Modified Method for Generating B-Spline Curves of Degree Three and Their Controlling

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ABSTRACT
The reaction between designer, and design needs modified methods to control the design. This paper presents modified mathematical technique for controlling the generation of the 2D designs of third degree, by using modified Gallier of Bezier curves. The paper discuses a polynomial in terms, of polar forms, with respect to the parameter. The modified method has resulted in good starting point, to generate which 2D design, algorithm which allows the designer to produce a design in combinational way allows him to get the shape that he has in his mind keeping the four control points for 2D design. The method shows a great flexibility in 2D design controlling area with changing. There is no need to change the control points of the design; moreover efficiency in designs is obtained in comparison with that needed for conventional methods.

Keywords: Gallier of Bezier curves**,** B-Spline Curves.

تطوير طريقة لتوليد منحنيات (B-Spline) من الدرجة الثالثة والسيطرة عليها

الخلاصة
تطوير التفاعل بين المصمم والتصميم يحتاج الى تطوير طرق التحكم بالتصميم وهذا البحث يعرض تقنية رياضيه مطورة للسيطرة على توليد التصاميم ثنائية ألأبعاد من الدرجة الثالثة. بأسد دام تطوير Gallier لمنحني B-Spline . تم مناقشة منحني متعددة الحدود في ألأستقطاب لحدوده المفصلية بالنسبة الى المتغيرات. وهذا السبب يعطي نقطة بداية جيده لأقتراح تقنية لتوليد التصاميم ثنائية ألأبعاد والذي يتيح للمصمم توليدالمنحني وتطويره بطريقة تفاعلية تمكنة من الحصول على الشكل الذي كونة في مخيلتة مع الاحتفاظ بتوافقه مع نقاط السيطرة الاربعه الهيكلية للمخطط ثنائية ألأبعاد.
أثبتت الطريقة المقترحة مرونة عالية في مجال التحكم بالتصميم ثنائية ألأبعاد من دون تغيير نقاط السيطرة للتصميم. كذلك أثبتنا الطريقة أنها أكثر كفاءة في التصاميم بالمقارنة مع مايوجد من طرق سابقة.

INTRODUCTION
This paper takes up the study of a polynomial curve. Polynomial curve is defined in terms, of polar forms. Natural way to polarize polynomial curve. The approach yields polynomial curve .It is shown versions of the de-Boor algorithm can be turned into subdivisions, by giving an efficient method of performing subdivision. It is also shown that it is easy to compute a new control net

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from given net. Intuitively this depends on the parameters, this is one of indications that deals with curve. The affine frame, for simplicity of notation is denoted as $F(u)$. Intuitively, a polynomial curve is obtained by bending the real affine curve using a polynomial map. This present method a different method, for controlling and generating the 2D design. The arithmetical technique is used to generate Gallier cubic B-Spline curve by using de- Boor algorithm, as given in [1], [2].This new work is modified for controlling and generate curves with no need to change the control points of B-Spline curve.

Polar Form of a Polynomial Curve. [1], [2].

A method of specifying polynomial curves that yield very nice geometric constructing the curves is used the polar polynomial form.

Consider a polynomial of degree two as

$$F(X) = aX^2 + bX + c.$$

The function of two variables

$$f_1(x_1,x_2) = ax_1x_2 + bx_1 + c.$$

The polynomial $F(X)$ on the diagonal, in the sense that $F(X)=f_1(X, X)$, for all $X\in R$, f_1 is affine in each of $x_1$and x_2. It would be tempting to say that f_1 is linear in each of x_1 and x_2, this is not true, due to the presence of the term bx_1 and of the constant c, and f_1 is only biaffine. Not that

$$f_2(x_2, x_1) = ax_2x_1 + bx_2 + c.$$

Also affine and $F(X)=f_2(X, X)$ for all $X\in R$. To find a unique biaffine function f such that, $F(X)=f(X, X)$, for all $X\in R$, and of course such a function should satisfy some addition property. It turns out that requiring f to be symmetric. The function f of two arguments is symmetric

$$f(x_1,x_2)=f(x_2,x_1), \text{for all } x_1,x_2,$$

to make f_1(and f_2) symmetric simply form :

$$\begin{aligned} f(x_1,x_2) &= [f_1(x_1,x_2)+f_2(x_2,x_1)]/2 \\ &= [a(x_1x_2)+ a(x_2x_1)+ bx_1+bx_2 +c+c]/2 \\ &= a(x_1x_2)+ [bx_1+bx_2]/2+c. \end{aligned}$$

This called the polar form of quadratic polynomial of F. Given a polynomial of degree three as

$$F(X) = aX^3 + bX^2 + cX + d.$$

The polar form of F is a symmetric affine function

$f: A^3 \rightarrow A$ that takes the same value for all permutations of x_1, x_2, x_3; that is

$$f(x_1, x_2, x_3)=f(x_2, x_1, x_3)$$

$$=f(x_1, x_3, x_2)=f(x_2, x_3, x_1)$$
$$=f(x_3, x_1, x_2)=f(x_3, x_2, x_1).$$

Which is affine in each argument and such that

$F(X)=f(X, X, X)$, for all $X\in R$, by same way of second degree as

$$f(x_1, x_2, x_3)=ax_1x_2x_3 +[x_1x_2+ x_1x_3+x_2x_3]/3+c[x_1+x_2+x_3]/3 +d.$$

This called the polar form of cubic polynomial of F.

Example 1:

Consider the polynomial of degree two given by

$F_1(t)=10t,$
$F_2(t)= t^2-2t.$
The polar forms of $F_1=x(t)$ and
$F_2=y(t)$ are
$f_1(t_1, t_2)= 5(t_1+t_2),$
$f_2(t_1, t_2)=t_1t_2-(t_1+t_2).$
Note that $f_1(t_1, t_2)$ is the polar form of $F_1=x(t)$, and $f_2(t_1, t_2)$ is the polar form of $F_2=y(t)$.

Example 2:

Consider a plane cubic which is defined as follows
$F_1(t)=30t,$
$F_2(t)= 3t^3-3t.$
The polar forms of $F_1= x(t)$ and
$F_2 = y(t)$ are
$f_1(t_1, t_2, t_3)= 10(t_1+t_2+t_3),$
$f_2(t_1, t_2, t_3)= 3t_1t_2t_3-(t_1+t_2+t_3).$
Also notice that $f_1(t_1, t_2, t_3)$ is the polar form of, $F_1= x(t)$, and $f_2(t_1, t_2, t_3)$ is polar form of $F_2 = y(t)$
.

THE DE BOOR MODIFICATION OF DE CASTELJAU ALGORITHM

Consider one more generalization of the de Casteljau algorithm. This generalization will be useful when deal with spline, such a version will be called the progressive version for reasons that will become clear shortly when dealing with spline.

Definition [1], [2].

Consider control points in the form $f(u_{k-2+i}, u_{k-1+i}, u_{k+i})$, $0 \le i \le 3$, suppose the degree three (m=3) where u_i are real numbers, $k\in Z$), (where Z is integer number), taken from the sequence, $\{u_{k-2+i}, u_{k-1+i}, u_{k+i}, u_{k+1+i}, u_{k+2+i}\ u_{k+3+i}\}$, of length (2m= 6) satisfying certain inequality conditions. The sequence $\{u_{k-2+i}, u_{k-1+i}, u_{k+i}, u_{k+1+i}, u_{k+2+i}\ u_{k+3+i}\}$, is said to be progressive iff the inequalities indicated in the following array hold:

$u_{k-2}\quad \neq$
$u_{k-1}\quad \neq \quad \neq$
$u_k\quad \neq \quad \neq \quad \neq$

$u_{k+1} \quad u_{k+2} \quad u_{k+3}$

Is obtaining as following:-

At stage 1 $u_{k-2} \neq u_{k+1}$,

$u_{k-1} \neq u_{k+2}$, and $u_k \neq u_{k+3}$.

This corresponds to the inqualities on main descending diagonal of the array of inequality conditions.

At stage 2 $u_{k-1} \neq u_{k+1}$, $u_k \neq u_{k+2}$.This corresponds to the inequalities on second descending diagonal of the array of inequality conditions. At stage 3 $u_k \neq u_{k+1}$.This corresponds on third-lowest descending diagonal of the array of inequality conditions.

For example at k=3, and m=3. Consider control points in the form

$f(u_{1+i}, u_{2+i}, u_{3+i}), 0 \leq i \leq 3$, and suppose u_i taken from the sequence, $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, of length (2m= 6) satisfying certain inequality conditions. The sequence $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, is said to be progressive iff the inequalities indicated in the following array hold:

$$\begin{array}{cccc} u_1 & \neq & & \\ u_2 & \neq & \neq & \\ u_3 & \neq & \neq & \neq \\ & u_4 & u_5 & u_6 \end{array}$$

Is obtaining as following;

At stage 1 $u_1 \neq u_4$, $u_2 \neq u_5$, $u_3 \neq u_6$

At stage 2 $u_2 \neq u_4$, $u_3 \neq u_5$.

At stage 3 $u_3 \neq u_4$.

The four control points are:

$f(u_1, u_2, u_3)$, $f(u_2, u_3, u_4)$, $f(u_3, u_4, u_5)$, $f(u_4, u_5, u_6)$. The points are obtained from the sequence $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, by sliding a window of length 3 over the sequence from left to right, this explains the term ``progressive ‘‘.

DE BOOR ALGORITHM OF DEGREE THREE

From de Boor algorithm, the following cases will be analyzed: [1], [2].

Case: m=1

The progressive sequence is $\{u_1, u_2\}$, and the control points $f(u_1)$ and. $f(u_2)$.

Observe that these points are obtained from the sequence $\{u_1, u_2\}$, by sliding a window of length 1 over the sequence from left to right. Number of stages is one.

Let us begin with straight lines. Given any on interval $[u_1, u_2]$ for which

$u_1 \neq u_2$ for $t \in R$, can be written uniquely at

$$t= [1- \lambda]\, u_1 + \lambda u_2 = u_1 + \lambda [u_2 - u_1].$$

And can find that:

$$\lambda = \frac{t-u_1}{u_2-u_1}, \; 1-\lambda = \frac{u_2-t}{u_2-u_1}.$$

These sequences turn out to define two de Boor control points for the curve segment $F(t)$ associated with the interval $[u_1, u_2]$, if $f(t)$ is the polar form of segment $F(t)$ these de Boor points are the polar value.

$f(t)=f[(1-\lambda)u_1 + \lambda u_2]$
$\quad =(1-\lambda)f(u_1) + \lambda f(u_2)$
$f(t)= (1-\lambda) f(u_1)+ \lambda f(u_2),$
$F(t) =F[(1-\lambda) u_1+ \lambda u_2]$
$\therefore F(t) = (1-\lambda) F(u_1) +\lambda F(u_2).$

De Boor algorithm uses two control points say $F(u_1)$ and $F(u_2)$. See fig.1.

Substitution $\lambda = \frac{t-u_1}{u_2-u_1}$, $1-\lambda = \frac{u_2-t}{u_2-u_1}$. gives

$$F(t)= \frac{u_2-t}{u_2-u_1} f(u_1)+ \frac{t-u_1}{u_2-u_1} f(u_2).$$

As said already every t belong to R can be expressed uniquely as a bray center combination of u_1 and u_2 say that interpolation at $t=u_1$ then

$F(t)=f(u_1)$, and at $t=u_2$ then
$F(t)=f(u_2)$.

Case: m=3.

Table.1, fig.2, and inequalities in the progressive array still hold
The point $f(t, t, t)$ is computed as follows, see fig.2.

$$f(t, t, t)= f(t, t, (1-\lambda_6) u_3 + \lambda_6 u_4)$$

$$= (1-\lambda_6)f(t,t,u_3)+\lambda_6 f(t,t,u_4), \quad \ldots. (1)$$

Where

$$f(t, t, u_3)=(1-\lambda_4) f(t, u_2, u_3) +\lambda_4 f(t, u_3, u_4) \quad \ldots.(2)$$

$$f(t, t, u_4) = (1-\lambda_5) f(t, u_3, u_4) + \lambda_5 f(t, u_4, u_5) \quad \ldots..(3)$$

and

$$f(t, u_2, u_3)= (1-\lambda_1)f(u_1, u_2, u_3) + \lambda_1 f(u_2, u_3, u_4) \quad \ldots.. (4)$$

$$f(t, u_3, u_4)= (1-\lambda_2)f(u_2, u_3, u_4) +\lambda_2 f(u_3, u_4, u_5) \quad \ldots\ldots(5)$$

$$f(t, u_4, u_5) = (1-\lambda_3)f(u_3, u_4, u_5) +\lambda_3 f(u_4, u_5, u_6) \quad \ldots\ldots(6)$$

Substitution of Eqs {2, 3, 4, 5,and6} in (1) gaves

$F(t)=f(t,t,t)$
$\quad =(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)f(u_1, u_2,u_3) + [\lambda_1(1-\lambda_6)(1-\lambda_4)+(1-\lambda_2)(1-\lambda_6)\lambda_4+(1-\lambda_2)(1-\lambda_5)\lambda_6]$
$f(u_2, u_3, u_4) +[\lambda_2 \lambda_4(1-\lambda_6) + \lambda_6 (1-\lambda_5)\lambda_2)+ \lambda_5 \lambda_6(1-\lambda_3)] f(u_3, u_4, u_5) + \lambda_3\lambda_5\lambda_6 f(u_4,u_5,u_6)$
….. (7)

Where :

$$\lambda_1=\frac{t-u_1}{u_4-u_1},\ 1-\lambda_1=\frac{u_4-t}{u_4-u_1}$$

$$\lambda_2=\frac{t-u_2}{u_5-u_2},\ 1-\lambda_2=\frac{u_5-t}{u_5-u_2}$$

$$\lambda_3=\frac{t-u_3}{u_6-u_3},\ 1-\lambda_3=\frac{u_6-t}{u_6-u_3}$$

$$\lambda_4=\frac{t-u_2}{u_4-u_2},\ 1-\lambda_4=\frac{u_4-t}{u_4-u_2}$$

$$\lambda_5=\frac{t-u_3}{u_5-u_3},\ 1-\lambda_5=\frac{u_5-t}{u_5-u_3}$$

$$\lambda_6=\frac{t-u_3}{u_4-u_3},\ 1-\lambda_6=\frac{u_4-t}{u_4-u_3}$$

Substitute $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ in Eq (7) gives

$$F(t)=f(t,t,t)=\{\frac{[u_4-t]^3}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}\}$$

$$f(u_1,u_2,u_3)]$$

$$+\{\frac{[t-u_1][u_4-t]^2}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}+\frac{u_5-t}{u_5-u_2}\,\frac{t-u_2}{u_4-u_2}\,\frac{u_4-t}{u_4-u_3}$$

$$+\frac{[u_5-t]^2[t-u_3]}{(u_5-u_2)(u_5-u_3)(u_4-u_3)}\}$$

$$f(u_2,u_3,u_4)\qquad +\{\frac{[t-u_2]^2[u_4-t]}{(u_5-u_2)(u_4-u_2)(u_4-u_3)}+\frac{t-u_2}{u_5-u_2}\,\frac{u_5-t}{u_5-u_3}\,\frac{t-u_3}{u_4-u_3}$$

$$+\frac{[u_6-t][t-u_3]^2}{(u_6-u_3)(u_5-u_3)(u_4-u_3)}$$

$$\}f(u_3,u_4,u_5)+\{\frac{[t-u_3]^3}{(u_6-u_3)(u_5-u_3)(u_4-u_3)}\}$$

$$f(u_4,u_5,u_6) \qquad \ldots..(8)$$

Which is a cubic polynomial in *t*?

GALLIER MODIFIED CUBIC B-SPLINE CURVES [1], [2].
For a modified cubic B-Spline, m=3, the sequence of (2m=6) consecutive knots:
$[u_{k-m+1}, u_{k-m+2}, u_{k-m+3}, u_{k-m+4}, u_{k-m+5}, u_{k-m+6}]$
$=[u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3})$, which yield 4 sequence of consecutive knots
$(u_{k-2+i}, u_{k-1+i}, u_{k+i})$, each of length 3 where $0 \leq i \leq 3$.
These sequence turn out to define 4 de Boor control points for the curve segment F_k associated with the middle interval $[u_k. u_{k+1}]$, if f_k is the polar form of segment F_k these de Boor points are the polar value.
Given a knot sequence $\{u_k\}$, and a set of de Boor control points d_k, where $d_k = f(u_{k+1}, \ldots, u_{k+m})$ for every k such that $u_k < u_{k+1}$. For every $t \in [u_k, u_{k+1}]$ the B-spline curve $F_k(t)$ will take the form defined by the following :

$$F_k(t)=\sum_j B_{j,m+1,k}(t)d_j=\sum_j B_{j,m+1,k}(t)f(u_{j+1},\ldots, u_{j+m}),$$

where the B`s are to be defined later. The polar form $f_k(t_1,\ldots, t_m)$ of $F_k(t)$ is
$f_i(t\ldots, t)=$

$$\sum_j b_{j,m+1,k}(t_1,\ldots,t_m)f(u_{j+1},\ldots, u_{j+m}).$$

Where $b_{j,m+1,k}$ is the polar form of $B_{j,m+1,k}$. The polar form f_k is influenced by the $m+1$ de Boor control points b_j for $j \in [k-m, k]$. The $b_{j,m+1,k}(t_1,\ldots, t_m)$ are computed from the recurrence relation :-

$$b_{j,m+1,k}(t_1,\ldots, t_m)=\frac{t_m-u_j}{u_{j+m}-u_j}b_{j,m,k}(t_1,\ldots,t_{m-1})+\frac{u_{j+m+1}-t_m}{u_{j+m+1}-u_{j+1}}b_{j+1,m,k}(t_1,\ldots, t_{m-1}), \quad \ldots\ldots(9)$$

Where

$$b_{j,m+1,k}(\,)=d_{j,k}, \quad \ldots..(10)$$

and
$d_{j,k}=1$ iff $j=k$, and
$d_{j,k}=0$ otherwise,
$d_{j,k}$ is called the Kronecker delta. Putting all $t_j=t$, and drop the subscript k, that get the standard recurrence relation defining the B-splines [de Boor, and Cox 78]. [3], [4].
Let now;

$$B_{j,1}(t)=\begin{cases}1 & \text{if } t\in[u_j, u_{j+1}\\ 0 & \text{otherwise}\end{cases}.$$

$$B_{j,m+1}(t)=\frac{t-u_j}{u_{j+m}-u_j}B_{j,m}(t)+\frac{u_{j+m+1}-t}{u_{j+m+1}-u_{j+1}}B_{j+1,m}(t).$$

As a special case, the previous work will be calculated for cubic spline where $m=3$ corresponding to the interval (u_3, u_4), $\{t\in [u_3, u_4]\}$ where $k=3$ and $j\in [0, 3]$.Now

$$f_3(t_1, t_2, t_3) = b_{0,4,3}(t_1, t_2, t_3)\, d_0+b_{1,4,3}(t_1, t_2, t_3)\, d_1+ b_{2,4,3}(t_1, t_2, t_3)\, d_2+b_{3,4,3}(t_1,t_2,t_3)d_3 \quad .....(11)$$

For $j=0$

$$b_{0,4,3}(t_1, t_2, t_3)= \frac{t_3-u_0}{u_3-u_0}\, b_{0,3,3}(t_1, t_2) + \frac{u_4-t_3}{u_4-u_1}\, b_{1,3,3}(t_1, t_2).$$

From (9) and (10) $b_{0,3,3}(t_1, t_2)=0$, and

$$b_{1,3,3}(t_1, t_2)= \frac{t_2-u_1}{u_3-u_1}\, b_{1,2,3}(t_1) + \frac{u_4-t_2}{u_4-u_2}\, b_{2,2,3}(t_1),$$

Where ; $b_{1,2,3}(t_1)=0$, and

$$b_{2,2,3}(t_1) = \frac{t_1-u_2}{u_3-u_1}\, b_{2,1,3}(\,) + \frac{u_4-t_1}{u_4-u_3}\, b_{3,1,3}(\,).$$

Where $b_{2,1,3}(\,)=0$,
and $b_{3,1,3}(\,)=1$.

Also from *(9)* and *(10):*
$b_{2,1,3}(\,)=0$, $b_{3,1,3}(\,)=1$.

$$b_{2,2,3}(t_1) = \frac{u_4-t_1}{u_4-u_3}$$

$$b_{1,3,3}(t_1,t_2)=\frac{u_4-t_2}{u_4-u_2}\,\frac{u_4-t_1}{u_4-u_3}$$

Hence $b_{0,4,3}(t_1,t_2,t_3)= \frac{u_4-t_3}{u_4-u_1}\,\frac{u_4-t_2}{u_4-u_2}\,\frac{u_4-t_1}{u_4-u_3}$.

If $t_j=t$, then

$$b_{0,4,3}(t_1,t_2,t_3)= \frac{[u_4-t]^3}{(u_4-u_1)(u_4-u_2)(u_4-u_3)} \quad ....(12)$$

At $j=1$

$$b_{1,4,3}(t_1, t_2, t_3)= \frac{t_3-u_1}{u_4-u_1}\, b_{1,3,3}(t_1, t_2)$$

$$+\frac{u_5-t_3}{u_5-u_2}\; b_{2,3,3}(t_1,t_2),$$

$b_{2,3,3}(t_1,t_2)=$

$$\frac{t_2-u_2}{u_4-u_2}\; b_{2,2,3}(t_1)$$

$$+\frac{u_5-t_2}{u_5-u_3}\; b_{3,2,3}(t_1),$$

Where

$$b_{3,2,3}(t_1)=\frac{t_1-u_3}{u_4-u_3}\; b_{3,1,3}(\,)+0$$

$$=\frac{t_1-u_3}{u_4-u_3}$$

$$b_{2,3,3}(t_1,t_2)=\frac{t_2-u_2}{u_4-u_2}\frac{u_4-t_1}{u_4-u_3}$$

$$+\frac{u_5-t_2}{u_5-u_3}\frac{t_1-u_3}{u_4-u_3}$$

$$b_{1,4,3}(t_1,t_2,t_3)=\frac{t_3-u_1}{u_4-u_1}\frac{u_4-t_2}{u_4-u_2}\frac{u_4-t_1}{u_4-u_3}+\frac{u_5-t_3}{u_5-u_2}\frac{t_2-u_2}{u_4-u_2}\frac{u_4-t_1}{u_4-u_3}$$

$$+\frac{u_5-t_3}{u_5-u_2}\frac{t_2-u_2}{u_4-u_2}\frac{u_4-t_1}{u_4-u_3}+\frac{u_5-t_3}{u_5-u_2}\frac{u_5-t_2}{u_5-u_3}\frac{t_1-u_3}{u_4-u_3}.$$

If $t_j=t$, then

$$b_{1,4,3}(t,t,t)=\frac{[t-u_1][u_4-t]^2}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}+\frac{u_5-t}{u_5-u_2}\frac{t-u_2}{u_4-u_2}\frac{u_4-t}{u_4-u_3}$$

$$+\frac{[u_5-t]^2[t-u_3]}{(u_5-u_2)(u_5-u_3)(u_4-u_3)}. \qquad \text{.....(13)}$$

At j=2

$b_{2,4,3}(t_1,t_2,t_3)=$

$$\frac{t_3-u_2}{u_5-u_2}\; b_{2,3,3}(t_1,t_2)$$

$$+\frac{u_6-t_3}{u_6-u_3}\; b_{3,3,3}(t_1,t_2).$$

Where

$b_{3,3,3}(t_1,t_2)=$

$$\frac{t_2 - u_3}{u_5 - u_3} b_{3,2,3}(t_1) + 0$$

$$= \frac{t_2 - u_3}{u_5 - u_3} \frac{t_1 - u_3}{u_4 - u_3}$$

$$\therefore b_{2,4,3}(t_1,t_2,t_3) = \frac{t_3 - u_2}{u_5 - u_2} \frac{t_2 - u_2}{u_4 - u_2} \frac{u_4 - t_1}{u_4 - u_3} + \frac{t_3 - u_2}{u_5 - u_2} \frac{u_5 - t_2}{u_5 - u_3} \frac{t_1 - u_3}{u_4 - u_3}$$

$$+ \frac{u_6 - t_3}{u_6 - u_3} \frac{t_2 - u_3}{u_5 - u_3} \frac{t_1 - u_3}{u_4 - u_3}.$$

If $t_j = t$, then

$$b_{2,4,3}(t,t,t) = \frac{[t - u_2]^2 [u_4 - t]}{(u_5 - u_2)(u_4 - u_2)(u_4 - u_3)} + \frac{t - u_2}{u_5 - u_2} \frac{u_5 - t}{u_5 - u_3} \frac{t - u_3}{u_4 - u_3}$$

$$\frac{[u_6 - t][t - u_3]^2}{(u_6 - u_3)(u_5 - u_3)(u_4 - u_3)}. \quad \text{..... (14)}$$

At j=3

$b_{3,4,3}(t_1, t_2, t_3)$

$$= \frac{t_3 - u_3}{u_6 - u_3} b_{3,3,3}(t_1, t_2) + 0$$

$\therefore b_{3,4,3}(t_1,t_2,t_3) =$

$$\frac{t_3 - u_3}{u_6 - u_3} \frac{t_2 - u_3}{u_5 - u_3} \frac{t_1 - u_3}{u_4 - u_3}$$

If $t_j = t$, then

$$\therefore b_{3,4,3}(t,t,t) = \frac{[t - u_3]^3}{(u_6 - u_3)(u_5 - u_3)(u_4 - u_3)} \quad \text{......(15)}$$

Substituting (12), (13), (14) and (15) in (11) gives: -

$F_k\,(t)=$

$$\left\{ \frac{[u_4-t]^3}{(u_4-u_1)(u_4-u_2)(u_4-u_3)} \right\} d_0 + \left\{ \frac{[t-u_1][u_4-t]^2}{(u_4-u_1)(u_4-u_2)(u_4-u_3)} + \frac{u_5-t}{u_5-u_2}\,\frac{t-u_2}{u_4-u_2}\,\frac{u_4-t}{u_4-u_3} + \frac{[u_5-t]^2[t-u_3]}{(u_5-u_2)(u_5-u_3)(u_4-u_3)} \right\} d_1$$

$$+\left\{ \frac{[t-u_2]^2[u_4-t]}{(u_5-u_2)(u_4-u_2)(u_4-u_3)} + \frac{t-u_2}{u_5-u_2}\,\frac{u_5-t}{u_5-u_3}\,\frac{t-u_3}{u_4-u_3} + \frac{[u_6-t][t-u_3]^2}{(u_6-u_3)(u_5-u_3)(u_4-u_3)} \right\} d_2 + \left\{ \frac{[t-u_3]^3}{(u_6-u_3)(u_5-u_3)(u_4-u_3)} \right\} d_3 \quad \ldots.(16)$$

The Eq (16) is called a cubic B-Spline.

In the case $k=0$ then $t \in [u_0, u_1]$ and
$(u_1, u_2, u_3, u_4, u_5, u_6)=(u_{-2}, u_{-1}\, u_0, u_1\, u_2, u_3)$.
For the special case $k=0$, let $(u_{-2}, u_{-1}\, u_0, u_1\, u_2, u_3)=(-2, -1, 0, 1, 2, 3)$, then $t \in [0,1]$, and $t_1=t_2=t_3=t$. and(16) becomes:

$$F_0(t)=\frac{1}{6}(1-t)^3 b_0+\frac{1}{6}\{3t^3-6t^2+4\}b_1+\frac{1}{6}\{-3t^3+3t^2+3t+1\}b_2+\frac{1}{6}t^3 b_3 \quad \ldots(17)$$

Eq (17) called original B-Spline curve dependent on interval $[0, 1]$. [1], [2], [3]. [4], [5], [6], [7]. [8].

DEVELOPED CUBIC B-SPLINE CURVES

To construct the new cubic B-Spline. Take the case m=3. The sequence [2m=6] consecutive knots
$[u_{k-m+1}, u_{k-m+2}, u_{k-m+3}, u_{k-m+4}, u_{k-m+5}, u_{k-m+6}]$
$=[u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}]$, yields 4 sequences of consecutive knots ($u_{k-2+i}, u_{k-1+i}, u_{k+i}$), each of length 3 where $0 \le i \le 3$, and these sequences turn out to define 4 de Boor control points for the curve segment f_k associated with the middle interval [u_k. u_{k+1}]. f_k is the polar form of segment F_k. Then the de Boor control points $d_{k+i} = f_i(u_{k-2+i}, u_{k-1+i}, u_{k+i})$, where i= 0, 1, 2, 3, are given by:

$d_k = f_0(u_{k-2}, u_{k-1}, u_k)$ at $i=0$
$d_{k+1} = f_1(u_{k-1}, u_k, u_{k+1})$ at $i=1$
$d_{k+2} = f_2(u_k, u_{k+1}, u_{k+2})$ at $i=2$
$d_{k+3} = f_3(u_{k+1}, u_{k+2}, u_{k+3})$ at $i=3$.

Observe that these points are obtained from the sequence $\{u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}\}$, by using the new de Boor algorithm for calculating $f(t, t, t)$ at each value of t where $u_k \le t \le u_{k+1}$ is given in Table 2 below.

From stage 3, $f(t, t, t_6)$ is given by :

$$f(t, t, t_3)= (1-\lambda_6)f(t, t_2, u_k) +\lambda_6 f(t, t_2, u_{k+1}) \qquad \text{....(18)}$$

From stage 2, $f(t, t_2, u_k)$ and $f(t, t_5, u_{k+1})$ is given by:

$$f(t, t_2, u_k)=(1-\lambda_4 f(t_1, u_{k-1}, u_k) +\lambda_4 f(t_1, u_k, u_{k+1}), \qquad \text{..... (19)}$$

$$f(t, t_2,u_{k+1})= (1-\lambda_5)f(t_1, u_k, u_{k+1}) + \lambda_5 f(t_1, u_{k+1}, u_{k+2}). \qquad \text{.... (20)}$$

From stage 1, $f(t_1, u_{k-1}, u_k)$, $f(t_2, u_k .u_{k+1})$, and $f(t_3, u_{k+1},u_{k+2})$are given by:

$$f(t_1, u_{k-1}, u_k)= (1-\lambda_1) f(u_{k-2}, u_{k-1}, u_k) + \lambda_1 f(u_{k-1}, u_k, u_{k+1}), \qquad \text{.... (21)}$$

$$f(t_1, u_k,u_{k+1}) = (1-\lambda_2)f(u_{k-1}, u_k, u_{k+1}) +\lambda_2 f(u_k, u_{k+1}, u_{k+2}) \qquad \text{...... (22)}$$

$$f(t_3, u_{k+1},u_{k+2}) =(1-\lambda_3) f(u_k, u_{k+1}, u_{k+2}) +\lambda_3 f(u_{k+1}, u_{k+2}, u_{k+3}). \qquad \text{.... (23)}$$

Substitution of Eqs {19, 20, 21, 22, and 23} in (18) gives

$$F_k=f_k(t_1, t_2, t_3)=(1-\lambda_6)(1-\lambda_4)(1-\lambda_1) [f(u_{k-2}, u_{k-1}, u_k)] +\{(1-\lambda_4)(1-\lambda_6)\lambda_1+(1-\lambda_2)(1-\lambda_6) \lambda_4$$

$$+(1-\lambda_2)(1-\lambda_5) \lambda_6\} f(u_{k-1}, u_k, u_{k+1}) +\{\lambda_2\lambda_4(1-\lambda_6)+\lambda_6(1-\lambda_5)\lambda_2+(1-\lambda_3)\lambda_5\lambda_6\}f(u_k,u_{k+1}, u_{k+2}) + \lambda_3\lambda_5 \lambda_6 f(u_{k+1}, u_{k+2}, u_{k+3}). \qquad \text{.... (24)}$$

Suppose $f(u_{k-2}, u_{k-1}, u_k)]$, $f(u_{k-1}, u_k, u_{k+1})$, $f(u_k,u_{k+1}, u_{k+2})$, $f(u_k,u_{k+1}, u_{k+2})= d_0,d_1 ,d_2 ,d_3$ are control points and equation 24 becomes.

$$\boldsymbol{F_k=f_k(t_1,t_2,t_3)=(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)d_0+\{(1-\lambda_4)(1-\lambda_6)\lambda_1+(1-\lambda_2)(1-\lambda_6)\lambda_4+(1-\lambda_2)(1-\lambda_5) \lambda_6\}d_1+\{\lambda_2\lambda_4(1-\lambda_6)+\lambda_6(1-\lambda_5)\lambda_2+(1-\lambda_3)\lambda_5\lambda_6\}d_2 +\lambda_3\lambda_5\lambda_6 d_3.} \qquad \text{..... (25)}$$

Eq (25) thé formula of a **new cubic B-Spline.**

Treat the coordinates of each point as a two-component vector and using the symbols d_0, d_1, d_2 and d_3 for control points. Let as given as, [5], [6], [10]. [11].

$d_i=(x_i, y_i)$ for i=0,1,…,m,

and

$$d_i=\begin{pmatrix} x_i \\ \\ y_i \end{pmatrix},$$

The set of points, in parametric form is

$$d(t)=\begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix}, u_k \le t \le u_{k+1}$$

Where

$$\lambda_1=\frac{t-u_{k-2}}{u_{k+1}-u_{k-2}},\ 1-\lambda_1=\frac{u_{k+1}-t}{u_{k+1}-u_{k-2}},$$

$$\lambda_2=\frac{t-u_{k-1}}{u_{k+2}-u_{k-1}},\ 1-\lambda_2=\frac{u_{k+2}-t}{u_{k+2}-u_{k-1}},$$

$$\lambda_3=\frac{t-u_k}{u_{k+3}-u_k},\ 1-\lambda_3=\frac{u_{k+3}-t}{u_{k+3}-u_k},$$

$$\lambda_4=\frac{t-u_{k-1}}{u_{k+1}-u_{k-1}},\ 1-\lambda_4=\frac{u_{k+1}-t}{u_{k+1}-u_{k-1}},$$

$$\lambda_5=\frac{t-u_k}{u_{k+2}-u_k},\ 1-\lambda_5=\frac{u_{k+2}-t}{u_{k+2}-u_k},$$

$$\lambda_6=\frac{t-u_k}{u_{k+1}-u_k},\ 1-\lambda_6=\frac{u_{k+1}-t}{u_{k+1}-u_k}.$$

The formula of a new equation of cubic B-Spline in (25) becomes

$$F_k(t)=\left\{\frac{[u_{k+1}-t]^3}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})(u_{k+1}-u_k)}\right\}\frac{(u_{k+2}-t)(u_{k+1}-t)(t-u_{k-1})}{(u_{k+1}-u_k)(u_{k+2}-u_{k-1})(u_{k+1}-u_{k-1})}+$$

$$\frac{[u_{k+2}-t]^2[t-u_k]}{(u_{k+1}-u_k)(u_{k+2}-u_k)(u_{k+2}-u_{k-1})}\ \}\ d_1+\left\{\frac{[t-u_{k-1}]^2[u_{k+1}-t]}{(u_{k+1}-u_k)(u_{k+1}-u_{k-1})(u_{k+2}-u_{k-1})}\right.$$

$$+\frac{t-u_k}{u_{k+1}-u_k}\,\frac{u_{k+2}-t}{u_{k+2}-u_k}\,\frac{t-u_{k-1}}{u_{k+2}-u_{k-1}}+\frac{[u_{k+3}-t][t-u_k]^2}{(u_{k+1}-u_k)(u_{k+2}-u_k)(u_{k+3}-u_k)}\ \}\ d_2$$

$$+\left\{\frac{[t-u_k]^3}{(u_{k+1}-u_k)(u_{k+2}-u_k)(u_{k+3}-u_k)}\right\}d_3. \qquad \text{...... (26)}$$

For the case $k=3$ then $t\in [u_3, u_4]$

F_k (t)=

$$\left\{\frac{[u_4-t]^3}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}\right\}d0\ \frac{[t-u_1][u_4-t]^2}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}$$

$$+\ \{$$

$$+\frac{u_5-t}{u_5-u_2}\,\frac{t-u_2}{u_4-u_2}\,\frac{u_4-t}{u_4-u_3}+\frac{[u_5-t]^2[t-u_3]}{(u_5-u_2)(u_5-u_3)(u_4-u_3)}\ \}\ d_1$$

$$+\{ \frac{[t-u_2]^2[u_4-t]}{(u_5-u_2)(u_4-u_2)(u_4-u_3)} + \frac{t-u_2}{u_5-u_2}\,\frac{u_5-t}{u_5-u_3}\,\frac{t-u_3}{u_4-u_3} + \frac{[u_6-t][t-u_3]^2}{(u_6-u_3)(u_5-u_3)(u_4-u_3)} \} d_2 + \frac{[t-u_3]^3}{(u_6-u_3)(u_5-u_3)(u_4-u_3)} \{\} d3 \quad .....(27)$$

The Eq (27) is identical with Eq (16)

In the case $k=0$ then $t\in [u_0, u_1]$,and

$(u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3})=(u_{-2}, u_{-1} u_0, u_1 u_2, u_3)$.

For the special case k=0, let $(u_{-2}, u_{-1} u_0, u_1 u_2, u_3)=(-2, -1, 0, 1, 2, 3)$, then $t\in [0, 1]$, and $t_1=t_2=t_3=t$. and (27) becomes:

$$F_0(t) = \frac{1}{6}(1-t)^3 d_0 + \frac{1}{6}\{3t^3 - 6t^2 + 4\} d_1 + \frac{1}{6}\{-3t^3 + 3t^2 + 3t + 1\} d_2 + \frac{1}{6} t^3 d_3 \quad ...(28)$$

The Eq (28) is identical with Eq (17)

To explain the above new formula of B-Spline curve (25), the following example is given

Example: -

Given the following control points:

$d_0=(x_0, y_0) = 100,300,$

$d_1=(x_1, y_1) = 100, 50,$

$d_2=(x_2, y_2) = 300, 50,$

$d_3=(x_3, y_3) = 300,300,$

$d_4=(x_4, y_4) = 100,300,$

$d_5=(x_5, y_5) = 100, 50,$

$d_6=(x_6, y_6) = 300, 50,$

In the case $k=0$ then $t\in [u_0, u_1]$,and

$(u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3})=(u_{-2}, u_{-1} u_0, u_1 u_2, u_3)$.

For the special case k=0, let $(u_{-2}, u_{-1} u_0, u_1 u_2, u_3) =(-2, -1, 0, 1, 2, 3)$, then $t\in [0, 1]$.

The following cases will be studied

Case 1:

Taking the **special case** $u_k = u_0 = 0$, and $u_{k+1} = u_1 = 1$, and $t_1=t_2=t_3=t$. where $t\in [0,1]$, then Eq (25), reduces to original, Eq(17). The **change** of the curve take only when change the control points. See fig.3.

Case 2:

The parameter $(+\lambda_1)$ is taken to be increased with step different from that of the parameter t in piecewise or in all piecewise is starting with $t= u_k$ *and* ending with $t= u_{k+1}$. The value of λ_1 (i.e. t_1) is taken according to the desire of the designer when he wants the design to be changed in **exterior manner**. This can be seen in fig. 4 the changes took place with **no change** among the control points.

Case 3:

The parameter $(-\lambda_1)$ is taken to be decreased with step different from that of the parameter t in piecewise or in all piecewise is starting with $t= u_k$ and ending with $t=$

u_{k+1}. The value of $-\lambda_1$ (i.e. t_1) is taken according to the desire of the designer when he wants the design to be changed in **interior manner**. This can be seen in fig. 5. The changes took place with **no change** among the control points.

Case 4:

$(1-\lambda_1)$ is taken to be varied with step, say h. In this case it is found that the design can be moved **upward** with **no need to change** any of the control points. See fig. 6.

Case 5:

The parameter - $(1-\lambda_1)$ is taken to be decreased with step different from that of the parameter t is starting with $t= u_k$ and ending with $t= u_{k+1}$. The value of

$-(1-\lambda_1)$ (i.e. t_1) is taken according to the desire of the designer when he wants the design to be changed in **interior manner**. This

can be seen in fig. 7. The changes took place with **no change** among the control points.

{Eq (25) is built on new mathematical procedure. A procedure that can be developed by mathematicians and designers in the future to give other new properties}.

Case 6:

The parameter $(+\lambda_1)$ is taken to be increased with step different from that of the parameter t in **all piecewise** is starting with $t= u_k$ and ending with $t= u_{k+1}$. The value of λ_1 (i.e. t_1) is taken according to the desire of the designer when he wants the design to be changed in **exterior manner**. This can be seen in fig. 8. The changes took place with **no change** among the control points.

Case 7:

The parameter $(-\lambda_1)$ is taken to be decreased with step different from that of the parameter t in **all piecewise** is starting with $t= u_k$ and ending with $t= u_{k+1}$. The value of $(-\lambda_1)$ (i.e. t_1) is taken according to the desire of the designer when he wants the design to be changed in **interior manner**. This can be seen in fig. 9. The changes took place with **no change** among the control points

CONCLUSIONS

In this work conclude the following points:

1-The developed B-Spline equation is based on a mathematical procedure depending on the linear construction of polynomials and following

de-Casteljau and de Boor algorithms. This led to a general procedure that can be used easily.

2-A constriction of a modified formula for B-Spline curve has been achieved through a procedure following de-Boor algorithm. The procedure has been developed in a sequential and mathematical way as it is obvious in formula (25). It has the following advantages:

a-The modified linear mathematical construction of the equation gives the

Designer move choice to reach and modify any segment of the design.

b-The modified formula works on much more flexible real values of the parameter t, that is on a sequence of the form $\ldots, u_0$, u_1, u_2, …, rather than the special familiar frame $[0, 1]$.

c-The designer has advantage of cotvolnj and modifying all or part of the design through the values of the parameter t without changing any of the control points.

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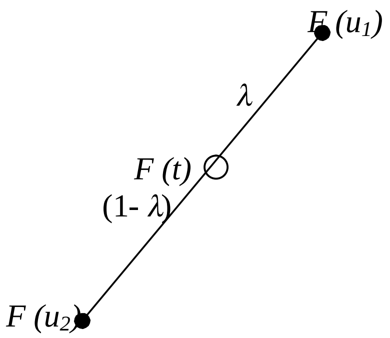


Figure (1) Linear Curve Interpolation at t= u_1 then, $F(t) = F(u_1)$, and, at t= u_2 then $F(t) = F(u_2)$

Table (1) (i, j)	de Boor 0	algorithm(m=3) 1	2	3
0	$f(u_1,u_2,u_3)$	$f(t,u_2,u_3)$		
1	$f(u_2,u_3,u_4)$	$f(t,u_3,u_4)$	$f(t,t,u_3)$	$f(t,t,t)$
2	$f(u_3,u_4,u_5)$	$f(t,u_4,u_5)$	$f(t,t,u_4)$	
3	f(u4, u5,u6)			

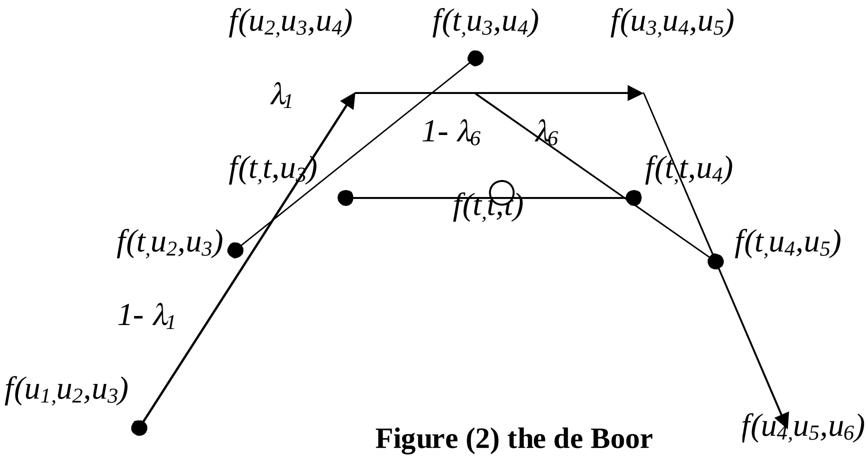
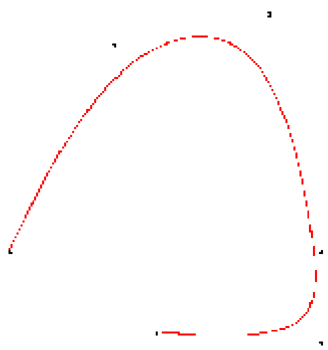
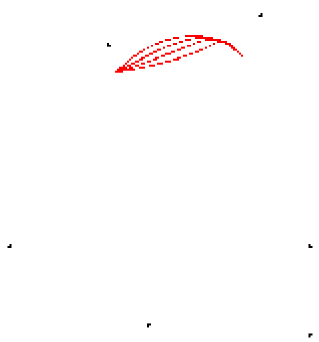


Figure (2) the de Boor Algorithme

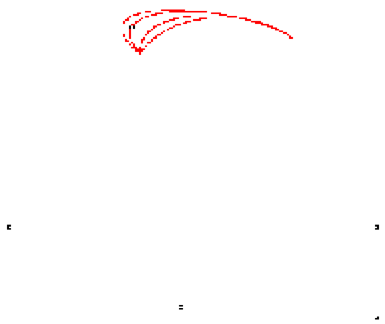
Table 2 *(i, j)*	de Boor *0*	algorithm*(m=3)* *1*	*2*	*3*
0	$f(u_{k-2}, u_{k-1},u_k)$	$f(t_1, u_{k-1}, u_k)$		
1	$f(u_{k-1}, u_k,u_{k+1})$	$f(t_2, u_k, u_{k+1})$	$f(t, t_4, u_k)$	$f(t, t, t)$
2	$f(u_k,u_{k+1},u_{k+2})$	$f(t_3,u_{k+1},u_{k+2})$	$f(t, t_5, u_{k+1})$	
3	$f(u_{k+1},u_{k+2},u_{k+3})$			



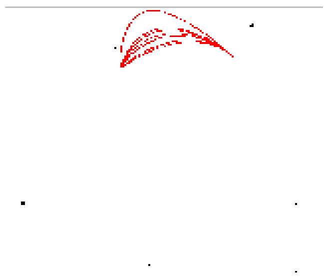
Figure(3) Cubic B-Spline special case $u_k = u_0 = 0$, and $u_{k+1} = u_1 = 1$, *then* $t \in [0,1]$. $u_k = u_0 = 0$, and $u_{k+1} = u_1 = 1$, where $t \in [0,1]$.



Figure(4) Cubic B-Spline when take *(+ λ_1)* changed in exterior manner. The changes took place with no change among the control points.



Figure(5) Cubic B-Spline when take *(-λ_1)* changed in interior manner. The changes took place with no change among the control points.



Figure(6) Cubic B-Spline when take *(1-λ_1)* moved upward with no need to change any of the control points.

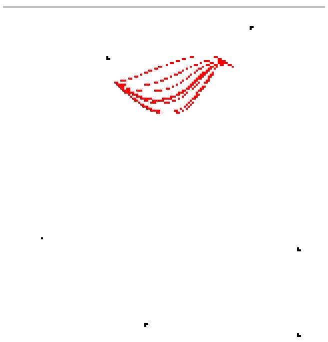


Figure (7) Cubic B-Spline when take -*(1- λ_1)* changed in interior manner. The changes took place with no change among the control points.

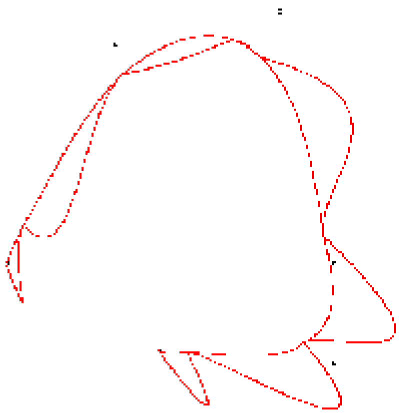


Figure (8) Cubic B-Spline. When (+ λ_1) is increases with step different from that of the parameter t in all piecewise, changed in exterior manner. The changes took place with no change among the control points

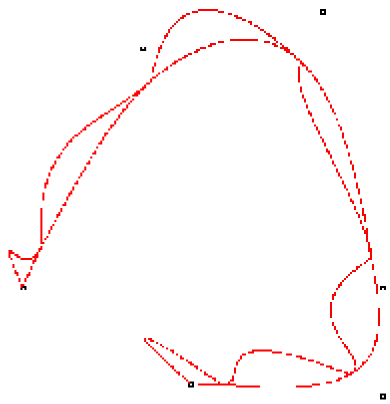


Figure (9) Cubic B-Spline. When *($-\lambda_1$)* is decreases with step different from that of the parameter t in all piecewise, changed in interior manner. The changes took place with no change among the control points.