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Insertion of Rotational Effects to the Calculation of J-Integrals Using Finite and Boundary Element Methods

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ABSTRACT

The calculation of J-integrals for cases with rotational effects using the finite or boundary elements needs special treatment. In this work, an attempt is made to derive the J-integral expressions for cases with rotational loading conditions. New useful ideas for reducing domain integrals to boundary integrals and improving the accuracy of J-integrals are developed. The new ideas are implemented in finite and boundary element programs which were developed and validated in this work by taking a case study of central-cracked rotating disc with known analytical solution. The results of the developed programs show good agreements with the analytical solution.

أدخال التأثيرات الدورانية على حسابات تكامل- J باستخدام نظرية العناصر المحددة ونظرية العناصر الحدودية

الخلاصة

ان حسابات تكامل-J لحالات تحتوي على تأثيرات دورانية باستخدام نظرية العناصر المحددة او نظرية العناصر الحدودية تحتاج الى معالجة خاصة.

في هذا البحث تم اشتقاق معادلات تكامل-J لحالات تحتوي على احمال دورانية باستخدام فكرة جديدة ومفيدة في التقليل من التكاملات المجالية وتحويلها الى تكاملات حدودية وكذلك في التحسين من دقة الحسابات. هذه الفكرة الجديدة تم احتوائها في برنامجي عناصر محددة وعناصر حدودية تم اعدادهما في هذا البحث. البرامج المعدة تم التأكد من صحتها ودقتها باستخدام حالة دراسة لقرص دوار يحتوي على شق مركزي. نتائج البرامج المعدة تم مقارنتها مع النتائج التحليلية لحالة الدراسة المستخدمة ووجدت انها في تطابق جيد معها.

NOTATIONS

a_i, b_i = Constants
T_x, T_y = Traction in the x and y directions (N/mm^2)
u, v = Displacement in the x and y directions (mm)
x, y = Coordinates of a point in the structure (mm)
X, Y = Domain loading intensities (N/unit volume)
γ_{xy} = Engineering shear strain
Γ = Boundary of the structure
Ω = Domain of the structure
μ = Modulus of Rigidity (N/mm^2)

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ν = Possion's ratio

INTRODUCTION

It is clear from the literature that Eshelby [1] was the first to derive a number of contour integrals including the so-called J-integral. Cherepanov [2] and Rice [3,4] were apparently the first to apply such an integral to crack problems. The basic advantage of the J-integral is that it is independent of the integration path, and hence it can be evaluated over to avoid singularities and nonlinearities often encountered in the vicinity of the crack tip. Unfortunately, many of the algorithms suggested in the literature, for the estimation of J-integral values are either crude or lack the generality. In this work an attempt has been made to derive the J-integral expressions for cases with rotational effects in a form which includes some useful procedures for reducing domain integrals to boundary integrals and/or improving the accuracy of the J-integrals. These useful procedures are implemented in finite element and boundary element programs developed in this work. The finite element and boundary element formulations for cases with domain type loading can be reviewed in Reference [5].

DERIVATION OF J-INTEGRAL EXPRESSION

The J-integral expression for the case with rotational loading condition can be written as follows

$$J = \int_{\Gamma} W dy - \int_{\Gamma} \left(T_x \frac{\partial u}{\partial x} + T_y \frac{\partial v}{\partial x} \right) ds - \iint_{\Omega} \left(X \frac{\partial u}{\partial x} + Y \frac{\partial v}{\partial x} \right) dxdy \quad \text{………..…..… (1)}$$

Defining the rotational loading term as

$$R_L = \iint_{\Omega} \left(X \frac{\partial u}{\partial x} + Y \frac{\partial v}{\partial x} \right) dxdy \quad \text{…………..…. (2)}$$

Using integration by-parts theorems, equation (2) can be written as

$$R_L = \int_{\Gamma} (X\,u + Y\,v)\, dy - \iint_{\Omega} \left(u \frac{\partial X}{\partial x} + v \frac{\partial Y}{\partial x} \right) dxdy \quad \text{……...……. (3)}$$

The domain loading intensities due to rotational inertia can be expressed as follows

$$X = a_1\, x + a_2\, y + a_3$$

$$Y = b_1\, x + b_2\, y + b_3 \quad \text{…………….. (4)}$$

where a_i, b_i can be obtained from a vectorial equation given in Reference [6]. Hence, using equations (4), it can be deduced that

$$R_L = \int_{\Gamma} (X\,u + Y\,v)\, dy - \iint_{\Omega} (a_1\, u + b_1\, v)\; dxdy \quad \text{……….…… (5)}$$

The domain integration, in the above equation, is evaluated in terms of integration cells within the whole domain, including the crack tip. For most cases, the strain parameters $\partial u/\partial x$ and $\partial v/\partial x$, shown in equation (2), are singular at the crack tip, whilst the displacements u and v shown in equation (5), are not. Hence, it is clear that using equation (5) for the calculation of rotational loading term does not involve any singular parameters and should lead to more accurate results. Note also that for the special case of rotation with respect to the x-axis, a_1 = b_1 = 0, and the domain integral term in equation (5) vanish, leading to simple boundary integration, i.e.

$$R_L = \int_{\Gamma} (X\,u + Y\,v)\, dy \quad \text{…………..…. (6)}$$

For the case of a structure rotating about the z-axis, the corresponding domain intensities can be expressed as follows

$$X = a_1\, x + a_3$$

$$Y = b_2\, y + b_3 \quad \text{…………….. (7)}$$

Rewriting equation (2) as

$$R_L = \iint_\Omega \left(X \frac{\partial u}{\partial x}\right) dxdy + \iint_\Omega \left(Y \frac{\partial v}{\partial x}\right) dxdy \qquad \text{……………..… (8)}$$

Using integration by-parts theorems, it can be shown that

$$\iint_\Omega \left(Y \frac{\partial v}{\partial x}\right) dxdy = \int_\Gamma Y\, v\, dy - \iint_\Omega \left(v \frac{\partial Y}{\partial x}\right) dxdy \qquad \text{………..……… (9)}$$

And by using equations (7), it is clear that $\partial Y/\partial x$ = 0, which means that

$$\iint_\Omega \left(Y \frac{\partial v}{\partial x}\right) dxdy = \int_\Gamma Y\, v\, dy \qquad \text{………….. (10)}$$

The remaining problem is to reduce the term $\iint_\Omega \left(X \frac{\partial u}{\partial x}\right) dxdy$.

Considering the equilibrium equation in terms of displacements given as

$$\nabla^2 u + \frac{1}{1-2\nu'} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + \frac{X}{\mu} = 0 \qquad \text{…………….. (11)}$$

where,

$\nu' = \nu$ for plane strain,

$= \nu/ 1- \nu$ for plane stress.

Defining a function $f(x)$ such that

$$\frac{\partial f}{\partial x} = X$$

$$\frac{\partial f}{\partial y} = 0 \qquad \text{…………….. (12)}$$

The simplest form of such function can be shown as

$$f(x) = \frac{1}{2}\, a_1\, x^2 + a_3\, x \qquad \text{………………. (13)}$$

Hence, equation (11) can be written as

$$\nabla^2 u + \frac{1}{1-2\nu'} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \frac{1}{\mu}\frac{\partial f}{\partial x} = 0 \qquad \text{……………... (14)}$$

For more simplifications, the equilibrium equation may be written as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{2(1-\nu')} \frac{\partial}{\partial y}\left[\left(1-2\nu'\right)\gamma_{xy} + 2\nu' \frac{\partial v}{\partial x}\right] + \frac{1-2\nu'}{2(1-\nu')\mu}\frac{\partial f}{\partial x} = 0$$

Now, defining

$$C_1 = \frac{1-2\nu'}{4(1-\nu')\mu}, \qquad C_2 = \frac{1-2\nu'}{2(1-\nu')}, \qquad C_3 = \frac{\nu'}{1-\nu'}$$

Hence, the equilibrium equation can be expressed as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left[C_2\,\gamma_{xy} + C_3 \frac{\partial v}{\partial x}\right] + 2\,C_1 \frac{\partial f}{\partial x} = 0 \qquad \text{……………… (15)}$$

Using the weighted residual concept, and considering $f(x)$ be a weighting function, then

$$\iint_\Omega f(x)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(C_2\,\gamma_{xy} + C_3 \frac{\partial v}{\partial x}\right) + 2\,C_1 \frac{\partial f}{\partial x}\right] dxdy = 0$$

Expanding the above equation, gives

$$\iint_\Omega f(x) \frac{\partial^2 u}{\partial x^2}\, dxdy + \iint_\Omega f(x) \frac{\partial}{\partial y}\left(C_2\,\gamma_{xy} + C_3 \frac{\partial v}{\partial x}\right) dxdy +$$

$$\iint_\Omega C_1 \frac{\partial}{\partial x}\, [f(x)]^2\, dxdy = 0$$

Using integration by-parts theorems, it can be deduced that

$$\int_\Gamma f(x) \frac{\partial u}{\partial x}\, dy - \iint_\Omega \frac{\partial f}{\partial x}\frac{\partial u}{\partial x}\, dxdy - \int_\Gamma f(x)\left(C_2\,\gamma_{xy} + C_3 \frac{\partial v}{\partial x}\right) dx -$$

$$\iint_\Omega \frac{\partial f}{\partial y}\left(C_2\,\gamma_{xy} + C_3 \frac{\partial v}{\partial x}\right) dxdy + \int_\Gamma C_1\, f^2(x)\, dy = 0$$

Now, by considering equations (12), it can be proved that the above expression may be reduced to

$$\int_\Gamma f(x)\, \frac{\partial u}{\partial x}\, dy - \iint_\Omega X\, \frac{\partial u}{\partial x}\, dxdy - \int_\Gamma f(x)\, \left(C_2\, \gamma_{xy} + C_3\, \frac{\partial v}{\partial x}\right) dx + \int_\Gamma C_1\, f^2(x)\, dy \;= 0$$

Rearranging, gives

$$\iint_\Omega X\, \frac{\partial u}{\partial x}\, dxdy = \int_\Gamma f(x)\, \frac{\partial u}{\partial x}\, dy - \int_\Gamma f(x)\, \left(C_2\, \gamma_{xy} + C_3\, \frac{\partial v}{\partial x}\right) dx + \int_\Gamma C_1\, f^2(x)\, dy \;= 0$$

or

$$\iint_\Omega X\, \frac{\partial u}{\partial x}\, dxdy = \int_\Gamma f(x)\, \left[\frac{\partial u}{\partial x} + C_1\, f(x)\,\right] dy - \int_\Gamma f(x)\, \left(C_2\, \gamma_{xy} + C_3\, \frac{\partial v}{\partial x}\right) dx \quad \ldots\ldots\ldots\ldots\ldots (16)$$

Hence, from equations (8), (10), and (16), it can be shown that

$$R_L = \int_\Gamma f(x)\, \left[\frac{\partial u}{\partial x} + C_1\, f(x)\,\right] dy - \int_\Gamma f(x)\, \left(C_2\, \gamma_{xy} + C_3\, \frac{\partial v}{\partial x}\right) dx \;- \int_\Gamma Y\, v\, dy \quad \ldots\ldots\ldots\ldots\ldots. (17)$$

The above equation shows clearly that the domain integrals of equation (8) are reduced strictly to boundary integrals. Computationally, this neglect the need for two dimensional integration cells often encountered in the calculation of domain integrals which obviously reduce computer time and data preparation effort.

CENTRAL-CRACKED ROTATING DISC ANALYSIS

In order to verify the developed expressions and ideas of the calculation of J-Integrals with rotational loading conditions, a case study of central-cracked rotating disc, shown in Figure (1), has been analyzed. Due to the symmetry of the disc, only one quarter of its domain has been modeled with the same degrees of freedom for finite element and boundary element meshes. Using the developed finite element and boundary element programs, the J-integral values for this case study have been evaluated for different crack ratios and compared with analytical solution of Cartwright & Rooke [7], as shown in Figure (2). The calculated values prove that the developed procedure gives very accurate results, compared with the analytical solution, up to a crack ratio of $\frac{a}{R} = 0.35$, beyond which the analytical solution is no longer valid as stated in reference [7].

To study the effects of the rotational loading conditions and the crack ratio on the path independency of the J-integral, the J-values at different contours around the crack tip have been evaluated for different crack ratios as shown in figures (3-6). It is clear from these figures that the rotational loading conditions have no effect on the path independency of the J-integral. However, the crack ratio has some effect not on the path independency of the J-integral but also on the accuracy of the results. This fact is very clear in Figure (3) when the crack ratio is very small, and it can be seen that the best agreement with the analytical solution is gained when the crack ratio $\frac{a}{R} = 0.25$ as shown in Figure (4). Also it is clear from these figures that when the crack ratio exceeds the analytical solution limiting value of 0.35, the finite and boundary element J-integral values, although converging with each other, start to diverge away from the analytical solution, whilst the path independency of their J-values improves noticeably.

CONCLUSIONS

It is clear from the above analysis that the derived expressions and the developed ideas are verified for plane stress condition and proved to be very accurate and efficient for the calculation of J-integrals with rotational effects. Some conclusion remarks can be drawn as follows

-The developed finite and boundary element programs have given the same results as shown in Figures (2-6).

-The rotational loading conditions have no effect on the path independency of the J-integral as shown in Figures (3-6).

-The crack ratio has some effect on the path independency of the J-integral and also on the accuracy of the results as can be seen from Figures (3-6).

RECOMMENDATION

It is recommended to test and verified the developed ideas and procedure for plane strain condition by using a case study of rotating cylinder.

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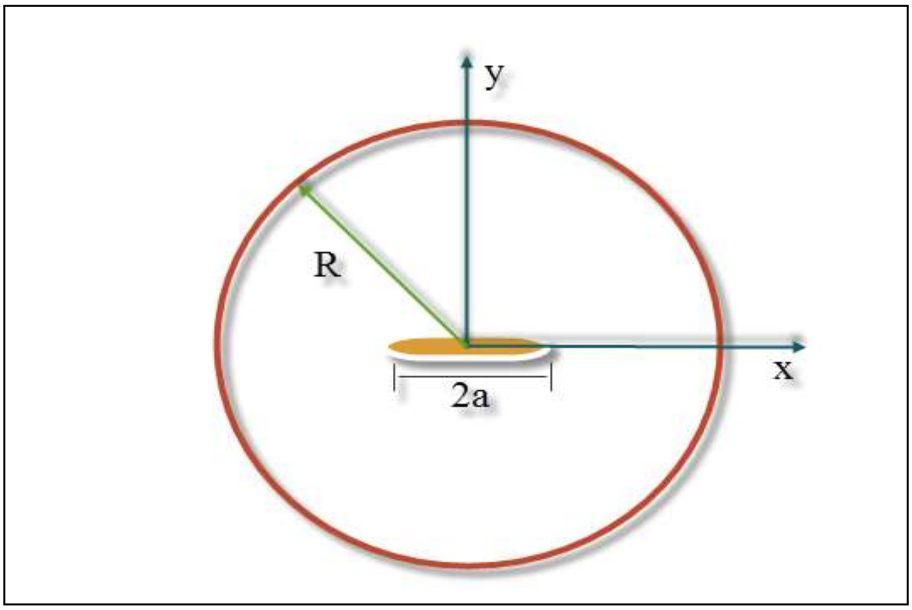

Possion's ratio = 0.3
Angular velocity = 20 rad/s
Material density = 0.025 kg/mm^3
Disc radius = 20 mm
Young Modulus= 10 GN/m^2

Figure (1) Central Cracked Rotating Disc.

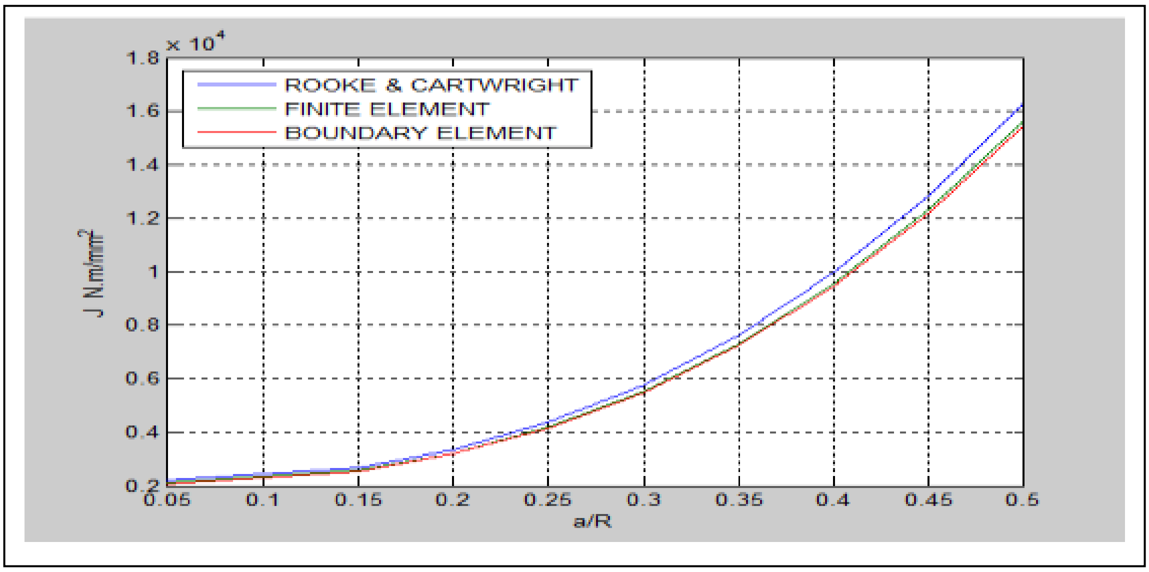

Figure (2) J-integral values calculated at different crack ratios.

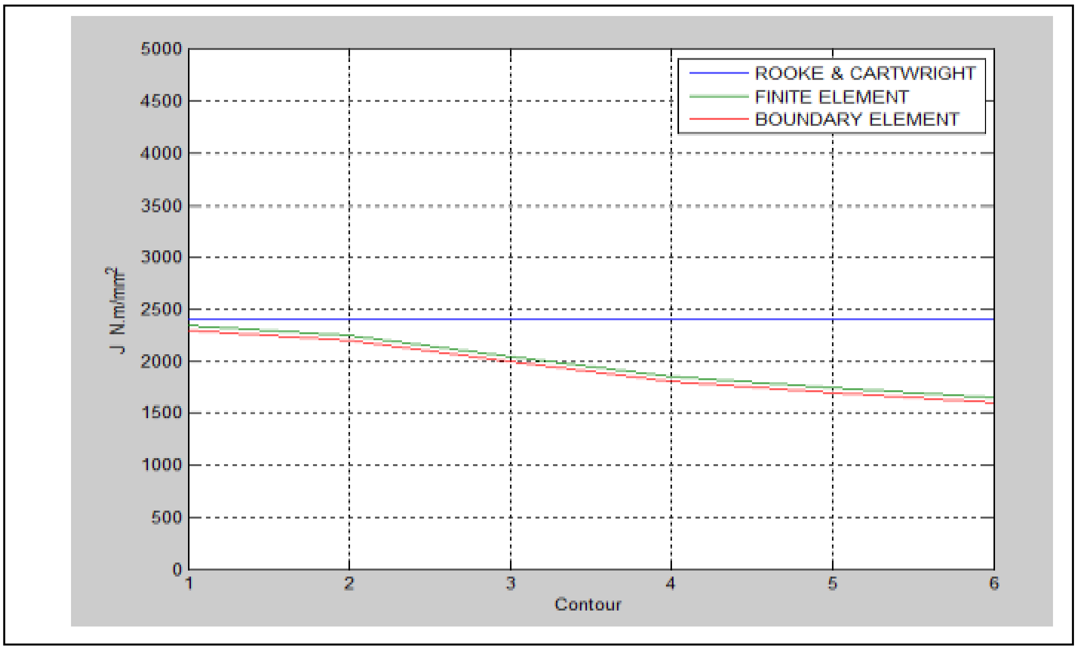

Figure (3) J-integral values calculated at different contours around the crack tip at a/R = 0.125.

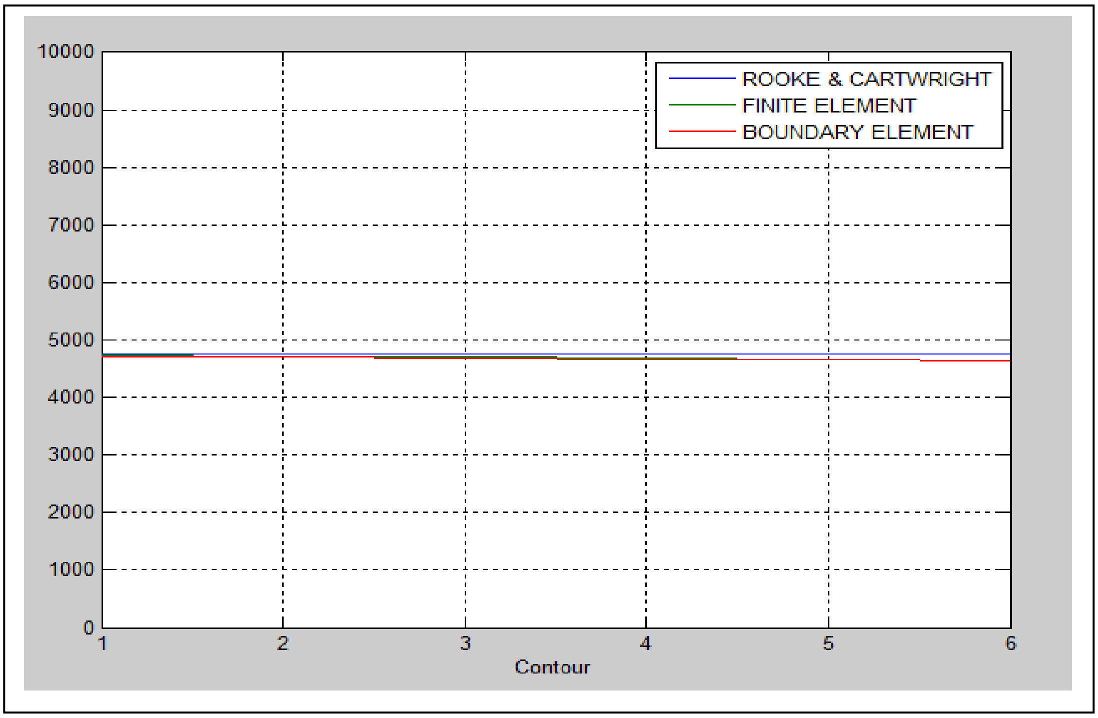

Figure (4) J-integral values calculated at different contours around the crack tip at a/R = 0.25.

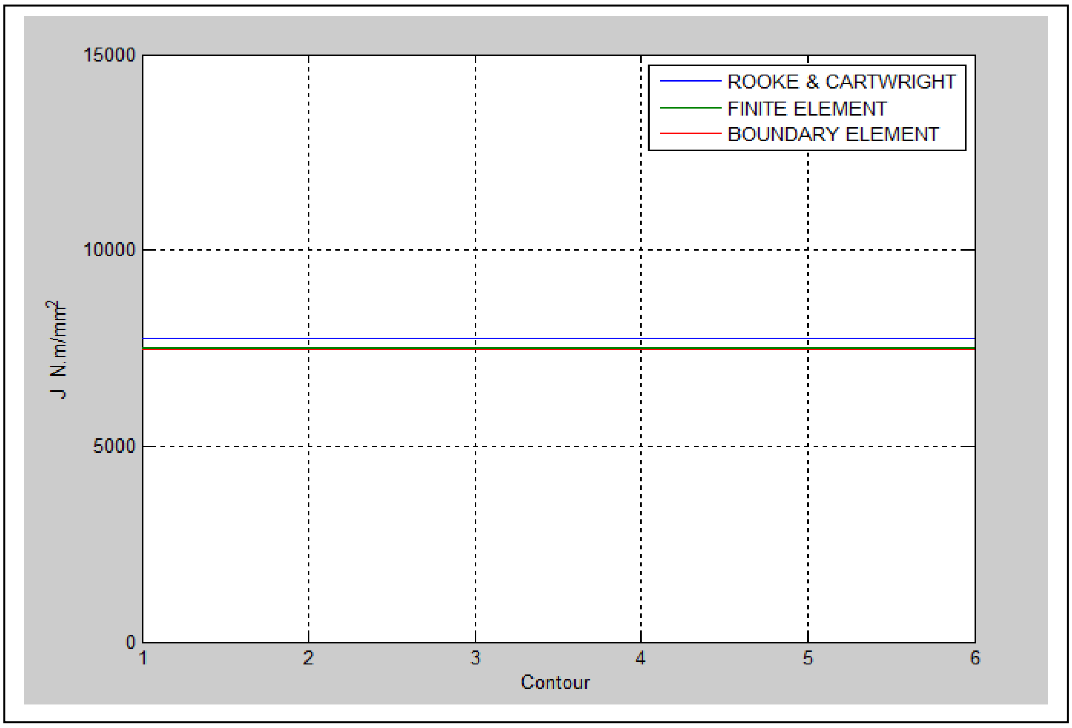

Figure (5) J-integral values calculated at different contours around the crack tip at a/R = 0.375.

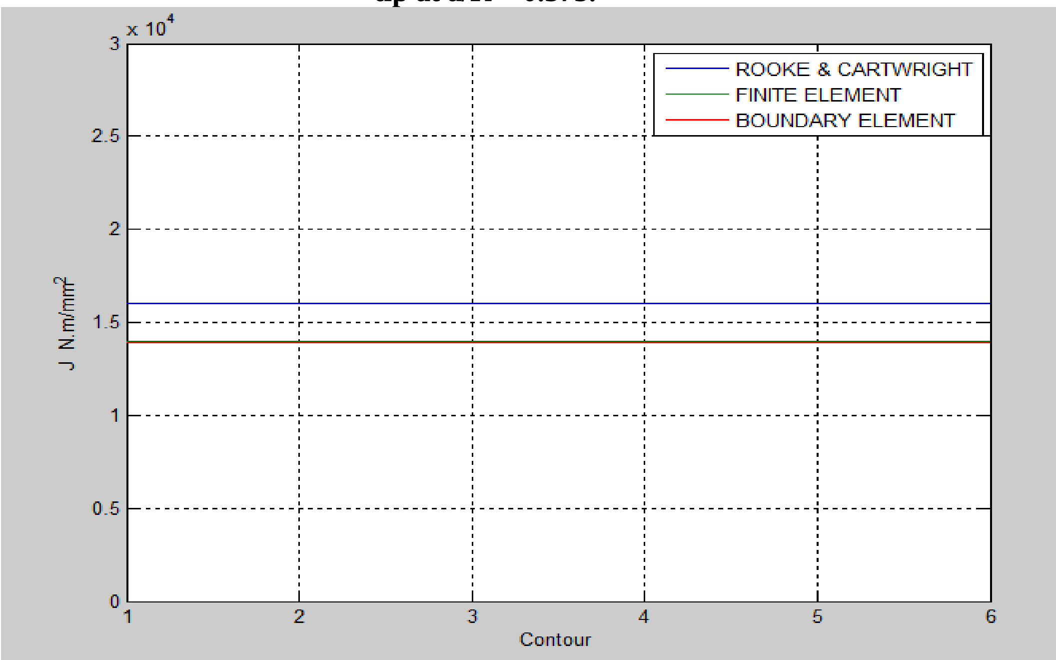

Figure (6) J-integral values calculated at different contours around the crack tip at a/R = 0.5.