi}_j(D)f\|_p^q, \end{aligned}$$

since for a given level j there are $\#L_j \asymp 2^{jK}$ polar cubes in the covering $\{\varphi_{j,k}\}$.

We now turn to the second embedding. Suppose $p \geq 1$ and $q < 1$. Since $\text{supp}(\psi_j) = \cup_{k \in L_j} \text{supp}(\tilde{\varphi}_{j,k})$ we have

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} (2^{\beta j} \|\psi_j(D)f\|_p)^q &= \sum_{j \in \mathbb{N}_0} \left(2^{\beta j} \left\| \psi_j(D) \sum_{k \in L_j} \tilde{\varphi}_{j,k}(D)f \right\|_p \right)^q \leq C \sum_{j \in \mathbb{N}_0} \left(2^{\beta j} \left\| \sum_{k \in L_j} \tilde{\varphi}_{j,k}(D)f \right\|_p \right)^q \\ &\leq C \sum_{j \in \mathbb{N}_0} \left(2^{\beta j} \sum_{k \in L_j} \|\tilde{\varphi}_{j,k}(D)f\|_p \right)^q \leq C \sum_{j \in \mathbb{N}_0} \sum_{k \in L_j} (2^{\beta j} \|\tilde{\varphi}_{j,k}(D)f\|_p)^q. \end{aligned}$$

The other cases are obtained by appropriate Hölder inequalities using the bound on the sum over k for a given level j . \square

7.3. Equivalence with second generation curvelets. The curvelet-type frames defined in Section 7 appear to be quite similar to the second generation curvelets of Candés and Donoho in the comparable case, i.e., for $d = 2$. In this section we compare the two type of frames on a quantitative basis and show that the sparseness spaces associated with the two type of frames are the same (and given by specific decomposition smoothness spaces). Thus, whenever a function has a sparse curvelet expansion, the function will have an equally sparse curvelet-type decomposition and vice versa.

Let us briefly recall the definition of the so-called second generation curvelets given by Candés and Donoho. We refer the reader to [7] for a much more detailed discussion on the curvelet construction.

7.3.1. Second generation curvelets. Assume ν is an even $C^\infty(\mathbb{R})$ window that is supported on $[-\pi, \pi]$ and its 2π -periodic extension obeys $|\nu(\theta)|^2 + |\nu(\theta - \pi)|^2 = 1$, for $\theta \in [0, 2\pi)$. Define $\nu_{j,\ell}(\theta) = \nu(2^{\lfloor j/2 \rfloor} \theta - \pi \ell)$ for $j \geq 0$ and $\ell = 0, 1, \dots, 2^{\lfloor j/2 \rfloor} - 1$. Assume that w is a smooth compactly supported function that obeys

$$|w_0(t)|^2 + \sum_{j \geq 0} |w(2^{-j}t)|^2 = 1, \quad t \in \mathbb{R},$$

with w_0 a smooth function supported in a neighborhood of the origin. For $j \geq 2$ and $\ell = 0, 1, \dots, 2^{\lfloor j/2 \rfloor} - 1$, put

$$(7.1) \quad \kappa_{j,\ell}(\xi) = w(2^{-j}|\xi|)(\nu_{j,\ell}(\theta) + \nu_{j,\ell}(\theta + \pi)), \quad \xi = |\xi| e^{i\theta}.$$

The support of $w(2^{-j}|\xi|)\nu(2^{\lfloor j/2 \rfloor}\theta)$ is contained in a rectangle $R_j = I_{1j} \times I_{2j}$ given by

$$I_{1j} = \{\xi_1, t_j \leq \xi_1 \leq t_j + L_j\}, \quad I_{2j} = \{\xi_2, 2|\xi_2| \leq l_j\},$$

where $L_j = \delta_1 \pi 2^j$ and $l_j = \delta_2 2\pi 2^{j/2}$ (δ_1 depends weakly on j , see [7, Sec. 2]). Now let $\tilde{I}_{1j} = \pm I_{1j}$, and define $\tilde{R}_j = \tilde{I}_{1j} \times I_{2j}$. The system

$$u_{j,k}(\xi_1, \xi_2) = \frac{2^{-3j/4}}{2\pi\sqrt{\delta_1\delta_2}} \cdot e^{i(k_1+1/2)2^{-j}\frac{\xi_1}{\delta_1}} \cdot e^{i\frac{k_2 2^{-j/2}\xi_2}{\delta_2}}, \quad k_1, k_2 \in \mathbb{Z},$$

is then an orthonormal basis for $L_2(\tilde{R}_j)$. Finally, define

$$(7.2) \quad \hat{\gamma}_{\mu'}(\xi) = \kappa_J(\xi)u_{j,k}(R_J^\top \xi), \quad \mu' = (j, \ell, k),$$

where R_J is rotation by the angle $\pi 2^{-\lfloor j/2 \rfloor} \ell$, and let $\hat{\gamma}_{\mu_1}(\xi) = 2\pi \cdot \kappa_1(\xi)u_k(\xi)$, where $\kappa_1^2(\xi) = w_0^2(|\xi|) + w^2(|\xi|) + w^2(|\xi|/2)$, and $u_k(\xi) = (2\pi\delta_0)^{-1} \cdot e^{i(k_1\xi_1/\delta_0 + k_2\xi_2/\delta_0)}$ for a constant $\delta_0 > 0$. The curvelet system $\{\gamma_\mu\}_\mu$ is a tight frame for $L_2(\mathbb{R}^2)$.

7.3.2. Equivalence of curvelet-type frames and second generation curvelets. Now we prove that the second generation curvelets and the curvelet-type frames of Section 7 yield the same sparseness spaces for $d = 2$. In this case we have an explicit structured family of affine transformations. We let $\mathcal{T} = \{T_1, T_{j,\ell}\}_{j \geq 2, \ell=0,1,\dots,2^{\lfloor j/2 \rfloor+1}-1}$ be given by $T_{j,\ell} := R_{j,\ell}D_j$, where $R_{j,\ell}$ is the Rotation by the angle $\pi 2^{-\lfloor j/2 \rfloor} \ell$, and $D_j = \text{diag}(2^j, 2^{j/2})$.

Lemma 7.5. *Let $\kappa_{j,\ell}$ and $\gamma_{j,\ell,k}$ be given by (7.1) and (7.2) respectively. Then we have*

- (1) $\kappa_{j,\ell}(D)$ is a bounded operator on $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, uniformly in j and ℓ .
- (2) There exist two constants $C, C' < \infty$ such that

$$\left(\sum_{k \in \mathbb{Z}^2} |\langle f, \gamma_{j,\ell,k} \rangle|^p \right)^{1/p} \leq C 2^{3/2j(\frac{1}{p}-\frac{1}{2})} \|\widetilde{\kappa_{j,\ell}^2}(D)f\|_{L_p}, \quad \text{and}$$

$$\|\kappa_{j,\ell}^2(D)f\|_{L_p} \leq C' 2^{3/2j(\frac{1}{2}-\frac{1}{p})} \left(\sum_{k \in \mathbb{Z}^2} |\langle f, \gamma_{j,\ell,k} \rangle|^p \right)^{1/p}.$$

for all $(j, \ell) \in \mathcal{J}$.

Proof. Let $\Omega_{j,k}$ be the support of $\kappa_{j,k}$. Notice that in polar coordinates

$$\Omega_{j,k} = \{(r, \theta) : a2^j \leq |r| \leq b2^j, |\theta - \pi 2^{-\lfloor j/2 \rfloor} \ell| \leq \pi 2^{-\lfloor j/2 \rfloor}\},$$

for some constants a and b satisfying $0 < 4a < b < \infty$.

Define $\lambda_{j,\ell}(\xi) := \kappa_{j,\ell}(T_{j,\ell}\xi)$. By simple estimates on sine and cosine we obtain

$$\text{supp}(\lambda_{j,\ell}) = T_{j,\ell}^{-1}\Omega_{j,\ell} \subset K := [-b, b] \times [-\sqrt{2}b\pi, \sqrt{2}b\pi].$$

In fact, $\text{supp}(\lambda_{j,\ell}) \subset [-b, -a/2] \cup [a/2, b] \times [-\sqrt{2}b\pi, \sqrt{2}b\pi]$ for $j \geq 4$. Furthermore, it can be verified that for any $\beta \in \mathbb{N}_0^2$ there exists a constant C_β such that

$$(7.3) \quad \partial_\beta \lambda_{j,\ell}(\xi) \leq C_\beta \chi_K(\xi),$$

Now, the same technique as in the proof of Proposition 3.1 yields the first statement of the lemma.

To prove the second statement, notice that we can write $\hat{\gamma}_{j,\ell,k}$ in terms of $\lambda_{j,\ell}$:

$$\hat{\gamma}_{j,\ell,k}(\xi) = \frac{2^{-3/4j}}{2\pi\sqrt{\delta_1\delta_2}} \lambda_{j,\ell}(T_{j,\ell}^{-1}\xi) e^{k' \cdot (T_{j,\ell}^{-1}\xi)},$$

where

$$k' := \begin{bmatrix} (k_1 + 1/2)/\delta_1 \\ k_2/\delta_2 \end{bmatrix}.$$

Thus

$$\gamma_{j,\ell,k}(x) = \frac{2^{3/4j}}{2\pi\sqrt{\delta_1\delta_2}} \mathcal{F}^{-1} \lambda_{j,\ell}(T_{j,k}^\top x + k').$$

Now, as in the proof of Lemma 4.1, the bound (7.3) implies

$$\sup_{x \in \mathbb{R}^d, \ell} \left\| \{ \gamma_{j,\ell,k}(x) \}_{k \in \mathbb{Z}^2} \right\|_{\ell_p} \leq C_p 2^{3j/4} \quad \text{and} \quad \sup_{k \in \mathbb{Z}^d, \ell} \|\gamma_{j,\ell,k}\|_{L_p} \leq C'_p 2^{3/2j(\frac{1}{2} - \frac{1}{p})},$$

and the claim follows using the same arguments as in the proof of Lemma 4.1. \square

Let $\{\psi_1, \psi_{j,\ell}\}_{j,\ell}$ be a BAPU corresponding to $\{T_1, \mathcal{T}\}$, where T_1 is a suitable invertible affine transformation taking care of low frequencies. From the proof of Lemma 7.5 we have that we can choose the BAPU such that $\text{supp}(\varphi_{j,\ell}) \subset \text{supp}(\widetilde{\kappa_{j,\ell}})$, and

$$\text{supp}(\kappa_{j,\ell}) \subset \text{supp}(\widetilde{\varphi_{T_{j,\ell}}}) \cup \text{supp}(\widetilde{\varphi_{T_{j,\ell+2\lfloor j/2 \rfloor}}}).$$

Thus Lemma 7.5 implies

$$\begin{aligned} \|f\|_{G_{p,q}^\beta} &\asymp \left(\sum_{(j,\ell) \in \mathcal{J}} 2^{jq(\beta + \frac{3}{2}(\frac{1}{2} - \frac{1}{p}))} \left(\sum_k |\langle f, \gamma_{j,\ell,k} \rangle|^p \right)^{q/p} \right)^{1/q} \\ &\asymp \left(\sum_{(j,\ell) \in \mathcal{J}} |T_{j,\ell}|^{\frac{2q\beta}{3}} \left(\sum_k |\langle f, \gamma_{j,\ell,k}^p \rangle|^p \right)^{q/p} \right)^{1/q}, \end{aligned}$$

where $\gamma_{j,\ell,k}^p$ denotes the function $\gamma_{j,\ell,k}$ normalized in $L_p(\mathbb{R}^2)$.

7.3.3. Concluding remarks. The reader can verify that the result from the previous section imply that the curvelets on \mathbb{R}^2 satisfy the following Jackson estimate

$$\sigma_n(f, \{\gamma_\mu\}_\mu)_{L_2(\mathbb{R}^2)} \leq C \|f\|_{G_{2/3,2/3}^{3/2}(\mathbb{R}^2)} \cdot n^{-1}, \quad n \geq 0.$$

This Jackson estimate along with the careful analysis in [7] show that images (functions) with compact support that are C^2 except for discontinuities along piecewise C^2 -curves (see [7, Definition 1.1] for the precise definition of this function class) are contained in $G_{2/3,2/3}^{3/2+\varepsilon}(\mathbb{R}^2)$ for any $\varepsilon > 0$. For $G_{2/3,2/3}^{3/2+\varepsilon}(\mathbb{R}^2)$, Lemma 7.4 gives the embedding

$$B_{2/3,2/3}^{9/4+\varepsilon}(\mathbb{R}^2) \hookrightarrow G_{2/3,2/3}^{3/2+\varepsilon}(\mathbb{R}^2) \hookrightarrow B_{2/3,2/3}^{5/4+\varepsilon}(\mathbb{R}^2).$$

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