*Eng. & Tech. Journal, Vol.30, No.4, 2012*

# Numerical Solution of Linear Delay Fredholm Integral Equations by using Homotopy Perturbation Method

**Shymaa Hussain Salih**
Applied Sciences Department, University of Technology/Baghdad
**Isra'a Hadi Hassan**
Applied Sciences Department, University of Technology/Baghdad
**Fuad A. Al Heety**
Applied Sciences Department, University of Technology/Baghdad
Email:FuadHeety@yahoo.com

**Received on: 13/9 /2011 &Accepted on: 5/1/2012**

## ABSTRACT

The main purpose of this work is to propose the Homotopy perturbation method for solving first type Linear Delay Fredholm Integral Equations (LDFIE).A Comparison between the numerical solution and exact solution has been made depending on least squares errors. Results are presented in tables and figures. efficient and accuracy are appeared in applying of this the proposed method.

# الحل العددي لمعادلات فريدهولم التكاملية التباطؤية الخطية بأستخدام طريقة هوموتوبي المتخلخل

## الخلاصة

الغرض الأساسي في هذا العمل هو اقتراح طريقة هوموتوبي المتخلخل لحل النوع الأول من معادلات فريدهولم التكاملية التباطؤية الخطية . كما تم اجراء مقارنة بين الحل العددي والحل المضبوط باستخدام الاخطاء التربيعية. دونت النتائج في جداول ومثلت بيانيا والتي اظهرت نتائج كفؤة ودقيقة في تطبيق الطريقة المقترحة .

## 1-INTRODUCTION

Delay integral equation is an equation in which the unknown function appears under an integral sign .The Delay integral lies in ability to describe processes with retarded time. The importance of these equations in various branch of Technology, economics, biology, has been recognized recently and tersest. Many authors have studied the delay integral equations. For example [Burton T.],[Kurger R.],[AL-Shakhaly T.] have studied some basic concepts and facts about the Delay integral equations.

This work can classify the linear Fredholm delay integral equations as:-

1-*The first type*, when the delay appears in the unknown function $u(x)$ Inside the integral sign such that:

$$h(x)u(x) = f(x) + l\int_a^b k(x,t)u(t-t)dt \qquad ..... (1)$$

2-*The second type*, when the delay appears outside the integral sign such that:



https://doi.org/10.30684/etj.30.4.10
2412-0758/University of Technology-Iraq, Baghdad, Iraq

$$h(x)u(x-t) = f(x) + l\int_a^b k(x,t)u(t)dt \qquad \text{..... (2)}$$

3- *The third type*, when the two delays appear in the unknown function $u(x)$ inside and outside the integral sign such that:

$$h(x)u(x-t_1) = f(x) + l\int_a^b k(x,t)u(t-t_2)dt \ldots (3)$$

Where $h, f$ and $k$ are known function. The function $f$ is said to be the driving term and $k$ is said to be kernel function that depends on $x,t$ and $l$ is scalar parameter (in this work $l=1$), $a$ and $b$ known constants,
$t, t_1, t_2$ are positive constant numbers and $u$ is the unknown function that must be determined.
If $h(x)=0$ , the equation above are called Fredholm delay integral equation of the first kind otherwise when $(h(x)=1)$ it's called Fredholm delay integral equation of the second kind. [1, 2, 8]

## THE HOMOTOPY PERTURBATION METHOD

The Homotopy perturbation method was first proposed by Chinese mathematician He J. Huan for solving integral equations, linear and nonlinear problems has been developed by scientists and engineers, becomes this method continuously deforms the difficult equation under study into a simple equation, easy to solve [10] .In this method the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution [13].

## SOME BASIC CONCEPTS OF THE HOMOTOPY PERTURBATION METHOD

We give some basic concepts of the Homotopy perturbation method, we recall the following definitions**:-**

**Definition**: [13]

Let $X$ and $Y$ be two topological spaces. If $f$ and $g$ are continuous maps of the space $X \to Y$ , it is said that $f$ is Homotopic to $g$ , if there exists a continuous map $H : X \times [0,1] \to Y$, such that:

$$\begin{aligned} H(x,0) &= f(x) \\ H(x,1) &= g(x) \end{aligned} \qquad \forall x \in X$$

Then the map is called Homotopy between $f$ and $g$ .

***Theorem*** [3]

The Homotopy relation is an equivalence relation.

**Definition:**

Let $X$ and $Y$ be two topological spaces. Two continuous functions $f$ and $g$ are said Homotopic relative to $A \subseteq X$ if there exists a continuous function $H : X \times [0,1] \to Y$ such that:

$$\left.\begin{array}{l} H(x,0) = f(x) \\ H(x,1) = g(x) \end{array}\right. \quad \forall x \in X$$

$$H(a, p) = f(a) = g(a), \forall p \in [0,1], \forall a \in A$$

To illustrate the basic idea of the homotopy perturbation method, we considering the following integral equation:

$$A(u) - f(x) = 0 \quad , x \in \Omega \qquad \dots (4)$$

Where $A$ ageneral integral operator, f is is a known function of $x$ ,$\Omega = \{(x,t) \,|\, a \le x,t \le b\}$ and $u$ is the unknown function that must be determined.

The operator $A$ ,generally speaking, is divided into two parts $L$ *and* $N$, where $L$ is linear, and $N$ is nonlinear. Therefore equation (4) can be rewritten as follows:

$$L(u) + N(u) - f(x) = 0$$

By the homotopy technique [4, 5, 9, 11, 12], we construct a homotopy $v(x, p) : \Omega \times [0,1] \to R$ which satisfies the homotopy equation:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(x)] = 0 \qquad \dots (5)$$

Where $p \in [0,1]$ and $u_0$ is an initial approximation of the solution of equation (4).obviously from equation (5) we have

$$H(v,0) = L(v) - L(u_0) = 0$$

$$H(v,1) = A(v) - f(x) = 0$$

The changing process of $p$ from zero to unity is just that of $v(x, p)$ from $u_0(x)$ *to* $u(x)$.

In topology, these is called deformation, $L(v) - L(u_0)$, and $A(v) - f(x)$ are homotopic.

$$\text{Therefore } L(v) - L(u_0) \cong A(v) - f(x), \qquad x \in \Omega$$

Assume that the solution of equation (5) can be written as a power series in $p$ as follows:

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + \dots \qquad \dots (6)$$

By setting $p = 1$ in equation (6), one can get:

$$u(x) = \lim_{p \to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \dots \qquad \dots (7)$$

This is the solution of equation (4).

The series (7) is convergent for some cases.However; the convergent rate depends on $A(v)$ [6, 7].

## THE HOMOTOPY PERTURBATION METHOD FOR SOLVING LINEAR DELAY INTEGERAL EQUATIONS

In this work the homotopy perturbation is employed for solving linear delay integral equations. Consider the first type linear delay Fredholm integral equation:

$$u(x) = f(x) + l\int_a^b k(x,t)u(t-t)dt \qquad x \in [a,b] \qquad \dots (8)$$

we rewrite equation (8) as

$$A(u) - f(x) = 0$$

Where $A(u) = u(x) - l\int_a^b k(x,t)u(t-t)dt$

Then the integral operator $A$ can be divided into two parts $L$ and $N$ such that equation (8) becomes:

$$L(u) + N(u) - f(x) = 0$$

where $L(u) = u \ and \ N(u) = -l\int_a^b k(x,t)u(t-t)dt$

By using the homotopy technique, we can construct a homotopy $v(x,p):[a,b]\times[0,1] \to R$ which satisfies:

$$H(v,p) = (1,p)[v(x,p) - u_0(x)] + p[v(x,p) - l\int_a^b k(x,t)v(t-t,p)dt - f(x)] = 0 \qquad \dots (9)$$

where $p \in [0,1]$ and $u_0$ is the initial approximating to the solution of equation (8). By using equation (9) it follows that

$$H(v,0) = v(x,0) - u_0(x) = 0$$

$$H(v,1) = v(x,1) - l\int_a^b k(x,t)v(t-t,1)dt - f(x) = 0$$

and the changing process of $p$ from zero to unity is just that of $v(x,p)$ from $u_0(x)$ to $u(x)$ .In topology, this called Homotopic.

Therefore $v(x,0) - u_0(x) \cong v(x,1) - l\int_a^b k(x,t)v(t-t,1)dt - f(x) \qquad x \in [a,b]$

Next, we assume that the solution of equation (9) can be expressed as

$$v(x,p) = \sum_{i=0}^{\infty} p^i v_i(x) \qquad \dots (10)$$

Therefore the approximated solution of the integral equation (8) can be obtained as follows:

$$u(x) = \lim_{p \to 1} v(x, p) = \sum_{i=0}^{\infty} v_i(x) \qquad \ldots (11)$$

By substituting the approximated solution given by equation (10) into equation (9) one can get

$$\sum_{i=0}^{\infty} p^i v_i(x) - u_0(x) + pu_0(x) + p[-l\int_a^b k(x,t)\sum_{i=0}^{\infty} p^i v_i(t-t)dt - f(x)] = 0$$

Then by equating the terms with identical power of $p$ one can have

$$p^0 : v_o(x) - u_0(x) = 0 \qquad \ldots (12.a)$$

$$p^1 : v_1(x) + u_0(x) - f(x) - l\int_a^b k(x,t)v_0(t-t)dt = 0 \qquad \ldots (12.b)$$

$$p^j : v_j(x) - l\int_a^b k(x,t)v_{j-1}(t-t)dt = 0 \qquad j = 1,2,3,\ldots \qquad \ldots (12.c)$$

For simplicity we set $v_o(x) = u_0(x) = f(x)$,and then equation (12.a) is automatically satisfied. By substituting $v_o(x) = u_0(x) = f(x)$ into equation (12.b) one can get $v_1(x) = l\int_a^b k(x,t)\, v_0(t-t)dt$

and in general we have:

$$v_o(x) = f(x)$$

$$v_j(x) = l\int_a^b k(x,t)v_{j-1}(t-t)dt \qquad j = 1,2,3,\ldots$$

Then by substituting $v_i(x)$ $, i = 0,1,2,\ldots$ into equation (11) we can get the approximated solution of the integral equation (8).[12,14]

## NUMERICAL EXAMPLES

In this section we present two examples of the linear Fredholm delay integral differential equation that are solved by the Homotopy perturbation method (HPM).

***Example (1)***

Consider the first type Linear Fredholm Delay Integral Equation:

$$u(x) = x^2 + \frac{5}{3}x + \int_{-1}^{1} xt\, u(t-1)dt$$

Where $a = -1, b = 1$ , $f(x) = x^2 + \frac{5}{3}x$, and $k(x,t) = x\ t$,with the exact solution $u(x) = x^2 + x$ .

To solve this example via the homotopy perturbation method, the algorithm start with initial approximation $v_0(x) = f(x)$ to obtain a first approximation

$$v_0(x) = u_0(x) = f(x) = x^2 + \frac{5}{3}x\ ,$$

Then

$$v_1(x) = \int_{-1}^{1} k(x,t)\, v_0(t-1)dt = \int_{-1}^{1} xt\ ((t-1)^2 + \frac{5}{3}(t-1))\,dt = \frac{-2}{9}x$$

In this case, let N=1, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x) \Rightarrow u_1(x) = v_0(x) + v_1(x) = x^2 + \frac{13}{9}x \cong x^2 + 1.4444\,x$$

Next, we must find $v_2(x)$:

$$v_2(x) = \int_{-1}^{1} k(x,t)\, v_1(t-1)dt = \frac{-4}{27}x$$

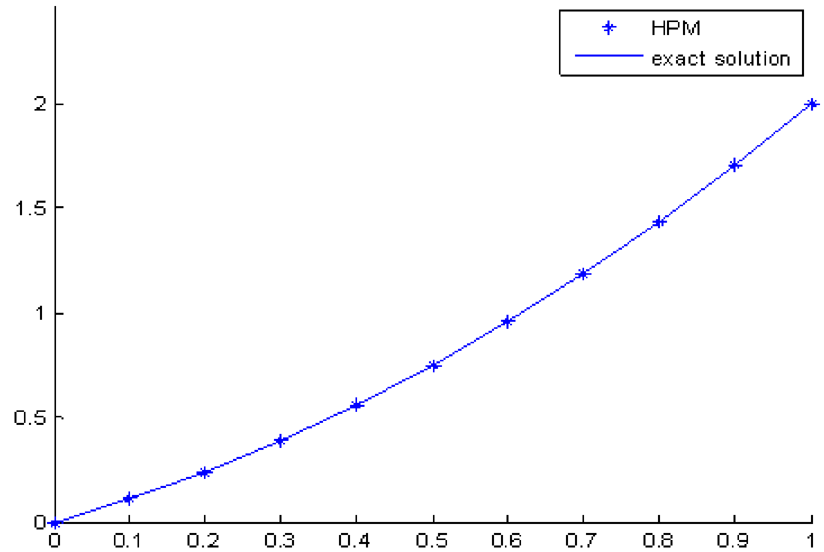
In the case of $N$=2,

$$u(x) \cong \sum_{i=0}^{N} v_i(x) \Rightarrow u_2(x) = v_0(x) + v_1(x) + v_2(x) = x^2 + \frac{35}{27}x \cong x^2 + 1.2963\,x$$

Table (1) presents the comparison between the exact and approximated solution by using homotopy perturbation method for N=21 with least square error (L.S.E) . Figure (1) shows the approximated solution by using homotopy perturbation method and the exact solution.

**Table (1): Numerical results for example (1)**

| ***x*** | ***Exact solution*** | ***Approximated solution using homomtopy perturbation method*** |
|---|---|---|
| **0** | **0** | **0** |
| **0.1** | **0.1100** | **0.1100** |
| **0.2** | **0.2400** | **0.2400** |
| **0.3** | **0.3900** | **0.3900** |
| **0.4** | **0.5600** | **0.5600** |
| **0.5** | **0.7500** | **0.7500** |
| **0.6** | **0.9600** | **0.9600** |
| **0.7** | **1.1900** | **1.1900** |
| **0.8** | **1.4400** | **1.4400** |
| **0.9** | **1.7100** | **1.7100** |
| **1** | **2** | **2** |
| | **L .S.E** | **0.0000** |




Figure(1)comparison between the exact solution and Homotopy perturbation method in Ex.(1)

Example (2):

Consider the first type of linear Fredholm delay integral differential equation:

$$u(x) = 5\cos(x) - \frac{629}{299}\cos(x) - \int_0^1 \sin(t)\cos(x)\,u(t-1)\,dt$$

Where $a = 0, b = 1$ *and* $f(x) = 5\cos(x) - \frac{629}{299}\cos(x)$ ,and

$k(x,t) = \sin(t)\cos(x)$, with the exact solution $u(x) = 5\cos(x)$ .

To solve this example via the homotopy perturbation method, the algorithm start with initial approximation $v_0(x)$ =f(x) to obtain a first approximation

$$v_0(x) = u_0(x) = f(x) = 5\cos(x) - \frac{629}{299}\cos(x)$$

Then

$$v_1(x) = \int_0^1 k(x,t)\,v_0(t-1)dt = \int_0^1 \sin(t)\cos(t)(5\cos(t-1) - \frac{629}{299}\cos(t-1)) = 1.21$$

86

In this case, let $N$=1, then

$$u(x) \cong \sum_{i=0}^{N} v_i(x)$$

$$u(x) \cong \sum_{i=0}^{N} v_i(x) \Rightarrow u_1(x) = v_0(x) + v_1(x) = 4.1149\cos(x)$$

Next, we must find $v_2(x)$:

$$v_2(x)=\int_0^1 k(x,t)\,v_1(t-1)dt = 0.5127\cos(\mathrm{x})$$

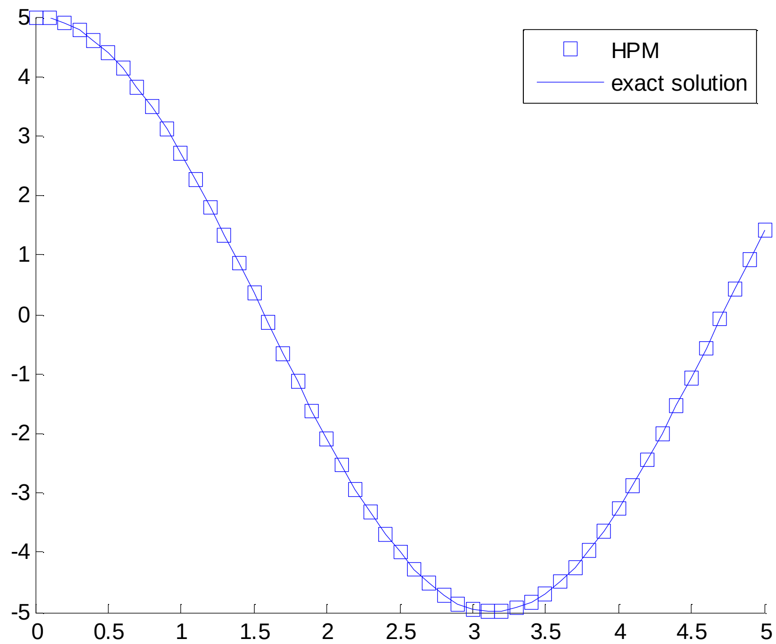
In the case of $N$=2,

$$u(x)\cong\sum_{i=0}^{N}v_i(x)\Rightarrow u_2(x)=v_0(x)+v_1(x)+v_2(x)=4.6276\cos(x)$$

Table (2) presents the comparison between the exact and approximated solution by using homotopy perturbation method for N=15 with least square error (L.S.E). Figure (2) shows the approximated solution by using homotopy perturbation method and the exact solution.

**Table (2): Numerical results for example (2)**

| *x* | *Exact solution* | *Approximated solution using homomtopy perturbation method* |
|---|---|---|
| 0 | 5.0000 | 5.0000 |
| 0.1 | 4.9730 | 4.9730 |
| 0.2 | 4.9003 | 4.9003 |
| 0.3 | 4.7767 | 4.7767 |
| 0.4 | 4.6053 | 4.6053 |
| 0.5 | 4.3879 | 4.3879 |
| 0.6 | 4.1267 | 4.1267 |
| 0.7 | 3.8242 | 3.8242 |
| 0.8 | 3.4835 | 3.4835 |
| 0.9 | 3.1080 | 3.1080 |
| 1 | 2.710 | 2.710 |
| | L .S.E | 0.0000 |




Figure(2)comparison between the exact solution and Homotopy perturbation method in Ex.(2)

## CONCLUSIONS

In this paper, we used Homotopy Perturbation Method for solving first type of linear delay Fredholm integral equation .the results show a marked improvement in the least square error (L.S.E).

From the present study, we can conclude the following points:

1- The Homotopy perturbation method gives more accurate results when the solutions of the problems are polynomials.

2- The good approximation depend on $N$, as $N$ increased, the error term is decreased.

3- The Homotopy perturbation method for solving any linear integral equation required the initial approximation; in this work we suppose the initial approximation $u_0(x) = f(x)$

## REFERENCES

[1] Al-Shakhaly, T., "About Delay Integral Equation" ,M.Sc.Thesis, college of Education ,Baghdad university ,2001.

[2] Burton T.,"Integral Equation with a Delay" , Aeta Math hung 72, No.3, pp. 233-242, 1998.

[3] Chatterje, D.,"Topology General & Algebraic", New Age International L(p) Ltd,2007.

[4] Farah Lateef, J.,On the Homotopy Perturbation method and its Applications, M.Sc.Thesis, college of science, AL-Nahrain University, 2011.

[5] He, J. H., Coupling method of a Homotopy Technique and a Perturbation Technique for non-linear problems , International Journal of Non-Linear Mechanics volume 35.No.1,pp. 37-43,2000.

[6] Jafari, H., Alipour M.and Tajadodi H., Convergence of Homotopy Perturbation method for solving Integral Equations, Journal of mathematics volume 8, No .3, pp.511-520, 2010.

[7] Kumar, S., Dixit S., Singh O., Homotopy Perturbation Method for Solving System of Generalized Abel's Integral Equations,An international Journal of Applications and Applied Mathematics volume 6, Issure11, pp.2009-2024, 2011.

[8] Krueger, R., Some Results on Linear Delay Integral Equations, Journal of Mathematics Analysis and Applied volume 67, pp.232-238, 1979.

[9] Liao, S., Comparison between the Homotopy Analysis method and Homotopy Perturbation method, Journal of Applied mathematics and computation volume 169, pp.1186-1194, 2005.

[10] Mohamed, I., Mahdy A., Farouk R., Numerical Solution of $12^{th}$ order Boundary value Problem by using Homotopy Perturbation method , Journal of Mathematics and computer science volume 1, No.1,pp.14-27,2010.

[11] He, J.H., Homotopy perturbation technique, Journal of compute Applied Mechanics Engineering volume 178, No.34, pp.257-262.

[12] Rwayda, Hazim H., Numerical Treatments of the Multi-dimensional Integral Equations, ph.D.Thesis, University of Technology, 2010.

[13] Shama, P. and Methi G., Applications of Homotopy perturbation method to Partial Differential Equations ,Asian Journal of Mathematics and statistics volume 3,No.3,pp. 140-150, 2011.

[14] Tavassoli, K., Ghasemi M., Babolian E., Application of He's homotopy perturbation method to nonlinear integro-differential equation ,Journal of Applied Mathematics and computation volume 188,pp.538-548,2007.