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On Generalized Permuting Left 3-Derivations of Prime Rings

Abstract-*Let $\mathcal{R}$ be an associative ring. Park and Jung introduced the concept of permuting 3-derivation and they are studied this concept as centralizing and commuting. The main intent of this work is to generalize Park and Jung's results by introducing the concept of generalized permuting left 3-derivation on Lie ideal.*

Keywords- *Permuting 3-derivation, Lie ideal, Prime ring, Left derivation, Commuting, Centralizing.*

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1. Introduction

Throughout this paper, $\mathcal{Z}(\mathcal{R})$ is the center of associative ring $\mathcal{R}$ and $[x,y] = xy - yx$ will represent the commutator of $x, y \in \mathcal{R}$, [1]. A ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = (0)$ then $a = 0$ or $b = 0$ where $a, b \in \mathcal{R}$, [2]. Algebra, functional analysis and quantum physics are related with concept of derivation. An additive mapping $d: \mathcal{R} \to \mathcal{R}$ such that $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$ is called left derivation of $\mathcal{R}$, [3] it is clear that the concepts of derivation and left derivation are identical whenever $\mathcal{R}$ is commutative. In 1987 the concept of a symmetric bi-derivation has been introduced by Maksa in [4], a bi-additive mapping $d: \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is said to be bi-derivation if $d(xy,z) = d(x,z)y + xd(y,z)$, $d(x,yz) = d(x,y)z + yd(x,z)$, for all $x, y, z \in \mathcal{R}$. In 1989 J. Vukman [5,6] investigated symmetric bi-derivations on prime rings. A ring $\mathcal{R}$ is said to be n-torsion-free if whenever $na = 0$ where n is a non-zero integer with $a \in \mathcal{R}$, then $a = 0$, [2].

An additive subgroup U of $\mathcal{R}$ is called Lie ideal if whenever $u \in U$, $r \in \mathcal{R}$ then $[U, r] \in U$, [2]. A Lie ideal U of $\mathcal{R}$ is called a square closed Lie ideal of $\mathcal{R}$ if $u^2 \in U$, for all $u \in U$, [3]. A map $d: \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called commuting (resp. centralizing) on U if $[d(x), x] = 0$ for all $x \in U$ (resp. $[d(x), x] \in \mathcal{Z}(\mathcal{R})$, for all $x \in U$), [1]. A square closed Lie ideal U of $\mathcal{R}$ such that $U \not\subseteq \mathcal{Z}(\mathcal{R})$ is called an admissible Lie ideal of $\mathcal{R}$, [7]. In 2007, park and Jung introduced the concept of permuting 3-derivation and they are studied this concept as centralizing and commuting, [1]. The history of centralizing and commuting mapping is due to Divinsky 1955, [8]. Posner initiated several aspects of a study of commuting and centralizing derivations on prime ring, [7]. Many papers have expressed their interests of the permuting 3-derivation [9], [10]. In this paper we extend the concept of permuting 3-derivation to new concept which is called the concept of generalized permuting 3-derivation and study the commuting and centralizing of this concept and the commutatively of Lie ideal under certain conditions. Throughout this paper U will represent a Lie ideal of $\mathcal{R}$.

2. Basic concepts

Key finding the results is starting by the following facts:

Lemma (2.1), [11]:
Let $\mathcal{R}$be a 2-torsion free prime ring and $aUb = 0$ such that $U \not\subseteq \mathcal{Z}(\mathcal{R})$, then either $a = 0$ or $b = 0$, whenever $a, b \in \mathcal{R}$.

Lemma (2.2), [12]:
Let U be a nonzero an admissible of a 2-torsion free prime ring $\mathcal{R}$. Then U contains a nonzero ideal of $\mathcal{R}$.

Lemma (2.3), [13]:
Let U be a commutative Lie ideal of a 2-torsion free prime ring $\mathcal{R}$. Then $U \subseteq \mathcal{Z}(\mathcal{R})$.

Definition (2.4), [1]:
A map $d: \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is said to be permuting if for all $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{R}$ and for every permutation $\{\Pi(1), \Pi(2), \Pi(3)\}$, the equation $d(\alpha_1, \alpha_2, \alpha_3) = d(\alpha_{\Pi(1)}, \alpha_{\Pi(2)}, \alpha_{\Pi(3)})$ is hold.

Definition (2.5), [1]:A 3 -derivation $d: \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called permuting 3-derivatoin if d is

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permuting. That is, the following equations are equal to each other:
$d(\alpha\gamma,\beta,\upsilon) = d(\alpha,\beta,\upsilon)\gamma + \alpha\, d(\gamma,\beta,\upsilon)$,
$d(\alpha,\beta\gamma,\upsilon) = d(\alpha,\beta,\nu)\gamma + \beta d(\alpha,\gamma,\upsilon)$ and
$d(\alpha,\beta,\upsilon\gamma) = d(\alpha,\beta,\upsilon)\gamma + \upsilon d(\alpha,\beta,\gamma)$, for all $\alpha,\beta,\upsilon,\gamma \in \mathcal{R}$.

Definition (2.6), [1]:
The trace δ_d of $d : \mathcal{R}\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ is defined by $\delta_d(\alpha) = d(\alpha,\alpha,\alpha)$, $\forall\, \alpha\in\mathcal{R}$.

Theorem (2.7), [14]:
Let U be an admissible of a 6-torsionfree prime ring $\mathcal{R}$. If there exists a permuting 3-derivation $d : U\times U\times U\to\mathcal{R}$ such that the trace δ_d of d is commuting on U. Then $d = 0$ on $\mathcal{R}\times\mathcal{R}\times\mathcal{R}$.

Now, we offer the concepts of left 3-derivation and generalized permuting 3-derivatoin.

Definition (2.8):
A 3-additivve map $d: U\times U\times U\to\mathcal{R}$ is called a 3-left derivation if the following equations are satisfied:
$d(\alpha_1\alpha_4,\alpha_2,\alpha_3) = \alpha_4\, d(\alpha_1,\alpha_2,\alpha_3)+\alpha_1\, d(\alpha_4,\alpha_2,\alpha_3)$,
$d(\alpha_1,\alpha_2\alpha_4,\alpha_3) = \alpha_4 d(\alpha_1,\alpha_2,\alpha_3) + \alpha_2\, d(\alpha_1,\alpha_4,\alpha_3)$
and
$d(\alpha_1,\alpha_2,\alpha_3\alpha_4) = \alpha_4 + \alpha_3\, d(\alpha_1,\alpha_2,\alpha_4)$,
$\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in U$.
If d is permuting, then d is called permuting 3-left derivation.

Definition (2.9):
A 3-additive map $\mathcal{F}: U\times U\times U\to\mathcal{R}$ is called a generalized 3-left derivation if there exists a left 3 – derivation $d: U\times U\times U\to\mathcal{R}$ such that:
$\mathcal{F}(\alpha_1\alpha_4,\alpha_2,\alpha_3) = \alpha_4\,\mathcal{F}(\alpha_1,\alpha_2,\alpha_3) + \alpha_1\, d(\alpha_4,\alpha_2,\alpha_3)$,
$\mathcal{F}(\alpha_1,\alpha_2\alpha_4,\alpha_3) = \alpha_4\mathcal{F}(\alpha_1,\alpha_2,\alpha_3) + \alpha_2\, d(\alpha_1,\alpha_4,\alpha_3)$
and
$\mathcal{F}(\alpha_1,\alpha_2,\alpha_3\alpha_4) = \alpha_4\mathcal{F}(\alpha_1,\alpha_2,\alpha_3) + \alpha_3\, d(\alpha_1,\alpha_2,\alpha_4)$,
$\forall\,\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in U$.

Definition (2.10):
A generalized left 3-derivation map $\mathcal{F}: U\times U\times U\to\mathcal{R}$ is said to be a generalized permuting left 3-derivation if there exist a permuting left 3-derivation $d : U\times U\times U\to\mathcal{R}$ suth that the equations in definition (1.9) are equal to each other. That is $\mathcal{F}(\alpha\gamma,\beta,\upsilon) = \gamma\,\mathcal{F}(\alpha,\beta,\upsilon) + \alpha\, d(\gamma,\beta,\upsilon)$, $\forall\,\beta,\alpha,\upsilon,\alpha_4 \in U$.

Example (2.11):
Let S be a ring, $\mathcal{R} = \{\begin{pmatrix}\alpha_1 & \alpha_2\\ 0 & 0\end{pmatrix}, \alpha_1,\alpha_2 \in S\}$ and $U = \{\begin{pmatrix}0 & \alpha_1\\ 0 & 0\end{pmatrix}, \alpha_1,\alpha_2 \in S\}$.
Define $\mathcal{F}: U\times U\times U\to\mathcal{R}$ by
$\mathcal{F}(\begin{pmatrix}0 & \alpha_1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_2\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_3\\ 0 & 0\end{pmatrix}) = \begin{pmatrix}0 & \alpha_1\alpha_2\alpha_3\\ 0 & 0\end{pmatrix}$,
$\forall \begin{pmatrix}0 & \alpha_1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_2\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_3\\ 0 & 0\end{pmatrix}\in U$. Then $\mathcal{F}$ is generalized left permuting 3-derivation, because there exist a left permuting 3-derivation $d: U\times U\times U\to\mathcal{R}$ defined by
$d(\begin{pmatrix}0 & \alpha_1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_2\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_3\\ 0 & 0\end{pmatrix}) = \begin{pmatrix}0 & \alpha_1\alpha_2\alpha_3\\ 0 & 0\end{pmatrix}$,
$\forall \begin{pmatrix}0 & \alpha_1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_2\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & \alpha_3\\ 0 & 0\end{pmatrix}\in U$.

3.The Main Results

Lemma (3.1):
Let $\delta_{\mathcal{F}}$ be the trace of permuting 3-additive map $\mathcal{F} : U\times U\times U\to\mathcal{R}$. Then $[\delta_{\mathcal{F}}(\alpha+2\gamma),\alpha+2\gamma] + [\delta_{\mathcal{F}}(\alpha),\alpha] =$
$2[\delta_{\mathcal{F}}(\alpha+\gamma),\alpha+\gamma] + 6[\mathcal{F}(\alpha,\alpha,\gamma),\gamma] + 18[\mathcal{F}(\alpha,\gamma,\gamma),\gamma] + 14[\delta_{\mathcal{F}}(\gamma),\gamma] + 6[\mathcal{F}(\alpha,\gamma,\gamma),\alpha] + 6[\delta_{\mathcal{F}}(\gamma),\alpha] + 2[\delta_{\mathcal{F}}(\gamma),\gamma]$, $\forall\,\alpha,\gamma\in U$.

Proof:
Let $T = [\mathcal{F}(\alpha+\gamma,\alpha+\gamma,\alpha+\gamma),\alpha+\gamma]\,\gamma$. Then
$T = [\mathcal{F}(\alpha,\alpha+\gamma,\alpha+\gamma) + \mathcal{F}(\gamma,\alpha+\gamma,\alpha+\gamma),\alpha+\gamma]$
$= [\mathcal{F}(\alpha,\alpha,\alpha+\gamma) + \mathcal{F}(\alpha,\gamma,\alpha+\gamma) + \mathcal{F}(\gamma,\alpha,\alpha+\gamma) + \mathcal{F}(\gamma,\gamma,\alpha+\gamma),\alpha+\gamma]$
$= [\mathcal{F}(\alpha,\alpha,\alpha) + \mathcal{F}(\alpha,\alpha,\gamma) + \mathcal{F}(\alpha,\gamma,\alpha) + \mathcal{F}(\alpha,\gamma,\gamma) + \mathcal{F}(\gamma,\alpha,\alpha) + \mathcal{F}(\gamma,\alpha,\gamma) + \mathcal{F}(\gamma,\gamma,\alpha) + \mathcal{F}(\gamma,\gamma,\gamma),\alpha+\gamma]$
$+[\mathcal{F}(\alpha,\alpha,\alpha),\alpha] + [\mathcal{F}(\alpha,\alpha,\alpha),\gamma]$
$+[\mathcal{F}(\alpha,\alpha,\gamma),\alpha] + [\mathcal{F}(\alpha,\alpha,\gamma),\gamma]$
By Jacobi's identity, the last equation reduced to
$T = [\delta_{\mathcal{F}}(\alpha),\alpha] + [\delta_{\mathcal{F}}(\alpha),\gamma] + [\delta_{\mathcal{F}}(\gamma),\alpha] + 3[\mathcal{F}(\alpha,\alpha,\gamma)),\alpha] + 3[\mathcal{F}(\alpha,\gamma,\gamma),\gamma] + 3[\mathcal{F}(\alpha,\alpha,\gamma),\gamma] + 3[\mathcal{F}(\alpha,\gamma,\gamma),\alpha] + [\delta_{\mathcal{F}}(\gamma),\gamma]$, $\forall\,\alpha,\gamma\in U$. (1)

Since Ƒ is an odd mapping, then if put $\alpha = -\alpha$ in Eq. (1) and comparing the results, we get
$[\delta_{\mathcal{F}}(\alpha+\gamma),\alpha+\gamma] + [\delta_{\mathcal{F}}(\gamma),\alpha-\gamma]$
$=2[\delta_{\mathcal{F}}(\alpha),\alpha] + 6[\mathcal{F}(\alpha,\alpha,\gamma),\gamma] + 6[\mathcal{F}(\alpha,\gamma,\gamma),\alpha] + 2[\delta_{\mathcal{F}}(\gamma),\gamma]$, $\forall\,\alpha,\gamma\in U$. (2)

Replace α by $\alpha+\gamma$ in Eq. (2) and use Eq. (1)) and Eq. (2) to get the required result.

The following proposition follows immediately from Lemma (3.1).

Proposition (3.2):
Let $\mathcal{R}$ be 6-torsionn free and $\delta_{\mathcal{F}}$ bee the trace of permuting left 3-derivation map $\mathcal{F}: U\times U\times U\to\mathcal{R}$. Then (1) If $\delta_{\mathcal{F}}$ is commuting on U, $3[\mathcal{F}(\alpha_1,\alpha_2,\alpha_2),\alpha_2] + [\delta_{\mathcal{F}}(\alpha_2),\alpha_1] = 0$, $\forall\alpha_2,\alpha_2\in U$.
(2) If $\delta_{\mathcal{F}}$ is centralizing on U, $3[\mathcal{F}(\alpha_1,\alpha_2,\alpha_2),\alpha_2] + [\delta_{\mathcal{F}}(\alpha_2),\alpha_1] \in \mathcal{Z}(\mathcal{R})$, $\forall\alpha_1,\alpha_2\in U$.

Proposition (3.3):
Let $\delta_{\mathcal{F}}$ be a trace of permuting left 3-derivation $\mathcal{F}: U\times U\times U\to\mathcal{R}$, U be an admissible Lie ideal of a 2-torsion free prime ring $\mathcal{R}$ and $a\in\mathcal{R}$ such that $a\,\delta_{\mathcal{F}}(\alpha) = 0, \forall\alpha\in U$.Then either $a = 0$ or $\delta_{\mathcal{F}}(x) = 0$, where $\delta_{\mathcal{F}}$ is the trace of $\mathcal{F}$.

Proof:
Since $a\,\delta_{\mathcal{F}}(\alpha) = 0, \forall\alpha\in U$, (3)
Linearize Eq. (1) on α to get

$0 = a(\delta_{\mathcal{F}}(\alpha) + 3\mathcal{F}(\alpha,\beta,\beta) + 3\mathcal{F}(\alpha,\alpha,\beta) + \delta_{\mathcal{F}}(\beta))$.

By using Eq. (3), we get

$0 = a\mathcal{F}(\alpha,\beta,\beta) + a\mathcal{F}(\alpha,\alpha,\beta)$ (4)

Put $\alpha = 2\beta\alpha$ in Eq. (4), we get

$0 = 2a\alpha\,\delta_{\mathcal{F}}(\beta) + a\alpha^2\,\delta_{\mathcal{F}}(\beta)$ (5)

Replace α by (-α) in Eq. (5) and comparing the results with Eq. (5), we get either $\alpha = 0$ or the trace of $\mathcal{F}$ is zero on U.

The following results are generalization of [1].

Theorem (3.4):

Let $\mathcal{R}$ be a 6- torsion free prime ring and U be an admissible of $\mathcal{R}$. If there exists a generalized permuting left 3- derivation $\mathcal{F}: U \times U \times U \to \mathcal{R}$ associated with permuting left 3-derivation d such that the traces $\delta_{\mathcal{F}}$ of $\mathcal{F}$ and δ_d of d are commuting on U, then $\mathcal{F} = 0$ on $\mathcal{R}\times\mathcal{R}\times\mathcal{R}$.

Proof:

Since δ_F is commuting on U, then by using proposition (3.2), we get

$0 = [\delta_{\mathcal{F}}(\beta),\alpha] + 3[\mathcal{F}(\alpha,\beta,\beta),\beta], \forall\alpha,\beta \in U.$ (6)

Putting 2βα instead of α in Eq. (6), then

$0 = \delta_{\mathcal{F}}(\beta), 2\beta\alpha] + 3[\mathcal{F}(2\beta\alpha,\beta,\beta),\beta]$

$= 2[\delta_{\mathcal{F}}(\beta),\beta]\alpha + 2\beta[\delta_{\mathcal{F}}(\beta),\alpha]$

$+ 6[\alpha\mathcal{F}(\beta,\beta,\beta) + \beta\, d(\alpha,\beta,\beta),\beta]$

Since $\mathcal{F}$ is commuting on U, then the last equation reduced to $2\beta\,\delta_{\mathcal{F}}(\beta),\alpha] + 6[\alpha,\beta]\,\delta_{\mathcal{F}}(\beta) + 6\beta[d(\alpha,\beta,\beta),\beta] = 0$.

Since $\mathcal{R}$ is 6- torsion free, the last equation becomes $\beta[\delta_{\mathcal{F}}(\beta),\alpha] + 3[\alpha,\beta]\,\delta_{\mathcal{F}}(\beta) + 3\beta[d(\alpha,\beta,\beta),\beta] = 0.$ (7)

Multiply Eq. (6) by β from left and compare the result with Eq. (7) and by applying ([14], Theorem (2.4)), we get

$3[\alpha,\beta]\,\delta_F(\beta) - 3\beta[\mathcal{F}(\alpha,\beta,\beta),\beta] = 0, \forall\alpha,\beta \in U.$ (8)

Replace α by 2αβ in Eq. (8) and using Eq. (8) and (Theorem (2.4), [14]), we get

$0 = [\alpha,\beta]\beta\,\delta_{\mathcal{F}}(\beta) - \beta^2[\delta_{\mathcal{F}}(\alpha,\beta,\beta),\beta]$

$= [\alpha,\beta]\beta\,\delta_{\mathcal{F}}(\beta) - \beta[\alpha,\beta]\,\delta_{\mathcal{F}}(\beta)$

$= [[\alpha,\beta],\beta]\,\delta_{\mathcal{F}}(\beta)$

By Proposition (3.3) we get either $[[\alpha,\beta],\beta] = 0$ or $\delta_{\mathcal{F}}(\beta) = 0, \forall\alpha,\beta \in U$.

If $[[\alpha,\beta],\beta] = 0$, then replace α by αγ, we get.

$0 = [\alpha[\gamma,\beta] + [\alpha,\beta]\gamma,\beta]$ (9)

$= [\alpha,\beta][\gamma,\beta] + \alpha[[\gamma,\beta],\beta] + [\alpha,\beta][\gamma,\beta] + [[\alpha,\beta],\beta]\gamma = 2[\alpha,\beta][\gamma,\beta]$ (10)

Replace γ by 2γυ in Eq. (10) and using Eq. (10), then $[\alpha,\beta]\gamma[\gamma,\beta] = 0$, and this implies that U is commutative and this is a contradiction. Hence, $\delta_{\mathcal{F}}(\beta) = 0, \forall\beta \in U.$ (11)

Linearize Eq. (11) on β we get $0 = \delta_F(\alpha) + \delta_{\mathcal{F}}(\alpha) + \delta_{\mathcal{F}}(\beta) + 3\mathcal{F}(\alpha,\alpha,\beta) + 3\mathcal{F}(\alpha,\beta,\beta)$.

By Eq. (11) and Since $\mathcal{R}$ is 6-torsion free, the last equation can be reduced to

$\mathcal{F}(\alpha,\alpha,\beta) + \mathcal{F}(\alpha,\beta,\beta) = 0, \forall\alpha,\beta \in U.$ (12)

Again linearize equation (8) on β and since $\mathcal{R}$ is 6-torsion free, then $0 = \mathcal{F}(\alpha,\beta,\gamma), \forall\alpha,\beta,\gamma \in U$. (13) Since U is an admissible Lie ideal, by Lemma (2.2), U contains a non zero ideal I of U. Therefore, $\mathcal{F}(\alpha,\beta,\gamma) = 0, \forall\alpha,\beta,\gamma \in I.$ (14)

Replace α by υα, $\upsilon \in \mathcal{R}$, in Eq. (14) to get

$0 = \mathcal{F}(\upsilon\alpha,\beta,\gamma) = \alpha\mathcal{F}(\upsilon,\beta,\gamma) + \upsilon\, d(\alpha,\beta,\gamma)$.

By ([14], Theorem (2.4)), the last equation reduce to $0 = \alpha\mathcal{F}(\upsilon,\beta,\gamma), \forall\alpha,\beta \in I, \upsilon \in \mathcal{R}$ since I is ideal and $\mathcal{R}$ is prime, then $\mathcal{F}(\upsilon,\beta,\gamma) = 0$ $\forall\upsilon \in \mathcal{R}, \alpha,\beta,z \in I.$ (15)

Set $\beta = \delta\beta$, $\delta \in R$ in Eq. (15) to get

$0 = \mathcal{F}(\upsilon,\delta\beta,\gamma) = \beta\mathcal{F}(\upsilon,\delta,\gamma) + \delta\, d(\upsilon,\beta,\gamma)$

By([14], Theorem (2.4)),the last equation reduce to $0 = \beta\mathcal{F}(\upsilon,\delta,\gamma), \forall\gamma \in I, \upsilon,\delta \in \mathcal{R}$ and this implies that $F(\upsilon,\delta,\gamma) = 0.$ (16)

Replace γ by γt, $t \in \mathcal{R}$ in Eq. (16) to get

$0 = \mathcal{F}(\upsilon,\delta,t\gamma) = \beta\mathcal{F}(\upsilon,\delta,t) + t\, d(\upsilon,\delta,\gamma)$

By ([14], Theorem (2.4)), the last equation reduce to $0 = \gamma\mathcal{F}(\upsilon,\delta,t), \forall\gamma \in I, \upsilon,\delta,t \in \mathcal{R}$ and this lead us to $\mathcal{F}(\upsilon,\delta,t) = 0, \forall\upsilon,\delta,t \in \mathcal{R}$.

Corollary (3.5):

Let $\mathcal{R}$ be a 6- torsion free prime ring and $\mathcal{F}: \mathcal{R}\times\mathcal{R}\times\mathcal{R} \to \mathcal{R}$ be a generalized permuting left 3-derivation associated with permuting left 3-derivation d such that the trace $\delta_{\mathcal{F}}$ of $\mathcal{F}$ and δ_d are commuting then $\mathcal{F} = 0$.

Theorem (3.6):

Let $\mathcal{R}$ be a 5!-torsion free prime ring and U be an admissible Lie ideal of $\mathcal{R}$. If $\mathcal{F}: U\times U\times U \to \mathcal{R}$ be a generalized permuting left 3- derivation associated with a permuting left 3-derivation d such that the trace $\delta_{\mathcal{F}}$ of $\mathcal{F}$ and the trace δ_d of d are centralizing on U. Then $\delta_{\mathcal{F}}$ is commuting on U.

Proof

Since $\delta_{\mathcal{F}}$ is centralizing on U, then $[\delta_{\mathcal{F}}(\alpha),\alpha] \in \mathcal{Z}(\mathcal{R}), \forall\alpha \in U.$ (17)

By using Proposition (3.2), we get $[\delta_{\mathcal{F}}(\beta),\alpha] + 3[\mathcal{F}(\alpha,\beta,\beta),\beta] \in \mathcal{Z}(\mathcal{R})$ (18)

$\alpha = 2\beta^2$ in Eq. (18), then

$\mathcal{Z}(\mathcal{R}) \ni 2[\delta_{\mathcal{F}}(\beta),\beta^2] + 6[\mathcal{F}(\beta^2,\beta,\beta),\beta]$

$= [\delta_{\mathcal{F}}(\beta),\beta]\beta + 2\beta[\delta_{\mathcal{F}}(\beta),\beta] +$

$12[\beta\,\delta_{\mathcal{F}}(\beta) + \beta\,\delta_d(\beta),\beta]$

$= 4[\delta_{\mathcal{F}}(\beta),y]\beta + 6\beta[\delta_{\mathcal{F}}(\beta),\beta] +$

$6\beta[\delta_d(\beta),\beta]$

$= 10[\delta_{\mathcal{F}}(\beta),\beta]\beta + 6\beta[\delta_d(\beta),\beta]$ (19)

By ([14], Theorem (2.4)), Eq. (19) reduce to

$10[\delta_{\mathcal{F}}(\beta),\beta]\beta \in \mathcal{Z}(\mathcal{R})$

That is , $0 = 10[[\delta_{\mathcal{F}}(\beta),\beta]\beta,\alpha] =$

$10[\delta_{\mathcal{F}}(\beta),\beta][\beta,\alpha] + 10[[\delta_{\mathcal{F}}(\beta),\beta],\alpha]\beta$

$= 10[\delta_{\mathcal{F}}(\beta),\beta][\beta,\alpha]$ (20)

Let $\alpha = 2\alpha\gamma$ in Eq. (20), we get $20\,[\delta_{\mathcal{F}}(\beta),\beta]\alpha\,[\beta,\gamma] + 20\,[\delta_{\mathcal{F}}(\beta),\beta][\beta,\alpha]\gamma = 0$.
By using Eq. (20) and hypothesis, we get
$0 = [\delta_{\mathcal{F}}(\beta),\beta]\alpha\,[\,\beta,\gamma],\ \forall \alpha,\beta,\gamma \in U$. .
By Lemma (2.1) and since U is not commutative we get $\delta_{\mathcal{F}}$ is commuting on U.

Corollary (3.7):

Let $\mathcal{R}$ be a 5!-torsion free prime ring and $\mathcal{F}: \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a generalized permuting left 3-derivation associated with permuting left 3-derivation d such that the trace $\delta_{\mathcal{F}}$ of F and the trace δ_d of d are centralizing on $\mathcal{R}$. Then $\delta_{\mathcal{F}}$ is commuting on $\mathcal{R}$.

Theorem (3.8):

Let $\mathcal{R}$ be a 5!- torsion free prime ring and U be a square closed Lie ideal of $\mathcal{R}$. If there exist a nonzero generalized permuting 3-left derivation $\mathcal{F}: U \times U \times U \to \mathcal{R}$ such that the trace $\delta_{\mathcal{F}}$ of $\mathcal{F}$ centralizing on U. Then U is commutative.

Proof:

Suppose that, U is commutative, then by Lemma (2.3) and Theorem (2.6) $\delta_{\mathcal{F}}$ is commuting and by Theorem (3.4) $\mathcal{F} = 0$ this contradiction with our hypothesis then U is commutative.

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