On aNew Class of Meromorphically Univalent Functions with Applications to Geometric Functions

M.H.Lafta
Department of Statistical ,
College of Administration and Economics,
University of Sumer Al-Refae Thi-qar, Iraq.
mmohamnedhadi@gmail.com

K. A. AL- Zubaidy
Department of Engineering Mechanics,
College of Engineering,
University of Baghdad.
Khawla.mech2016@gmail.com

Abstract- *In this work, we inform a new class of meromorphic univalent function. We derive basic properties such ascoefficient estimates, convex set, extreme points, radius of starlikeness and convexity, hadamard product, integral operator,ρ- neighborhoods and distortion and growth theorem.*

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1. Introduction

Analytical functions could be studied using certain or complex analysis nominated or Geometric functions. Geometric function is characterized by compromising between geometry and analysis. During recent decades, algebraic geometrical methods and theatrical function on compact Rieamann surface have been used in finite- gap, solution concerning non-linear integral system and constructing, **[8]**. The method is also connected through growing specialized area of mathematics to mathematical physics. Early string theory models is utilized for computation Veneziano amplitudes **[12]**. The new progress in approach of constructing to problems of linear and non-linear value and initial value lead to a role for geometric function by using spectral analysis **[10]** Geometric function could be considered as a classical subject.

Assume **M** institute the class of all functions of the form:

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty}k_s n^s \qquad \text{........(1)}$$

Who is analytic and meromorphic univalent in punctured unit disk

$$\xi^*=\{n\in\mathbb{C}: 0<|n|<1\}=\xi\backslash\{0\}$$

Consider subclass T of functions of the form:
$\mathbb{N}=\{1,2,\ldots\}$ ....(2)

A function $h\in$ T is meromorphic univalent starlike function of order $\delta(0\le\delta<1)$ if

$$-\mathrm{Re}\left\{\frac{nh'(n)}{h(n)}\right\}>\delta,\delta(0\le\delta<1;n\in\zeta^*) \qquad \text{......(3)}$$

A function $h\in$ T is meromorphic univalent convex function of order $\delta(0\le\delta<1)$ if

$$-\mathrm{Re}\left\{1+\frac{nh''(n)}{h(n)}\right\}>\delta,\delta(0\le\delta<1;n\in\zeta^*) \qquad \text{....(4)}$$

The convolution of two functions, h is shown in (2) and

$$c(n)=\frac{1}{n}+\sum_{s=1}^{\infty}w_s n^s,(w_s\ge 0,s\in\mathbb{N}=\{1,2,\ldots\}) \quad \text{.....(5)}$$

Is defined by

$$(h*c)(n)=\frac{1}{n}+\sum_{s=1}^{\infty}k_s w_s n^s .$$

Definition (1): Let $h\in$ T be shown in (2). The class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ is defined by

$$\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)=\left\{h\in T\left|\frac{\varsigma n^2h''(n)+\varsigma(1-\eta)nh'(n)-\frac{2\varsigma}{n}}{\frac{\varsigma(1-\eta)-\upsilon}{n}-\upsilon nh'(n)}\right|<\vartheta\right\};$$

$$(0<\upsilon\le 1,0<\varsigma\le 1,0<\vartheta\le 1,0\le\eta\le 1) \qquad \text{......(6)}$$

Different authors executed other class, like, Aouf **[2, 3]**, Aouf and Shammarky **[4]**, Atshan **[5]**, Atshan and Joudah **[6]**, Atshan and Kulkarni **[7]** and Cho, Owa, Lee and O. Altintas **[8]**.

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

2- Coefficient inequality

The first theorem, we get coefficient estimates for h to be in $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

Theorem (1): Let $h \in$ T. Then $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ if and only if

$$\sum_{s=1}^{\infty} s\left[\varsigma(s-\eta)+\upsilon\vartheta\right]k_s \le \varsigma(1+\vartheta)(1-\eta),$$
$$(0<\upsilon\le 1, 0<\varsigma\le 1, 0<\vartheta\le 1, 0\le\eta\le 1) \qquad \text{......(7)}$$

For the following function the result is acute

$$h(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s\left[\varsigma(s-\eta)+\upsilon\vartheta\right]}n^s,\ (s\ge 1)$$

Proof: Presume that the inequality (7) satisfy and postulate $|n|$ = 1. Then from (6), we get

$$\left|\varsigma n^3 h''(n)+\varsigma(1-\eta)n^2 h'(n)-2\varsigma\right|-\vartheta\left|\varsigma(1-\eta)-\upsilon-\upsilon n^2 h'(n)\right|$$

$$\left|\sum_{s=1}^{\infty} s\varsigma(s-\eta)k_s n^{s+1}-\varsigma(1-\eta)\right|-\vartheta\left|\varsigma(1-\eta)-\sum_{s=1}^{\infty} s\upsilon k_s n^{s+1}\right|$$

$$\le\sum_{s=1}^{\infty} s[\varsigma(s-\eta)+\upsilon\vartheta]k_s-\varsigma(1+\vartheta)(1-\eta)\le 0$$

by presumption.
Thus, using the principle of maximum modulus, we obtain $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$

Conversely, assume that h which is defined by (2) content in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.
Hence

$$\left|\frac{\varsigma n^2 h''(n)+\varsigma(1-\eta)nh'(n)-\frac{2\varsigma}{n}}{\frac{\varsigma(1-\eta)-\upsilon}{n}-\upsilon nh'(n)}\right|<\vartheta$$
$$=\left|\frac{\sum_{s=1}^{\infty} s\varsigma(s-\eta)k_s n^{s+1}-\varsigma(1-\eta)}{\varsigma(1-\eta)-\sum_{s=1}^{\infty} s\upsilon k_s n^{s+1}}\right|\le\vartheta$$

Since $\mathrm{Re}(n)\le |n|$ for all n, we have

$$\mathrm{Re}\left\{\frac{\sum_{s=1}^{\infty} s\varsigma(s-\eta)k_s n^{s+1}-\varsigma(1-\eta)}{\varsigma(1-\eta)-\sum_{s=1}^{\infty} s\upsilon k_s n^{s+1}}\right\}\le\vartheta \qquad \text{..........(8)}$$

Upon clearing divisor in (8) and letting $n\longrightarrow 1^-$, for real values, so we can rewrite (8) as follows

$$\sum_{s=1}^{\infty} s[\varsigma(s-\eta)+\upsilon\vartheta]k_s\le\varsigma(1+\eta)(1-\eta)$$

Finally sharpness follows if we take

$$h(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s\left[\varsigma(s-\eta)+\upsilon\vartheta\right]}n^s,\ (s\ge 1).$$

Corollary (1): Let $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then

$$k_s\le\frac{\varsigma(1+\vartheta)(1-\eta)}{s\left[\varsigma(s-\eta)+\upsilon\vartheta\right]},$$

where $(0<\upsilon\le 1, 0<\varsigma\le 1, 0<\vartheta\le 1, 0\le\eta\le 1)$.

3- Convex set

Next Orem, we get the convex set of the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

Theorem (2): Let the functions

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty} k_s n^s,\ (k_s\ge 0)$$
$$c(n)=\frac{1}{n}+\sum_{s=1}^{\infty} w_s n^s,\ (w_s\ge 0)$$

be in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then for 0 ≤ t ≤ 1 the function

$$d(n)=(1-t)h(n)+tc(n)=\frac{1}{n}+\sum_{s=1}^{\infty} u_s n^s \qquad \text{........(9)}$$

where $u_s=(1-t)k_s+tw_s\ge 0$
is also in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

Proof: presume that the functions h and d content in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.
Therefore, making use of **Theorem (1)**. We see that

$$\sum_{s=1}^{\infty} s[\varsigma(s-\eta)+\upsilon\vartheta]u_s$$
$$=(1-t)\sum_{s=1}^{\infty} s[\varsigma(s-\eta)+\upsilon\vartheta]k_s+t\sum_{s=1}^{\infty} s[\varsigma(s-\eta)+\upsilon\vartheta]w_s$$
$$\le(1-t)\varsigma(1+\vartheta)(1-\eta)+t\varsigma(1+\eta)(1-\eta)=\varsigma(1+\eta)(1-\eta)$$

which complete the proof of **Theorem (2)**.

4- Extreme points

In this section we present and prove new Theorem.

Theorem(3): Let $h_0=\frac{1}{n}$ and

$$h_s(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s\left[\varsigma(s-\eta)+\upsilon\vartheta\right]}n^s$$

For s = 1,2,3,… . Then $h\in\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ if and only if it can be expressed in the form

$h(n)=\sum_{s=0}^{\infty} d_s h_s(n)$ where $d_s\ge 0$ and $\sum_{s=0}^{\infty} d_s=1$.

Proof: suppose that $h(n)=\sum_{s=0}^{\infty} d_s h_s(n)$ where $d_s \geq 0$ and $\sum_{s=0}^{\infty} d_s = 1$. Then

$$h(n)=d_0h_0(n)+\sum_{s=1}^{\infty} d_s h_s(n)=d_0\frac{1}{n}+\sum_{s=1}^{\infty} d_s\left(\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}n^s\right)$$

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty}\frac{\varsigma(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}n^s=\frac{1}{n}+\sum_{s=1}^{\infty} k_s n^s$$

Where $u_s=\dfrac{\varsigma(1+\vartheta)(1-\eta)\,\mathrm{d}_s}{s[\varsigma(s-\eta)+\upsilon\vartheta]}$.

By **Theorem (1)**, we have $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ if and only if

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{s(1+\vartheta)(1-\eta)}u_s \leq 1$$

or

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty} u_s n^s$$

Hence

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{s(1+\vartheta)(1-\eta)}d_s\frac{s(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}=\sum_{s=1}^{\infty} d_s=1-d_0\leq 1$$

the proof is complete.

Conversely, assume $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then we show that h can be written in the form:

$$h(n)=\sum_{s=1}^{\infty} d_s h_s(n)$$

New $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, implies form **Theorem (1)**

$$k_s \leq \frac{s(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}$$

Setting

$$d_s=\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{s(1+\vartheta)(1-\eta)}k_s$$

and

$$d_0=1-\sum_{s=1}^{\infty} d_s$$

then

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty} u_s n^s=\frac{1}{n}+\sum_{s=1}^{\infty}\frac{s(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}n^s d_s$$

$$=\frac{1}{n}+\sum_{s=1}^{\infty}\left(h_s-\frac{1}{n}\right)d_s$$

$$=\frac{1}{n}(1-\sum_{s=1}^{\infty} d_s)+\sum_{s=1}^{\infty} d_s h_s=\frac{1}{n}d_0+\sum_{s=1}^{\infty} d_s h_s=\sum_{s=0}^{\infty} d_s h_s(n).$$

5- Radius of starlikeness and convexity

In the dependent theorems, we illustrate the radius Starlikeness and Convexity.

Theorem (4): If $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, Then h is univalent meromorphic Starlike of order δ (0 ≤ δ ≤ 1) in the disk $|n| < r_1$,

$$r_1=\inf\left\{\frac{s(1-\delta)[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(s+\delta)(1+\vartheta)(1-\eta)}\right\}^{\frac{1}{s+1}}$$

The outcome is severe for the function h shown in (2).

Proof: It is appropriate to show that

$$\left|\frac{nh'(n)}{h(n)}+1\right|\leq 1-\delta \quad \text{for} \quad |n| < r_1 \quad ......(10)$$

But $\left|\dfrac{nh'(n)+h(n)}{h(n)}\right|\leq 1-\delta$

$$\sum_{s=1}^{\infty}(s+1)k_s|n|^{s+1}\leq 1-\delta+\sum_{s=1}^{\infty}(1-\delta)k_s|n|^{s+1}$$

$$\frac{\sum_{s=1}^{\infty}(s+\delta)k_s|n|^{s+1}}{1-\delta}\leq 1$$

$$|n|\leq\left\{\frac{s(1-\delta)[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(s+\delta)(1+\vartheta)(1-\eta)}\right\}^{\frac{1}{s+1}}$$

Theorem (5): If $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, then h is univalent meromorphic convex of order δ(0 ≤ δ ≤ 1) in the disk $|n| < r_2$, where

$$r_2=\inf\left\{\frac{(1-\delta)[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(s-\delta+2)(1+\vartheta)(1-\eta)}\right\}^{\frac{1}{s+1}}$$

The score is intense for the function h shown in (2).

Proof: It is suitable to display that

$$\left|\frac{nh''(n)}{h(n)}+2\right|\leq 1-\delta \quad \text{for} \quad |n| < r_2 \quad ....(11)$$

$$\frac{\sum_{s=1}^{\infty}s(s+1)k_s n^{s+1}}{1-\sum_{s=1}^{\infty}sk_s n^{s+1}}\leq 1-\delta$$

$$\frac{\sum_{s=1}^{\infty}s(s-\delta+2)k_s|n|^{s+1}}{1-\delta}\leq 1$$

Theorem (6): Let h, $c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then $h*c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\ell$ for

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty}k_s n^s \quad , \quad c(n)=\frac{1}{n}+\sum_{s=1}^{\infty}w_s n^s$$

and

$$(h*c)(n)=\frac{1}{n}+\sum_{s=1}^{\infty}k_s w_s n^s$$

Where

$$\ell \quad \frac{\varsigma(1+\vartheta)^2(s-\vartheta)(1-\vartheta)-s[\varsigma(s-\eta)+\upsilon\vartheta]^2}{s[\varsigma(s-\eta)+\upsilon\vartheta]^2-\varsigma\upsilon(1+\vartheta)^2(1-\vartheta)}.$$

Proof: h, $c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ and so

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}k_s\leq 1 \quad ....(12)$$

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}w_s \leq 1 \qquad ....(13)$$

Now to calculate the smallest number ℓ as

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\ell}{\varsigma(1+\ell}\kappa_s w_s \geq 1 \qquad ....(14)$$

By Cauchy-Schwarz inequality

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}\sqrt{k_s w_s} \leq 1 \qquad ....(15)$$

$$|n| \leq \left\{\frac{(1-\delta)[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(s-\delta+2)(1+\vartheta)(1-\eta)}\right\}^{\frac{1}{s+1}}$$

which follows the result.

6- Hadamard product

In the subsidiary Theorem, We get the hadamard product of the **functions** h and $c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

Therefore it is enough to show that

$$\frac{s[\varsigma(s-\eta)+\upsilon\ell}{\varsigma(1+\ell}\kappa_s w_s \geq \sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}\sqrt{k_s w_s}$$

that is

$$\sqrt{k_s w_s} \leq \frac{[\varsigma(s-\eta)+\upsilon\vartheta](1+\ell}{[\varsigma(s-\eta)+\upsilon\ell} \qquad ....(16)$$

from (15)

$$\sqrt{k_s w_s} \leq \frac{\varsigma(1+\vartheta)(1-\ell}{s[\varsigma(s-\eta)+\upsilon\vartheta]}$$

Thus it is enough to show that

$$\frac{\varsigma(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]} \leq \frac{[\varsigma(s-\eta)+\upsilon\vartheta](1+\ell}{[\varsigma(s-\eta)+\upsilon\ell}$$

$$\ell \quad \frac{\varsigma(1+\vartheta)^2(s-\vartheta)(1-\vartheta)-s[\varsigma(s-\eta)+\upsilon\vartheta]^2}{s[\varsigma(s-\eta)+\upsilon\vartheta]^2-\varsigma\upsilon(1+\vartheta)^2(1-\vartheta)}$$

Theorem (7): Let $h, c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then

$$h(n)=\frac{1}{n}+\sum_{s=1}^{\infty}(k_s^2+w_s^2)n^s$$

belongs to $\mathrm{MK}(\eta,\upsilon,\varsigma,\ell$, where

$$\ell \quad \frac{2\varsigma^2(1+\vartheta)^2(1-\eta)(s-\eta)-s[\varsigma(s-\eta)+\upsilon\vartheta]^2}{s[\varsigma(s-\eta)+\upsilon\vartheta]^2-2\upsilon\varsigma(1+\vartheta)^2(1-\eta)}$$

Proof: Since $h, c \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. So **Theorem (1)** yields

$$\sum_{s=1}^{\infty}\left[\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}k_s\right]^2 \leq 1 \qquad \text{and}$$

$$\sum_{s=1}^{\infty}\left[\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}w_s\right]^2 \leq 1$$

From the two present inequalities we obtain

$$\sum_{s=1}^{\infty}\frac{1}{2}\left[\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}\right]^2(k_s^2+w_s^2) \leq 1 \qquad ....(17)$$

But $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\ell$ if and only if

$$\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}(k_s^2+w_s^2) \leq 1 .(18)$$

where $0 < \ell < 1$, however, (17) implies .....(18) if

$$\frac{s[\varsigma(s-\eta)+\upsilon\ell}{\varsigma(1+\ell} \geq \frac{1}{2}\left[\frac{\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}\right]^2$$

simplifying, we get

$$\ell \quad \frac{2\varsigma^2(1+\vartheta)^2(1-\eta)(s-\eta)-s[\varsigma(s-\eta)+\upsilon\vartheta]^2}{s[\varsigma(s-\eta)+\upsilon\vartheta]^2-2\upsilon\varsigma(1+\vartheta)^2(1-\eta)}$$

7- Integral Operators with some properties

Next, we consider some properties have been found on the another class in **[13]**.

Theorem (8): If $h \in \mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, then

$$H(n)=\frac{\tau}{n^{\tau+1}}\int_0^n o^{\tau}h(o)\,do,\ \tau>-1$$

Content in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta+1)$, the score is Sharp for the Function h shown in

$$f(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}n^s,\ (k\geq 1) \qquad ....(19)$$

Proof: By definition of $M(n)$, we get

$$M(n)=\frac{\tau}{n^{\tau+1}}\int_0^n o^{\tau}h(o)\,do=\frac{1}{n}+\sum_{s=1}^{\infty}\frac{\tau}{\tau+s+1}k_s n^s,\ \tau>-1$$

In view of **Theorem(1)**, it's enough to display that

$$\sum_{s=1}^{\infty}\frac{\tau s[\varsigma(s-\eta)+\upsilon(\vartheta+1]}{(\tau+s+1)\varsigma(2+\vartheta)(1-\eta)}k_s \leq 1 \qquad ....(20)$$

Since h $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta+1)$, then (20) satisfies if

$$\frac{\tau s[\varsigma(s-\eta)+\upsilon(\vartheta+1)]}{(\tau+s+1)\varsigma(2+\vartheta)(1-\eta)} \leq \frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}$$

or equivalently, when

$$\omega(s,\tau,\varsigma,\eta,\vartheta,\upsilon)=\frac{(1+\vartheta)\tau[\varsigma(s-\eta)+\upsilon(\vartheta+1)]}{(\tau+s+1)(2+\vartheta)[\varsigma(s-\eta)+\upsilon\vartheta]} \leq 1$$

since $\omega(s,\tau,\varsigma,\eta,\vartheta,\upsilon)$ is decreasing of $s(s \geq 1)$.

Then the proof is complete.

Theorem (9): Let the function h be shown in (2) in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then, the integral operator

$$L(n)=\mu\int_0^1 p^{\mu}h(pn)\,dp,\ (0<p\leq 1, 0<\mu<\infty) \qquad .....(21)$$

is in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ where

$$\sigma=\frac{\mu\upsilon\vartheta\varsigma}{(\mu+s+1)[\varsigma(s-\eta)+\upsilon\vartheta]-\mu\varsigma(s-\eta)}$$

The consequence is acute for the function

$$h(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{s[\varsigma(s-\eta)+\upsilon\vartheta]}n^{s}$$

Proof: Let $h(n)=n^{-1}+\sum_{s=1}^{\infty}k_s n^s$ in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then

$$L(n)=\mu\int_0^1 p^{\mu}h(pn)dp=\mu\int_0^1\left(\frac{p^{\mu-1}}{n}+\sum_{s=1}^{\infty}p^{\mu+s}k_s n^s\right)dp$$

$$=\frac{1}{n}+\sum_{s=1}^{\infty}\frac{\mu}{\mu+s+1}k_s n^s$$

It is enough to show that

$$\sum_{s=1}^{\infty}\frac{\mu s[\sigma(s-\eta)+\upsilon\vartheta]}{(\mu+s+1)\sigma(1+\vartheta)(1-\eta)}k_s\le 1 \qquad \text{...........(22)}$$

Since $h\in\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then by **Theorem (1)**. We have

$$\sum_{s=1}^{\infty}\frac{s[\sigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}k_s\le 1$$

Note that (22) is satisfied if

$$\frac{\mu s[\sigma(s-\eta)+\upsilon\vartheta]}{(\mu+s+1)\sigma(1+\vartheta)(1-\eta)}\le\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}$$

or equivalently

$$\sigma=\frac{\mu\upsilon\vartheta\varsigma}{(\mu+s+1)[\varsigma(s-\eta)+\upsilon\vartheta]-\mu\varsigma(s-\eta)}$$

8- *ρ*- neighborhoods

The above concept of ρ-neighborhoods was extended and applied recently to families of certain analytic functions with negative coefficients by Altinta¸s et al. [1] and to families of meromorphically multivalent functions by Liu and Song [13]. The main object of the present paper is to investigate the p-neighborhoods of several subclasses of the class T of normalized analytic functions in U with negative and missing coefficients, which are introduced below by making use of the Ruscheweyh derivatives.

<u>Definition(2):</u>Let

$(0<\upsilon\le 1,\ 0<\varsigma\le 1,\ 0<\vartheta\le 1,\ 0\le\eta<1)$ and $g\ge 0$

We define the ρ-neighborhoods of a function $h\in T$ and denote $\mathrm{N}_g(h)$ such that

$$\mathrm{N}_g(h)=\left\{g\in T: g(n)=\frac{1}{n}+\sum_{s=1}^{\infty}w_s n^s \text{ and } \sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}|k_s-w_s|\le g\right\}$$

…(23)

Goodman **[11]** , Ruscheweyh **[14]** , Altintas and Owa **[1]** have inspected neighborhoods for analytic univalent functions. We consider this notion for the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

<u>Theorem (10):</u>Let the function $h(n)$ defined by (2) be in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, for every complex number ℓ with $|\ell|<g$, $g\ge 0$, let $\frac{h(n)+\ell}{1+\ell}\in\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then $\mathrm{N}_g(h)\subset\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, $g\ge 0$.

Proof: Since $h\in\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$, h satisfies (7) and we can write for $j\in\mathbb{C}$, $|j|=1$, that

$$\frac{\varsigma n^s h''(n)+\varsigma(1-\eta)nh'(n)-\frac{2\varsigma}{n}}{\frac{\varsigma(1-\eta)-\upsilon}{n}-\upsilon nh'(n)}\ne j \qquad \text{.......(24)}$$

Equivalently, we must have

$$\frac{(h*\Im)(n)}{n^{-1}}\ne 0,\ n\in\zeta^*$$

.........(25)

Where

$$\Im(n)=\frac{1}{n}+\sum_{s=1}^{\infty}u_s n^s,$$

such that

$$u_s=\frac{js[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}$$

satisfying

$$|u_s|\le\frac{js[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)} \text{ and } s\ge 1$$

Since $\frac{h(n)+\ell}{1+\ell}\in\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ by .... (26)

$$\frac{1}{n^{-1}}\left(\frac{h(n)+\ell}{1+\ell}*\Im(n)\right)\ne 0 \text{ .(26)}$$

Now assume that $\left|\frac{(h*\Im)(n)}{n^{-1}}\right|<g$. Then, by (26), we have

$$\left|\frac{1}{1+\ell}\frac{(h*\Im)(n)}{}+\frac{\ell}{\ell}\right|\ge\frac{}{\ell}-\frac{}{\ell}\left|\frac{}{}\right|>\frac{\ell}{\ell}\ge 0$$

This is a contradiction as $|\ell|<g$ Therefore

$$\left|\frac{(h*\Im)(n)}{n^{-1}}\right|\ge g$$

Letting

$$g(n)=\frac{1}{n}+\sum_{s=1}^{\infty}w_s n^s\in\mathrm{N}_g(h).$$

Then

$$g-\left|\frac{(c*\Im)(n)}{n^{-1}}\right|\le\left|\frac{(h-c)*\Im(n)}{n^{-1}}\right|\le\left|\sum_{s=1}^{\infty}(k_s-w_s)u_s n^s\right|\le\sum_{s=1}^{\infty}|k_s-w_s||u_s||n|^s$$

$$<|n|^s\sum_{s=1}^{\infty}\frac{s[\varsigma(s-\eta)+\upsilon\vartheta]}{\varsigma(1+\vartheta)(1-\eta)}|k_s-w_s||u_s|\le g$$

Therefore $\frac{(c*\Im)(n)}{n^{-1}}\neq 0$, and we get $c(n)\in$ $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$ so $\mathrm{N}g(h)\subset\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.

9- Distortion and Growth Theorem

Next, we get the distortion and growth theorems for a function h to be belongs in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$

Theorem (11):-Let the Function $h(n)$ defined by (2) be in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$. Then for n $\in\xi^*$, we have

$$\frac{1}{|n|}-\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}|n|\le|h(n)|\le\frac{1}{|n|}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}|n|<1 \quad ....(27)$$

The score is squeaky for the function $h(n)$ specified by

$$h(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}n \quad ......(28)$$

Proof: It is easy to see from **Theorem (1)** that

$$[\varsigma(1-\eta)+\upsilon\vartheta]\sum_{s=1}^{\infty}k_s\le\sum_{s=1}^{\infty}s[\varsigma(s-\eta)+\upsilon\vartheta]k_s\le\varsigma(1+\vartheta)(1-\eta)$$

Then

$$\sum_{s=1}^{\infty}k_s\le\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]} \quad ....(29)$$

Making use of (29), we have

$$|h(n)|\ge\frac{1}{|n|}-|n|\sum_{s=1}^{\infty}k_s$$

$$|h(n)|\ge\frac{1}{|n|}-\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}|n|$$

and

$$|h(n)|\ge\frac{1}{|n|}+|n|\sum_{s=1}^{\infty}k_s$$

$$|h(n)|\le\frac{1}{|n|}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}|n|.$$

Theorem (12): presume the function $h(n)$ acquaint by (2) be in the class $\mathrm{MK}(\eta,\upsilon,\varsigma,\vartheta)$.Then for n $\in\xi^*$, we have

$$\frac{1}{|n|^2}-\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}\le|h'(n)|\le\frac{1}{|n|^2}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]},|n|<1 \quad ....(30)$$

with equality for

$$h(n)=\frac{1}{n}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}n$$

Proof: From (29) and **Theorem (1)** that

$$\sum_{s=1}^{\infty}sk_s\le\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}$$

Consequently, we have

$$|h'(n)|\ge\frac{1}{|n|^2}-\sum_{s=1}^{\infty}sk_s\ge\frac{1}{|n|^2}-\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}$$

and

$$|h'(n)|\le\frac{1}{|n|^2}+\sum_{s=1}^{\infty}sk_s\le\frac{1}{|n|^2}+\frac{\varsigma(1+\vartheta)(1-\eta)}{[\varsigma(1-\eta)+\upsilon\vartheta]}.$$

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