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An Optimized Half-Step Scheme Third derivative Methods for Testing Higher Order Initial Value Problems

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Abstract

An optimized half-step third derivative block scheme on testing third order initial value problems is presented in this article. This scheme suggests some certain points of evaluation which properly optimizes the truncation errors at point of formulas, the conditions that guarantee the properties of the method was considered and satisfied. However the develop scheme is used to test some third order optimized problems and the mathematical outcomes achieved confirms better calculation than the previous method we related with.

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1. Introduction

This article consider a direct solution of third order problems given by

$$y''' = f\left(x, y(x), y'(x)\right), y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1}$$

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on optimizing half-step hybrid method with the same off grid point. An efficient third derivative for solving optimized problems (1) is possible to yield a good accuracy and stability region for it to perform better according to [1]. Several areas of real live problems often arises in control and modeling of spread disease, vibration of strings, heat and mass transfer, flows of fluid, sciences and engineering, circuits electric models etc uses equation (1) in solving problems relating to them.

An optimized half-step block method proposed in this article for solving (1) where some researchers applied the reduction method before adopting it. However, this process can only compute the numerical solution at one point at a time and time constraint [2, 3]. Therefore, some scholars who newly applied the direct method to overcome the difficulties in reduction process in literature include [4, 5, 6].

Some researchers have proposed some methods in literature for solving (1), viz. [7] use interpolation and collocation procedure to develop a two-step continuous hybrid block method with two intra-step points, the optimization of local truncation error using two-step continuous block method was presented by [8], and [9] also adopt the uses of "optimization approach to form a two-step second derivative methods for solving of stiff systems".

An optimized half-step third derivative with equal equidistant points in this research was applied in block form through the collocation procedure to obtain the main scheme; the derivative will be evaluated at the last point of the block which is half.

The development of the article is as follows: the next section shows the methodological development of the optimized half-step. The basic conditions of the method are analyzed; these are convergence and stability region, numerical experiments. The effectiveness of the scheme is confirmed on some stiff mathematical samples and the result is discussed in Section 3. Section 4 is the conclusion.

2. Derivation of the Method

The continuous representation of half-step method which we shall derive will be used to generate the main method which we shall set up to obtain the block method. An estimate of power series given by

$$y(x) = \sum_{i=0}^{2} \psi_i y_{n+i} + h^3 \left[\sum_{j=0}^{1/2} \phi_j f_{n+j} + \phi_k f_{n+k} \right], \; k = u, v, w \tag{2}$$

is considered, where $\psi_i(t)$, $\phi_j(t)$, $\phi_k(\xi)$ are polynomials, $y_{n+j} = y(x_{n+j})$, $f_{n+j} = f\left(x_{n+j}, y_{n+j}\right)$, $\xi = \frac{x-x_n}{h}$. Equation (2) is obtained by using the power series approximate solution of the form

$$y(x) = \sum_{j=0}^{s+r-1} a_i \xi^j. \tag{3}$$

The interpolation point $r = 3$ and collocation points $s = 5$ were carefully considered to solve equation (1).

$$\begin{aligned} y_{n+j} &= y\left(x_{n+j}\right), \quad & j = 0, u, v \\ y'''\left(x_{n+j}\right) &= f_{n+j}, \quad & j = 0, u, v, w, \tfrac{1}{2} \end{aligned} \tag{4}$$

By differentiating (3) thrice we obtain

$$y'''(x) = \sum_{j=3}^{s+r-1} \frac{a_j j!}{h^3 (j-3)} \xi^{j-3} = f(x, y, y'). \tag{5}$$

Substituting (5) into (1) yield

$$f(x, y, y'') = \sum_{j=3}^{s+r-1} \frac{a_j j!}{h^3 (j-3)} \xi^{j-3} \tag{6}$$

Equation (6) is collocated at all points x_{n+s}, $s = 0, u, v, w, \frac{1}{2}$ and (5) is interpolated at x_{n+r}, $r = 0, u, v,$, to yield a system of non linear equation of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & u & v & w & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{u^2}{2!} & \frac{v^2}{2!} & \frac{w^2}{2!} & \frac{\left(\frac{1}{2}\right)^2}{2!} \\ 0 & 0 & 0 & 0 & \frac{u^3}{3!} & \frac{v^3}{3!} & \frac{w^3}{3!} & \frac{\left(\frac{1}{2}\right)^3}{3!} \\ 0 & 0 & 0 & 0 & \frac{u^4}{4!} & \frac{v^4}{4!} & \frac{w^4}{4!} & \frac{\left(\frac{1}{2}\right)^4}{4!} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_n' \\ y_n'' \\ f_n \\ f_{n+u} \\ f_{n+v} \\ f_{n+w} \\ f_{n+1/2} \end{bmatrix} \tag{7}$$

In order to find the unknown values u, v, w for b_i's, we solve equation (7) using Gaussian elimination method to yield a continuous hybrid linear multistep method of the form

$$p(x) = \sum_{j=0,u,v} \psi_j y_{n+j} + h^3 \left[\sum_{i=0}^{1/2} \phi_j f_{n+j} + \phi_k f_{n+k} \right], \ k = u, v, w. \tag{8}$$

The coefficient of y_{n+j}, $j = 0, u, v$ and f_{n+j}, $j = 0, u, v, w, \frac{1}{2}$ gives

$$y_{n+\xi} \sum_{i=0,u,v} (\psi_i(\xi) y_{n+i}) + h^3 \left[\phi_0(\xi) f_n + \phi_u(\xi) f_{n+u} + \phi_v(\xi) f_{n+v} + \phi_w(\xi) f_{n+w} + \beta_{\frac{1}{2}}(\xi) f_{n+\frac{1}{2}} \right] \tag{9}$$

where $\xi = \frac{x - x_{n+4}}{n}$, $\frac{d\xi}{dx} = \frac{1}{h}$

$$\psi_0 = 1 \tag{10}$$

$$\psi_u = \xi$$

$$\psi_v = \frac{1}{2}\xi^2$$

$$\phi_0 = \frac{1}{2520}\xi^3 \frac{\left(-105u\xi - 105v\xi + 42\xi^2 - 98\xi^3 + 64\xi^4 + 196u\xi^2 - 112u\xi^3 + 196v\xi^2 - 112v\xi^3 + 420uv - 490uv\xi + 224uv\xi^2\right)}{uv}$$

$$\phi_1 = \frac{1}{840u} \frac{\xi^4}{(u-v)(16u^2 - 14u + 3)} \left(42\xi - 105v + 196v\xi - 98\xi^2 + 64\xi^3 - 112v\xi^2\right)$$

$$\phi_2 = -\frac{1}{840v} \frac{\xi^4}{(u-v)(16v^2 - 14v + 3)} \left(42\xi - 105u + 196u\xi - 98\xi^2 + 64\xi^3 - 112u\xi^2\right)$$

$$\phi_3 = \frac{256}{315} \frac{\xi^4}{(8u-3)(8v-3)} \left(7\xi^2 - 14v\xi - 14u\xi - 8\xi^3 + 14u\xi^2 + 14v\xi^2 + 35uv - 28uv\xi\right)$$

$$\phi_4 = -\frac{1}{105} \frac{\xi^4}{(2u-1)(2v-1)} \left(21\xi^2 - 42v\xi - 42u\xi - 32\xi^3 + 56u\xi^2 + 56v\xi^2 + 105uv - 112uv\xi\right)$$

Evaluating the first and second derivative of (9) at all points and substituting the value of $w = 3/8$ we obtain equations (10) and (15) as shown in Tables 1 and 2, respectively.

We obtain the hybrid block multistep formula by substituting $w = 3/8$ and $\xi = 1/2$ in (9) to approximate the solution of (1) which then yield

$$y_{n+1/2} = y_n + \frac{1}{2}y_n' + \frac{1}{8}h^2 y_n'' + h^3 \left(\begin{array}{l} f_n \left(-\frac{1}{80640uv}(70u + 70v - 924uv - 9)\right) + f_{n+u}\left(-\frac{1}{26880u}\frac{70v-9}{(u-v)(16u^2-14u+3)}\right) \\ + f_{n+v}\left(\frac{1}{26880u}\frac{70u-9}{(u-v)(16v^2-14v+3)}\right) + f_{n+3/8}\left(-\frac{4}{315(8u-3)(8v-3)}(14u + 14v - 84uv - 3)\right) \\ + f_{n+1/2}\left(\frac{1}{6720(2u-1)(2v-1)}(28u + 28v - 196uv - 5)\right) \end{array} \right) \tag{11}$$

Differentiating (11) once and then substituting $w = 3/8$ and $\xi = 1/2$, we obtain the approximate formula of the first derivative as

$$y'_{n+1/2} = y'_n + \frac{1}{2}hy''_n + h^2 \left(\begin{array}{l} f_n \left(-\frac{1}{1440uv}(7u + 7v - 80uv - 1) \right) + f_{n+u} \left(-\frac{7v-1}{6720u^2v - 7680u^3v - 1440uv + 1440u^2 - 6720u^3 + 7680u^4} \right) \\ +f_{n+v} \left(\frac{7u-1}{6720uv^2 - 7680uv^3 - 1440uv + 1440v^2 - 6720v^3 + 7680v^4} \right) \\ +f_{n+3/8} \left(\frac{64u+64v-320uv-16}{1080u+1080v-2880uv-405} \right) + f_{n+1/2} \left(\frac{6u+6v-40uv-1}{480u+480v-960uv-240} \right) \end{array} \right) \quad (12)$$

Differentiating (9) once and then substituting $w = 3/8$ and $\xi = 1/2$, we obtain the approximate formula of the second derivative as

$$y''_{n+1/2} = y''_n + h \left(\begin{array}{l} f_n \left(-\frac{1}{1440uv}(20u + 20v - 200uv - 3) \right) + f_{n+u} \left(-\frac{20v-3}{6720u^2v - 7680u^3v - 1440uv + 1440u^2 - 6720u^3 + 7680u^4} \right) \\ +f_{n+v} \left(\frac{20u-3}{6720uv^2 - 7680uv^3 - 1440uv + 1440v^2 - 6720v^3 + 7680v^4} \right) \\ +f_{n+3/8} \left(\frac{320u+320v-1280uv-96}{1080u+1080v-2880uv-405} \right) + f_{n+1/2} \left(\frac{40uv-3}{240u+240v-480uv-120} \right) \end{array} \right) \quad (13)$$

Now we choose to optimize the local truncation error by determine the approximate values of u and v in the formulas (8), (9) and (10) which are related to the off-grid points of the hybrid multistep method. To advance to the next block, the values of $y''_{n+1/2}$, $y'_{n+1/2}$ and $y_{n+1/2}$ will be needed for the computation, therefore the local truncation error is consider using formula (8), (9) and (10) which yield

$$\begin{aligned} L[y(x); h] &= \frac{1}{103219200}(-18u - 18v + 140uv + 3) + O(h^9) \\ L[y'(x); h] &= \frac{1}{25804800}(-28u - 28v + 196uv + 5) + O(h^9) \\ L[y''(x); h] &= \frac{1}{18432200}(-6u - 6v + 40uv + 1) + O(h^9) \end{aligned} \quad (14)$$

Equating (15) to zero gives the principal term of the local truncation errors, hence we need to obtain the values of our unknown parameters which is u and v as in Ramos et al. (2015)

$$\begin{aligned} &-18u - 18v + 140uv + 3 = 0 \\ &-28u - 28v + 196uv + 5 = 0 \\ &-6u - 6v + 40uv + 1 = 0 \end{aligned} \quad (15)$$

Solving for u and v in (15), we obtain $u = \frac{1}{56}\sqrt{114} - \frac{1}{28}$ and $v = -\frac{7}{292}\sqrt{114} - \frac{29}{292}$ in which correspond to the plane curve that are symmetric with respect to the diagonal $u = v$ and has a unique constraint $0 < u < v < 3/8 < 1/2$, substituting these values into (11) gives three formulas, one for approximating the solution, and the other two for approximating the first and second derivative at point $x_{n+1/2}$, hence we shall have twelve unknown $\left(y_{n+j}, y'_{n+j}, y''_{n+j}, j = u, v, 3/8, 1/2\right)$. To obtain the half-step hybrid block method for solving (1), we need to consider the evaluation and at all the point to simplify the three formulas above which produces the following general equations in block form

$$A^{(0)}Y_m^{(i)} = \sum_{i=0}^{2} h^{(i)}y_n^{(i)} + h^{(3-i)}(g_i f(y_n) + p_i f(y_m)) \quad (16)$$

where

$$Y_m^{(i)} = \left[y^i_{n+u}\ y^i_{n+u}\ y^i_{n+3/8}\ y^i_{n+1/2} \right], \quad y_n^{(i)} = \left[y^i_{n-u}\ y^i_{n-u}\ y^i_{n-3/8}\ y^i_n \right]$$
$$f_m^{(i)} = \left[f^i_{n+u}\ f^i_{n+u}\ f^i_{n+3/8}\ f^i_{n+1/2} \right], \quad f_n^{(i)} = \left[f^i_{n-u}\ f^i_{n-u}\ f^i_{n-3/8}\ f^i_n \right]$$

and $A = 4 \times 4$ is identity matrix.

When $i = 0$,

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{3099}{20000} \\ 0 & 0 & 0 & -\frac{35527}{100000} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{2401}{20000} \\ 0 & 0 & 0 & -\frac{15777}{250000} \\ 0 & 0 & 0 & \frac{9}{128} \\ 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$g_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{677}{1562500} \\ 0 & 0 & 0 & -\frac{11953}{100000} \\ 0 & 0 & 0 & \frac{873}{250000} \\ 0 & 0 & 0 & \frac{62721}{10000000} \end{pmatrix}, \; p_0 = \begin{pmatrix} \frac{11383}{500000000} & -\frac{5749}{2000000000} & -\frac{53859}{10000000000} & \frac{7919}{500000000} \\ \frac{7287}{1000000} & -\frac{311}{500000} & -\frac{32509}{100000000} & \frac{2131}{2000000} \\ \frac{10849}{2000000} & -\frac{12851}{500000000} & -\frac{20143}{100000000} & \frac{99709}{1000000000} \\ \frac{13119}{1000000} & -\frac{37923}{1000000000} & \frac{7278}{5000000} & \frac{911}{40000000}. \end{pmatrix}$$

When $i = 1$,

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{3099}{20000} \\ 0 & 0 & 0 & -\frac{35527}{100000} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \; g_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{14603}{2000000} \\ 0 & 0 & 0 & \frac{1057}{10000} \\ 0 & 0 & 0 & \frac{1213}{62500} \\ 0 & 0 & 0 & \frac{25251}{1000000} \end{pmatrix}$$

$$p_1 = \begin{pmatrix} \frac{13959}{2500000} & -\frac{7323}{1250000000} & -\frac{5797}{5000000} & \frac{16883}{50000000} \\ -\frac{74549}{1000000} & \frac{88727}{10000000} & \frac{34507}{1000000} & -\frac{571}{50000} \\ \frac{9401}{200000} & -\frac{1071}{10000000} & \frac{42857}{10000000} & \frac{27911}{100000000} \\ \frac{15127}{200000} & -\frac{24339}{250000000} & \frac{24211}{1000000} & \frac{767}{5000000000}. \end{pmatrix}$$

When $i = 2$,

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{34039}{500000} \\ 0 & 0 & 0 & -\frac{11429}{20000} \\ 0 & 0 & 0 & \frac{21707}{500000} \\ 0 & 0 & 0 & \frac{50503}{1000000} \end{pmatrix}, \; p_2 = \begin{pmatrix} \frac{98231}{1000000} & -\frac{34753}{50000000} & -\frac{1869}{125000} & \frac{42879}{10000000} \\ \frac{47409}{100000} & -\frac{2627}{25000} & -\frac{11493}{50000} & \frac{7703}{100000} \\ \frac{4761}{20000} & -\frac{24381}{1000000000} & \frac{5253}{50000} & \frac{11769}{1000000} \\ \frac{137}{625} & -\frac{5693}{500000000} & \frac{19369}{100000} & \frac{36723}{1000000} \end{pmatrix}.$$

3. Analysis of Basic Properties of the Method

3.1. Order of the Block

Consider the linear operator $L\{y(x) : h\}$ associated with the discrete block method (16) be defined

$$L\{y(x) : h\} = A^{(0)} Y_m^i - \sum_{j=0}^{2} h^i e_i y_n^i - h^3 \left(g_i f(y_n) + p_i f(y_m) \right) \tag{17}$$

Using Taylor series to expand (17) and the coefficient of h are compared to give

$$L\{y(x) : h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots$$

Definition: Linear operator L and associated block formula are said to be of order p if $C_0 = C_1 = C_p = C_{p+1} = C_{p+2} = 0$, and $C_{p+3} \neq 0$. C_{p+3} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + O\left(h^{p+4}\right)$.

For our method, expanding (16) in Taylor series, and comparing the coefficients of h gives $C_0 = C_1 = C_2 = C_3 = \cdots = C_7 = 0$ and

$$\begin{bmatrix} -\frac{3133248341987641235360959768917}{2867200000000000000000000000000000000000}, & -\frac{23529667578922765111999345314414644817}{224000000000000000000000000000000000000000000000}, & -\frac{53250747273}{5872025600000000000}, \\ -\frac{1747797}{163840000000000}, & -\frac{816316798168710839015746641}{35840000000000000000000000000000000}, & \frac{1971759310784102449870384790006899}{1680000000000000000000000000000000000000000}, \\ -\frac{4640494293}{183500800000000000}, & -\frac{30110159}{4300800000000000}, & -\frac{173410329060433598636509502493843}{16788531968000000000000000000000000000}, \\ -\frac{60271262357346424710791152373}{7200000000000000000000000000000000000}, & -\frac{70522281}{262144000000000}, & \frac{205681815003036509}{37869881036063232000000000} \end{bmatrix}.$$

Hence our method is of order six (6).

3.2. Consistency

The optimized scheme (16) is consistent, since it has order more than or equal to one.

3.3. Zero Stability of Our Method

Definition 2:

A third derivative optimized scheme is said to be zero-stable, if the roots $z_i, i = u, v, 3/8, 1/2$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0$ satisfies $|z_i| \leq 1$ and for those roots with $z_i = 1$, multiplicity must not exceed two. Hence, our method is zero-stable.

3.4. Consistency

Theorem 3.1[6]

The compulsory and adequate terminologies for the optimized scheme to be convergent are that they must be consistent and zero-stable. Hence, the optimized scheme derived is convergent since all conditions are satisfied.

3.5. Linear Stability

The concept of A-stability according to Hairer and Wanner is discussed by applying the test equation

$$y^(k) = \lambda^{(k)} y \tag{18}$$

to yield

$$Y_m = \mu(z) Y_{m-1}, z = \lambda h \tag{19}$$

where $\mu(z)$ is the amplification matrix given by

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4\eta^{(0)}\right)^{-1} \left(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}\right). \tag{20}$$

The matrix $\mu(z)$ has eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for half-step optimize third derivative method with three offgrid hybrid points is given by

$$\xi = \frac{\left(3z^7 + 14z^6 + 143z^5 + 571z^4 + 2464z^3 + 8950z^2 + 20160z + 20160\right)}{(3z^6 + 51z^5 + 453z^4 + 2550z^3 + 9270z^2 - 20160z + 20160)}. \tag{21}$$

3.6. Regions of Absolute Stability

The stability polynomial of the optimized scheme is found to be

$$- h^{12}\left(\frac{30304968990455678983}{100000000000000000000000000000000}w^4 + \frac{16289978146044407871}{100000000000000000000000000000000}w^3\right)$$
$$- h^9\left(\frac{8199533764251297150317}{1000000000000000000000000000000}w^3 - \frac{54017378594461649}{1000000000000000000000000000}w^4\right)$$

3.7. Mathematical computation of the method

Sample I: The highly stiff system solved by [8] is given as

$$y''' = 3\sin g,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$$

Exact Solution: $y(g) = 3\cos g + \frac{g^2}{2} - 2, h = \frac{1}{10}$. The solution for Sample I are shown in Table 1.

Sample II the third order ODE solved by [5] is given by

$$y''' = -4y' + g,\ y(0) = 1,\ y''(0) = 1$$

Exact Solution: $y(g) = \frac{3}{16}(1 - \cos 2g) + \frac{g^2}{g}, h = \frac{1}{10}$. The solution for Sample I are shown in Table 1.

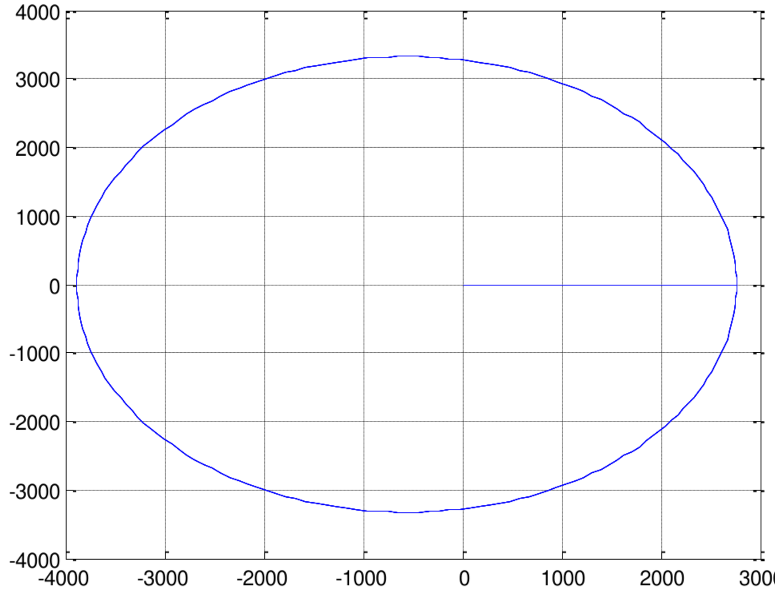

Figure 1. Showing the Region of Absolute Stability of our optimize scheme

Table 1. Showing the Result for sample 1

g-values	Error in our method	Error in [8]
0.1	2.1291 e-15	4.1078e-15
0.2	1.8981 e-15	1.6875e-14
0.3	1.9821 e-15	5.0848e-14
0.4	1.9639 e-14	1.1779e-13
0.5	1.7857 e-14	2.4081e-13
0.6	2.5981 e-14	4.3709e-13
0.7	2.4938 e-14	7.3708e-13
0.8	3.4447 e-14	1.1662e-12
0.9	4.5861 e-14	1.7587e-12
1.0	2.7518 e-14	2.5466e-12

Table 2. Showing the Result for sample 2

g-values	Error in our method	Error in [5]
0.1	2.55208 e-12	2.970e-08
0.2	3.64210e-12	1.988e-07
0.3	4.5313 e-12	6.508e-07
0.4	1.3406 e-12	1.5480e-06
0.5	3.28547 e-12	3.062e-06
0.6	4.59125 e-12	5.3625e-06
0.7	5.47318 e-12	8.6068e-06
0.8	1.96524 e-12	1.2926e-05
0.9	2.34526 e-12	1.8118e-05
1.0	2.55587e-12	2.5129e-05

4. Conclusion

The optimized half-step hybrid block third derivative method derived in this work was implemented efficiently. Maple 18 software was used for the implementation while the scientific workplace 5.5 version was used for the derivation of the optimize hybrid methods. The graphical representation was also generated with the aid of MATLAB 2021a programming language. The optimized results obviously converge quicker than the recent result of [5 and 8].

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