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Stability and Convergence of Explicit Difference Method for Solving the 3-Dimensional Two-Sided Fractional Diffusion Equation

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ABSTRACT

In this paper, a numerical solution of the 3-dimensional two-sided fractional diffusion equation has been presented. The algorithm for the numerical solution for this equation is based on explicit finite difference method. The consistency, conditional stability, and convergence of the fractional order numerical method are described.

The numerical method has been applied to solve a practical numerical example and the results have been compared with exact solution. The results were presented in tables using the MathCAD 12 software package when it is needed. The explicit finite difference method appeared to be effective and reliable in solving the 3-dimensional two-sided fractional Diffusion equation.

Keywords: Fractional derivative, explicit Euler method, fractional diffusion equation, stability, convergence.

الاستقرارية والتقارب لطريقة الفروق المنتهية الصريحة لحَلّ معادلة الانتشار الكسرية ذات الاتجاهين الثلاثية الابعادِ

الخلاصة

في هذا البحث قدمنا الحل العددي لمعادلة الانتشار الكسرية ذات الاتجاهين الثلاثية الأبعادِ . وان خوارزمية الحل العددي لتلك المعادلة قائمة على اساس الفروق المنتهية الصريحة.حيث تم مناقشة: الاتساق ،الاستقرارية الشرطية، والتقارب لطريقة العددية ذات الرتب الكسرية.

تم تطبيق الطريقة العددية لحل مثال عددي تطبيقي ومقارنة النتائج مع الحل المضبوط . تم عرض النتائج على شكل جداول باستخدام برنامج ماث كاد12عند الحاجة. لوحظ إن طريقة الفروق المنتهية الصريحة ظهرت لتكون ذات كفاءة ودقة عالية في حل معادلة الانتشار الكسرية ذات الاتجاهين الثلاثية الأبعادِ.

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INTRODUCTION

Fractional calculus is becoming a useful and, in some cases, a key tool in the analysis of scientific problems in abroad array of fields such as physics, engineering, biology, and economics. In particular, fractional partial differential equations have turned out to be especially relevant. For example, fractional diffusion equations have been successfully used to describe diffusion processes where the diffusion is anomalous [1, 2, 3, 4, 5, 6, 7, 11, 13], and fractional diffusion.

Difference methods and, in particular, explicit difference methods, are an important class of numerical methods for solving fractional and normal differential equations. The usefulness of the explicit methods and there as on why they are widely employed is based on their particularly attractive features [14,15]: flexibility, simplicity, scanty computational demand, and the possibility of easy generalization to spatial dimensions higher than1.

The method discussed in this paper is an explicit finite difference method designed for solving the 3-dimensional two-sided fractional diffusion where the fractional derivative is in the shifted Grunwald estimate form. The conditional stability and convergence of the explicit finite difference approximation are analyzed and finally, we will present example to show the efficiency of our numerical method.

EXPLICIT DIFFERENCE METHOD FOR SOLVING THE 3-DIMENSIONAL TWO-SIDED FRACTIONAL DIFFUSION EQUATION

In this section, we use the explicit finite difference method for solving the 3-dimensional two-sided fractional diffusion equation of the form:

$$\frac{\partial u(x,y,z,t)}{\partial t} = a(x,y,z)\left[(1-d)\frac{\partial^a u(x,y,z,t)}{\partial_- x^a} + d\frac{\partial^a u(x,y,z,t)}{\partial_+ x^a}\right] + b(x,y,z)\left[(1-e)\frac{\partial^b u(x,y,z,t)}{\partial_- y^b} + e\frac{\partial^b u(x,y,z,t)}{\partial_+ y^b}\right] + c(x,y,z)\left[(1-v)\frac{\partial^g u(x,y,z,t)}{\partial_- z^g} + v\frac{\partial^g u(x,y,z,t)}{\partial_+ z^g}+\right] q(x,y,z,t) \qquad \ldots(1)$$

In this problem initial and boundary conditions are considered which are:

$u(x,y,z,0)= j\ (x,y,z)$, for $x_0< x < x_R$, $y_0< y < y_R$ and $z_0< z< z_R$

$u(x_0,y,z,t)=0$, for $y_0< y< y_R$, $z_0< z < z_R$ and $0\le t \le T$

$u(x,y_0,z,t) = 0$, for $x_0 < x< x_R$, $z_0< z< z_R$ and $0\le t \le T$

$u(x,y,z_0,t) = 0$, for $x_0< x < x_R$, $y_0 < y< y_R$ and $0\le t \le T$

$u(x_R,y,z,t)=\psi_1(y,z,t)$, for $y_0< y< y_R$, $z_0<z < z_R$ and $0\le t \le T$

$$u(x,y_R,z,t)=\psi_2(x,z,t),\ \text{for } x_0< x< x_R,\ z_0<z< z_R \text{ and } 0\le t\le T$$
$$u(x,y,z_R,t)=\psi_3(x,y,t),\ \text{for } x_0< x < x_R,\ y_0< y< y_R \text{ and } 0\le t\le T$$

where a, b, c and j are known functions of x, y and z, and the weights $d, e, v, 1-d, 1-e, 1-v \in [0,1]$. ψ_1 is a known function of y ,z and t, ψ_2 is a known function of x,z and t, ψ_3 is a known function of x,y and t. a , b and g are given fractional number. q is a knwon function of x, y ,z and t.

The left–handed $\partial^a u/\partial_- x^a$, $\partial^b u/\partial_- y^b$, $\partial^g u/\partial_- z^g$ and the right–handed $\partial^a u/\partial_+ x^a$, $\partial^b u/\partial_+ y^b$, $\partial^g u/\partial_+ z^g$ fractional derivatives by the shifted Grunwald estimate formulae are [2, 7]:

$$\left.\begin{aligned}
\frac{\partial^a u(x,y,z,t)}{\partial_+ x^a} &= \frac{1}{(\Delta x)^a}\sum_{k=0}^{i+1} g_{a,k}u^s_{i-k+1,j,f} + O(\Delta x)\\
\frac{\partial^b u(x,y,z,t)}{\partial_+ y^b} &= \frac{1}{(\Delta y)^b}\sum_{k=0}^{j+1} g_{b,k}u^s_{i,j-k+1,f} + O(\Delta y)\\
\frac{\partial^a u(x,y,z,t)}{\partial_- x^a} &= \frac{1}{(\Delta x)^a}\sum_{k=0}^{n-i+1} g_{a,k}u^s_{i+k-1,j,f} + O(\Delta x)\\
\frac{\partial^b u(x,y,z,t)}{\partial_- y^b} &= \frac{1}{(\Delta y)^b}\sum_{k=0}^{m-j+1} g_{b,k}u^s_{i,j+k-1,f} + O(\Delta y)\\
\frac{\partial^g u(x,y,z,t)}{\partial_- z^g} &= \frac{1}{(\Delta z)^g}\sum_{k=0}^{p-f+1} g_{b,k}u^s_{i,j,f+k-1} + O(\Delta z)
\end{aligned}\right\}\ \ldots\ldots (2)$$

The finite difference method starts by dividing the x-interval $[x_0, x_R]$ into n subintervals to get the grid points $x_i = x_0 + iDx$, where $\Delta x = (x_R - x_0)/n$ and $i = 0,1,\ldots,n$. and we divide the y-interval $[y_0, y_R]$ into m subintervals to get the grid points $y_j = y_0+jDy$, where $\Delta y = (y_R - y_0)/m$ and $j=0,1,\ldots,m$. also we divide the z-interval $[z_0, z_R]$ into p subintervals to get the grid points $z_g = z_0 + f\,Dz$, where $\Delta z = (z_R - z_0)/p$ and $f = 0,1,\ldots,p$.

Also, the t-interval $[0,T\,]$ is divided into M subintervals to get the grid points $t_s = sDt$, $s = 0,\ldots, M$, where $\Delta t = T/M$.

Now, we evaluate eq.(1) at (x_i, y_j, z_f, t_s) and use the explicit Euler method to get

$$\frac{u(x_i,y_j,z_f,t_{s+1})-u(x_i,y_j,z_f,t_s)}{\Delta t}=$$
$$a(x_i,y_j,z_f)\left[(1-d)\frac{\partial^a u(x_i,y_j,z_f,t_s)}{\partial_- x^a}+d\frac{\partial^a u(x_i,y_j,z_f,t_s)}{\partial_+ x^a}\right]+$$
$$b(x_i,y_j,z_f)\left[(1-e)\frac{\partial^b u(x_i,y_j,z_f,t_s)}{\partial_- y^b}+e\frac{\partial^b u(x_i,y_j,z_f,t_s)}{\partial_+ y^b}\right]+$$
$$c(x_i,y_j,z_f)\left[(1-v)\frac{\partial^g u(x_i,y_j,z_f,t_s)}{\partial_- z^g}+v\frac{\partial^g u(x_i,y_j,z_f,t_s)}{\partial_+ z^g}\right]+q(x_i,y_j,z_f,t_s)$$

…. (3)

Use fractional derivative of the shifted Grunwald estimate eq.(2), to reduce eq.(3) to the following form:

$$\frac{u^{s+1}_{i,j,f}-u^{s}_{i,j,f}}{\Delta t}=a(x_i,y_j,z_f)\left[(1-d)\frac{1}{\Delta x^a}\sum_{k=0}^{n-i+1}g_{a,k}u^{s}_{i+k-1,j,f}+\right.$$
$$d\frac{1}{\Delta x^a}\sum_{k=0}^{i+1}g_{a,k}u^{s}_{i-k+1,j,f}+b(x_i,y_j,z_f)\left[(1-e)\frac{1}{\Delta y^b}\sum_{k=0}^{m-j+1}g_{b,k}u^{s}_{i,j+k-1,f}+\right.$$
$$\left.e\frac{1}{\Delta y^b}\sum_{k=0}^{j+1}g_{b,k}u^{s}_{i,j-k+1,f}\right]+c(x_i,y_j,z_f)\left[(1-v)\frac{1}{\Delta z^g}\sum_{k=0}^{p-f+1}g_{b,k}u^{s}_{i,j,f+k-1}+\right.$$
$$\left.v\frac{1}{\Delta z^g}\sum_{k=0}^{f+1}g_{b,k}u^{s}_{i,j,f-k+1}\right]+q(x_i,y_j,z_f,t_s)$$

$$\frac{u_{i,j,f}^{s+1}-u_{i,j,f}^{s}}{\Delta t}=(1-d)\frac{a_{i,j,f}}{\Delta x^{a}}\sum_{k=0}^{n-i+1}g_{a,k}u_{i+k-1,j,f}^{s}+$$

$$d\frac{a_{i,j,f}}{\Delta x^{a}}\sum_{k=0}^{i+1}g_{a,k}u_{i-k+1,j,f}^{s}+(1-e)\frac{b_{i,j,f}}{\Delta y^{b}}\sum_{k=0}^{m-j+1}g_{b,k}u_{i,j+k-1,f}^{s}+$$

$$(1-v)\frac{c_{i,j,f}}{\Delta z^{g}}\sum_{k=0}^{p-f+1}g_{g,k}u_{i,j,f+k-1}^{s}+v\frac{c_{i,j,f}}{\Delta z^{g}}\sum_{k=0}^{f+1}g_{b,k}u_{i,j,f-k+1}^{s}+$$

$$q_{i,j,f}^{s}\quad,\quad i=1,\dots,n-1,\ j=1,\dots,m-1,$$

$$f=1,\dots,p-1,\ s=0,\dots,M \qquad \dots (4)$$

Where $u_{i,j,f}^{s}=u(x_i,y_j,z_f,t_s)$,

$$a_{i,j,f}=a(x_i,y_j,z_f),\ b_{i,j,f}=b(x_i,y_j,z_f),\ c_{i,j,f}=c(x_i,y_j,z_f),\ q_{i,j,f}^{s}=q(x_i,y_j,z_f,t_s),$$

$$g_{a,k}=(-1)^{k}\frac{a(a-1)\,\mathbf{L}(a-k+1)}{k!},\ k=0,1,2,\dots,\ g_{b,k}=(-1)^{k}\frac{b(b-1)\,\mathbf{L}(b-k+1)}{k!},\ k=0,1,2,\dots$$

and $g_{g,k}=(-1)^{k}\dfrac{g(g-1)\,\mathbf{L}\,(g-k+1)}{k!}$, k=0,1,2,…

The resulting equation can be explicitly solved for $u_{i,j,f}^{s+1}$ to give

$$u_{i,j,f}^{s+1}=(1-d)a_{i,j,f}\frac{\Delta t}{\Delta x^{a}}\sum_{k=0}^{n-i+1}g_{a,k}u_{i-k+1,j,f}^{s}+$$

$$da_{i,j,f}\frac{\Delta t}{\Delta x^{a}}\sum_{k=0}^{i+1}g_{a,k}u_{i-k+1,j,f}^{s}+$$

$$(1-e)b_{i,j,f}\frac{\Delta t}{\Delta y^{b}}\sum_{k=0}^{m-j+1}g_{b,k}u_{i,j+k-1,f}^{s}+$$

$$eb_{i,j,f}\frac{\Delta t}{\Delta y^{b}}\sum_{k=0}^{j+1}g_{b,k}u_{i,j-k+1,f}^{s}+$$

$$(1-v)c_{i,j,f}\frac{\Delta t}{\Delta z^{g}}\sum_{k=0}^{p-f+1}g_{g,k}u^{s}_{i,j,f+k-1}+$$
$$vc_{i,j,f}\frac{\Delta t}{\Delta z^{g}}\sum_{k=0}^{f+1}g_{g,k}u^{s}_{i,j,f-k+1}+u^{s}_{i,j,f}+\Delta t q^{s}_{i,j,f} \qquad \textbf{.... (5)}$$

Also from the initial condition and boundary conditions one can get $u^{0}_{i,j,f}=j_{i,j,f}$, i=0,…, n, j=0,…,m and f = 0,1,…,p.

$u^{s}_{0,j,f}=0$, j=0,…, m, f =0,1,…,p and s=1,…,M

$u^{s}_{i,0,f}=0$, i=0,…, n , f =0,1,…,p and s=1,…,M

$u^{s}_{i,j,0}=0$,i=0,…,n, j=0,…,m and s=1,…,M

$u^{s}_{R,j,f}=y^{s}_{j,f}$, j=0,…,m, f=0,1,…,p and s=1,…,M

$u^{s}_{i,R,f}=y^{s}_{i,f}$, i=0,…,n, f=0,1,…,p and s=1,…,M

$u^{s}_{i,j,R}=y^{s}_{i,j}$, i=0,…, n, j=0,…, m and s=1,…,M.

Where $j_{i,j,f}=j(x_i,y_j,z_f)$,$y^{s}_{j,f}=y(y_j,z_f,t_s)$ and $y^{s}_{i,j}=y(x_i,y_j,t_s)$

STABILITY OF EXPLICIT DIFFERENCE METHOD 3-DIMENSIONAL TWO-SIDED FRACTIONAL DIFFUSION EQUATION

We define the following fractional partial difference operator:

$$w_{a,x}u^{s}_{i,j,f}=(1-d)a_{i,j,f}\frac{\Delta t}{\Delta x^{a}}\sum_{k=0}^{n-i+1}g_{a,k}u^{s}_{i+k-1,j,f}+ \qquad da_{i,j,f}\frac{\Delta t}{\Delta x^{a}}\sum_{k=0}^{i+1}g_{a,k}u^{s}_{i-k+1,j,f}$$

Where $O(\Delta x)$ is the approximation of the a th fractional derivative. Similarly, the following fractional partial difference operators are defined:-

$$w_{b,y}u^{s}_{i,j,f}=(1-e)b_{i,j,f}\frac{\Delta t}{\Delta y^{b}}\sum_{k=0}^{m-j+1}g_{b,k}u^{s}_{i,j+k-1,f}+$$
$$eb_{i,j,f}\frac{\Delta t}{\Delta y^{b}}\sum_{k=0}^{j+1}g_{b,k}u^{s}_{i,j-k+1,f}$$

$$w_{g,z}u_{i,j,f}^{s} = (1-v)c_{i,j,f}\frac{\Delta t}{\Delta z^{g}}\sum_{k=0}^{p-f+1} g_{g,k}u_{i,j,f+k-1}^{s} + vc_{i,j,f}\frac{\Delta t}{\Delta z^{g}}\sum_{k=0}^{f+1} g_{g,k}u_{i,j,f-k+1}^{s}$$

Where $O(\Delta y)$ and $O(\Delta z)$ are the approximation of the b and g -order Grunwald shifted fractional derivatives term, respectively.

With these definitions, the explicit difference scheme (5) may be written in the following compact form:

$$u_{i,j,f}^{s+1} = (1+\Delta tw_{a,x} + \Delta tw_{b,y} + \Delta tw_{g,z})u_{i,j,f}^{s} + q_{i,j,f}^{s}\Delta t \quad \ldots (6)$$

eq.(6) may be written in form

$$u_{i,j,f}^{s+1} = (1+\Delta tw_{a,x})(1+\Delta tw_{b,y})(1+\Delta tw_{g,z})u_{i,j,f}^{s} + \Delta tq_{i,j,f}^{s} \quad \ldots. (7)$$

Which introduces an additional perturbation error equal to

$$[(\Delta t)^2 w_{b,x}w_{a,y} + (\Delta t)^2 w_{b,y}w_{g,z} + (\Delta t)^2 w_{b,x}w_{a,z} + (\Delta t)^3 w_{b,x}w_{a,y}w_{a,z}]u_{i,j,f}^{s}$$

Then

$$\underline{U}^{s+1} = \underline{ETOU}^{s} + \underline{R}^{s} \quad \ldots. (8)$$

where

$$E = (1+\Delta tw_{a,x}),$$
$$T = (1+\Delta tw_{b,y}),$$
$$O = (1+\Delta tw_{g,z}),$$

and

$$\underline{U}^{s} = [u_{1,1,1}^{s}, u_{2,1,1}^{s}, \mathbf{K}, u_{n-1,1,1}^{s}, u_{1,2,1}^{s}, u_{2,2,1}^{s}, \mathbf{K}, u_{n-1,2,1}^{s}, \mathbf{K}, u_{1,m-1,1}^{s}, u_{2,m-1,1}^{s}, \mathbf{K}, u_{n-1,m-1,1}^{s}$$

$$
\begin{aligned}
& u^{s}_{1,1,2}, u^{s}_{2,1,2}, \mathbf{K}, u^{s}_{n-1,1,2}, u^{s}_{1,2,2}, u^{s}_{2,2,2}, \mathbf{K}, \\
& u^{s}_{n-1,2,2}, \mathbf{K}, u^{s}_{1,m-1,2}, u^{s}_{2,m-1,2}, \mathbf{K}, u^{s}_{n-1,m-1,2} \\
& \mathbf{L} \\
& u^{s}_{1,1,p-1}, u^{s}_{2,1,p-1}, \mathbf{K}, u^{s}_{n-1,1,p-1}, u^{s}_{1,2,p-1}, \\
& u^{s}_{2,2,p-1}, \mathbf{K}, u^{s}_{n-1,2,p-1}, \mathbf{K}, u^{s}_{1,m-1,p-1}, \\
& u^{s}_{2,m-1,p-1}, \mathbf{K}, u^{s}_{n-1,m-1,p-1}]^{T}
\end{aligned}
$$

$$
\begin{aligned}
\underline{U}^{s+1} = [& u^{s+1}_{1,1,1}, u^{s+1}_{2,1,1}, \mathbf{K}, u^{s+1}_{n-1,1,1}, u^{s+1}_{1,2,1}, u^{s+1}_{2,2,1}, \mathbf{K}, \\
& u^{s+1}_{n-1,2,1}, \mathbf{K}, u^{s+1}_{1,m-1,1}, u^{s+1}_{2,m-1,1}, \mathbf{K}, u^{s+1}_{n-1,m-1,1} \\
& u^{s+1}_{1,1,2}, u^{s+1}_{2,1,2}, \mathbf{K}, u^{s+1}_{n-1,1,2}, u^{s+1}_{1,2,2}, \\
& u^{s+1}_{2,2,2}, \mathbf{K}, u^{s+1}_{n-1,2,2}, \mathbf{K}, u^{s+1}_{1,m-1,2}, \\
& u^{s+1}_{2,m-1,2}, \mathbf{K}, u^{s+1}_{n-1,m-1,2} \\
& \mathbf{L} \\
& u^{s+1}_{1,1,p-1}, u^{s+1}_{2,1,p-1}, \mathbf{K}, u^{s+1}_{n-1,1,p-1}, \\
& u^{s+1}_{1,2,p-1}, u^{s+1}_{2,2,p-1}, \mathbf{K}, u^{s+1}_{n-1,2,p-1}, \mathbf{K}, \\
& u^{s+1}_{1,m-1,p-1}, u^{s+1}_{2,m-1,p-1}, \mathbf{K}, u^{s+1}_{n-1,m-1,p-1}]^{T}
\end{aligned}
$$

and the vector $\underline{R}^{s}$ is the forcing term Hence, we obtain the following fractional explicit scheme at time t_s :

$$u^{s+1}_{i,j,f} = (1 + \Delta t w_{a,x}) u^{s/3}_{i,j,f} + \Delta t q^{s}_{i,j,f} \qquad \ldots (9)$$

$$u^{s/3}_{i,j,f} = (1 + \Delta t w_{b,y}) u^{2s/3}_{i,j,f} \qquad \ldots (10)$$

and

$$u^{2s/3}_{i,j,f} = (1 + \Delta t w_{g,z}) u^{s}_{i,j,f} \qquad \ldots (11)$$

Thus, we require three steps to solve the third-dimensional two-sided fractional Diffusion equation in one time step.

Firstly: if (y_j, z_f) is fixed we will obtain an intermediate solution $u^{s/3}_{i,j,f}$ from (9).

Second: if (x_i, z_f) is fixed we will obtain an intermediate solution $u_{i,j,f}^{2s/3}$ from (10).

Third: if (x_i, y_j) is fixed we will obtain an intermediate solution from (11) using information compiled during **Second** step.

Now, we must prove that each one-dimensional explicit system defined by the linear difference eqs. (9), (10) and (11) is conditionally stable for all $1 < a , b , g < 2$.

Theorem: The explicit system defined by the linear difference eqs.(9), (10) and (11) with $1 < a , b , g < 2$ is conditionally stable if

$$\frac{\Delta t}{\Delta x^{a}} \le \frac{1}{a\left[(1-d)a_{\max} + da_{\max}\right]}$$

$$\frac{\Delta t}{\Delta y^{b}} \le \frac{1}{b\left[(1-e)b_{\max} + eb_{\max}\right]}$$

and

$$\frac{\Delta t}{\Delta z^{g}} \le \frac{1}{g\left[(1-v)c_{\max} + vc_{\max}\right]}$$

Proof:

At each grid point y_{k1}, for $k_1 = 1, \mathbf{K}, m-1$, and z_{k2} , for $k_2 = 1, \mathbf{K}, p-1$, the system of equations defined by (9). May be written as

$$\underline{U_{k_1,k_2}^{s+1}} = C_{k_1,k_2}\underline{U_{k_1,k_2}^{s/3}} + \Delta t \underline{Q_{k_1,k_2}^{s}}$$

where

$$\underline{U_{k_1,k_2}^{s+1}} = [u_{1,k_1,k_2}^{s+1}, u_{2,k_1,k_2}^{s+1}, \mathbf{K}, u_{n-1,k_1,k_2}^{s+1}]^T,$$

$$\underline{U_{k_1,k_2}^{s/3}} = [u_{1,k_1,k_2}^{s/3}, u_{2,k_1,k_2}^{s/3}, \mathbf{K}, u_{n-1,k_1,k_2}^{s/3}]^T,$$

$$\Delta t \underline{Q_{k_1,k_2}^{s}} = [\Delta t q_{1,k_1,k_2}^{s}, \Delta t q_{2,k_1,k_2}^{s}, \mathbf{K}, \Delta t q_{n-1,k_1,k_2}^{s}]^T$$

Therefore the resulting matrix entries $\underline{C_{i,j}}$ for $i = 1,2,\mathbf{K},n-1$ and $j = 1,\mathbf{K},n-1$ are defined by

$$C_{i,j}=\begin{cases}1+x_{i,k_1,k_2}g_{a,1}+h_{i,k_1,k_2}g_{a,1} & for \quad j=i\\ x_{i,k_1,k_2}g_{a,0}+h_{i,k_1,k_2}g_{a,2} & for \ j=i-1\\ x_{i,k_1,k_2}g_{a,2}+h_{i,k_1,k_2}g_{a,0} & for \ j=i+1\\ h_{i,k_1,k_2}g_{a,j+1} & for \ j<i+1\\ x_{i,k_1,k_2}g_{a,i-j+1} & for \ j>i+1\end{cases}$$

where the coefficients

$$x_{i,k_1,k_2}=(1-d)a_{i,k_1,k_2}\frac{\Delta t}{\Delta x^a}$$

and

$$h_{i,k_1,k_2}=da_{i,k_1,k_2}\frac{\Delta t}{\Delta x^a}$$

To illustrate this matrix pattern, we list the corresponding equations for the rows i =1, 2 and n-1:

$$u^{s+1}_{1,k_1,k_2}=(x_{1,k_1,k_2}g_{a,0}+h_{1,k_1,k_2}g_{a,2})u^{s/3}_{0,k_1,k_2}+(1+x_{1,k_1,k_2}g_{a,1}+h_{1,k_1,k_2}g_{a,1})u^{s/3}_{1,k_1,k_2}+(x_{1,k_1,k_2}g_{a,2}+h_{1,k_1,k_2}g_{a,0})u^{s/3}_{2,k_1,k_2}+x_{1,k_1,k_2}g_{a,3}u^{s/3}_{3,k_1,k_2}+\mathbf{L}+x_{1,k_1,k_2}g_{a,k}u^{s/3}_{K,k_1,k_2}+\Delta tq^{s}_{1,k_1,k_2}$$

$$u^{s+1}_{2,k_1,k_2}=h_{2,k_1,k_2}g_{a,3}u^{s/3}_{0,k_1,k_2}+(x_{2,k_1,k_2}g_{a,0}+h_{2,k_1,k_2}g_{a,2})u^{s/3}_{1,k_1,k_2}+(1+x_{2,k_1,k_2}g_{a,1}+h_{2,k_1,k_2}g_{a,1})u^{s/3}_{2,k_1,k_2}+(x_{2,k_1,k_2}g_{a,2}+h_{2,k_1,k_2}g_{a,0})u^{s/3}_{3,k_1,k_2}+\mathbf{L}+x_{2,k_1,k_2}g_{a,k}u^{s/3}_{K-1,k_1,k_2}+\Delta tq^{s}_{2,k_1,k_2}$$

$$u^{s+1}_{n-1,k_1,k_2}=h_{n-1,k_1,k_2}g_{a,n}u^{s/3}_{0,k_1,k_2}+\mathbf{L}+(x_{n-1,k_1,k_2}g_{a,0}+h_{n-1,k_1,k_2}g_{a,2})u^{s/3}_{n-2,k_1,k_2}+(1+x_{n-1,k_1,k_2}g_{a,1}+h_{n-1,k_1,k_2}g_{a,1})u^{s/3}_{n-1,k_1,k_2}+(x_{n-1,k_1,k_2}g_{a,2}+h_{n-1,k_1,k_2}g_{a,0})u^{s/3}_{n,k_1,k_2}+\mathbf{L}+x_{n-1,k_1,k_2}g_{a,k-n+2}u^{s/3}_{K,k_1,k_2}+\Delta tq^{s}_{n-1,k_1,k_2}$$

According to the Greshgorin theorem [9], the eigenvalues of the matrix <u>C</u> lie in the union of the circles centered at $c_{i,i}$ with radius $r_i=\sum_{\substack{l=0\\ l\neq i}}^{n}c_{i,l}$.

$$\text{Here we have } c_{i,i} = 1 + (x_{i,k_1,k_2} + h_{i,k_1,k_2}) g_{a,1}$$
$$= 1 - (x_{i,k_1,k_2} + h_{i,k_1,k_2}) a$$

and

$$r_i = \sum_{\substack{l=0 \\ l \neq i}}^{n} c_{i,l} = x_{i,k_1,k_2} \sum_{\substack{l=0 \\ l \neq i}}^{n-i+1} g_{a,i+l-1} + h_{i,k_1,k_2} \sum_{\substack{l=0 \\ l \neq i}}^{i+1} g_{a,i-l+1}$$
$$\leq x_{i,k_1,k_2} a + h_{i,k_1,k_2} a$$

and therefore $c_{i,i} + r_i \leq 1$. We also have

$$c_{i,i} - r_i \geq 1 - (x_{i,k_1,k_2} + h_{i,k_1,k_2}) a - (x_{i,k_1,k_2} + h_{i,k_1,k_2}) a$$
$$= 1 - 2(x_{i,k_1,k_2} + h_{i,k_1,k_2}) a$$
$$= 1 - 2\left[(1-d) a_{i,k_1,k_2} \frac{\Delta t}{\Delta x^a} + d q_{i,k_1,k_2} \frac{\Delta t}{\Delta x^a})\right] a \geq 1 - 2\left[(1-d) a_{\max} \frac{\Delta t}{\Delta x^a} + d a_{\max} \frac{\Delta t}{\Delta x^a})\right] a$$

Therefore, for the spectral radius of the matrix C to be at most one, it suffices to have

$$1 - 2\left[(1-a) d_{\max} \frac{\Delta t}{\Delta x^a} + a d_{\max} \frac{\Delta t}{\Delta x^a}\right] a \geq -1 \rightarrow \left[(1-a) d_{\max} \frac{\Delta t}{\Delta x^a} + a d_{\max} \frac{\Delta t}{\Delta x^a}\right] a \leq 1$$
$$\left[(1-a) d_{\max} a + a d_{\max} a\right] \frac{\Delta t}{\Delta x^a} \leq 1 \rightarrow \frac{\Delta t}{\Delta x^a} \leq \frac{1}{a\left[(1-a) d_{\max} + a d_{\max}\right]}$$

And with the same method above; the results of equations system, defined by (10), can be defined as:

$$\underline{U}_{k_1,k_2}^{s/3} = \underline{S_{k_1,k_2}} \underline{U}_{k_1,k_2}^{2s/3},$$

where

$$\underline{U}_{k_1,k_2}^{s/3} = [u_{k_1,1,k_2}^{s/3}, u_{k_1,2,k_2}^{s/3}, \mathbf{K}, u_{k_1,m-1,k_2}^{s/3}]^T,$$
$$\underline{U}_{k_1,k_2}^{2s/3} = [u_{k_1,1,k_2}^{2s/3}, u_{k_1,2,k_2}^{2s/3}, \mathbf{K}, u_{k_1,m-1,k_2}^{2s/3}]^T,$$

and $\underline{S_{k_1,k_2}}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix at the grid point x_{k1} for $k_1 = 1, \mathbf{K}, n-1$ *and* z_{k2} for $k_2 = 1, \mathbf{K}, p-1$. Therefore the resulting matrix entries $\underline{S_{i,j}}$ for $i = 1, 2, \mathbf{K}, m-1$ and $j = 1, \mathbf{K}, m-1$ are defined by

$$S_{i,j}=\begin{cases}1+V_{k_1,j,k_2}g_{b,1}+y_{k_1,j,k_2}g_{b,1} & for \quad j=i\\ V_{k_1,j,k_2}g_{b,0}+y_{k_1,j,k_2}g_{b,2} & for \quad j=i-1\\ V_{k_1,j,k_2}g_{b,2}+y_{k_1,j,k_2}g_{b,0} & for \quad j=i+1\\ y_{k_1,j,k_2}g_{b,j+1} & for \quad j<i+1\\ V_{k_1,j,k_2}g_{b,i-j+1} & for \quad j>i+1\end{cases}$$

where the coefficients

$$V_{k_1,j,k_2}=(1-e)b_{k_1,j,k_2}\frac{\Delta t}{\Delta y^b}$$

and

$$y_{k_1,j,k_2}=eb_{k_1,j,k_2}\frac{\Delta t}{\Delta y^b}$$

So, and by the same way, according to the Greshgorin theorem (cf. [9], pp. 135-136) we get

$$\frac{\Delta t}{\Delta y^b}\le\frac{1}{b\left[(1-b)e_{\max}+be_{\max}\right]}$$

Now, resulting the system of equations defined by (11) is then defined by:

$$\underline{U}^{2s/3}_{k_1,k_2}=\underline{A_{k_1,k_2}}\,\underline{U}^{s}_{k_1,k_2},$$

where

$$\underline{U}^{2s/3}_{k_1,k_2}=[u^{2s/3}_{k_1,k_2,1},u^{2s/3}_{k_1,k_2,2},\mathbf{K},u^{2s/3}_{k_1,k_2,m-1}]^T,$$
$$\underline{U}^{s}_{k_1,k_2}=[u^{s}_{k_1,k_2,1},u^{s}_{k_1,k_2,2},\mathbf{K},u^{s}_{k_1,k_2,m-1}]^T,$$

since $\underline{A_{k_1,k_2}}$ is x_{k1} for $k_1=1,\mathbf{K},n-1$ and y_{k2} for $k_2=1,\mathbf{K},m-1$. Therefore the resulting matrix entries $\underline{A_{i,j}}$ for $i=1,2,\mathbf{K},m-1$ and $j=1,\mathbf{K},m-1$ are defined by

$$A_{i,j}=\begin{cases}1+J_{k_1,k_2,f}g_{b,1}+s_{k_1,k_2,f}g_{b,1} & for \quad j=i\\ J_{k_1,k_2,f}g_{b,0}+s_{k_1,k_2,f}g_{b,2} & for \quad j=i-1\\ J_{k_1,k_2,f}g_{b,2}+s_{k_1,k_2,f}g_{b,0} & for \quad j=i+1\\ s_{k_1,k_2,f}g_{b,j+1} & for \quad j<i+1\\ J_{k_1,k_2,f}g_{b,i-j+1} & for \quad j>i+1\end{cases}$$

where the coefficients

$$J_{k_1,k_2,f} = (1-v)c_{k_1,k_2,f}\frac{\Delta t}{\Delta z^g}$$

and

$$s_{k_1,k_2,f} = vc_{k_1,k_2,f}\frac{\Delta t}{\Delta z^g}$$

So, according to the Greshgorin theorem (cf. [9], pp. 135-136) we get

$$\frac{\Delta t}{\Delta z^g} \le \frac{1}{g\left[(1-v)c_{\max} + vc_{\max}\right]}$$

CONSISTENCY AND CONVERGENT OF EXPLICIT DIFFERENCE METHOD 3-DIMENSIONAL TWO-SIDED FRACTIONAL DIFFUSION EQUATION

To obtain the consistency of the 3-dimensional the two-sided fractional Diffusion equation, note that the time difference operator in (8) has a local truncation error of order $O(\Delta t)$, and the three space difference operators in (8) have local truncation errors of orders $O(\Delta x)$, $O(\Delta y)$ and $O(\Delta z)$ respectively. Similar to Lemma 2.1 in paper of Meerschaert et al., (2006), we can obtain the following results:

$$\frac{\partial^a}{\partial x^a}\frac{\partial^b}{\partial y^b} f(x,y,z) = w_x^a w_y^b f(x,y,z) + \mathrm{O}(\Delta x + \Delta y)$$

$$\frac{\partial^a}{\partial x^a}\frac{\partial^g}{\partial z^g} f(x,y,z) = w_x^a w_z^g f(x,y,z) + \mathrm{O}(\Delta x + \Delta z)$$

$$\frac{\partial^b}{\partial y^b}\frac{\partial^g}{\partial z^g} f(x,y,z) = w_y^b w_z^g f(x,y,z) + \mathrm{O}(\Delta y + \Delta z)$$

then

$$\frac{\partial^a}{\partial x^a}\frac{\partial^b}{\partial y^b}\frac{\partial^g}{\partial z^g} f(x,y,z) = w_x^a w_y^b w_z^g f(x,y,z) + \mathrm{O}(\Delta x + \Delta y + \Delta z)$$

Which leads to the 3-dimensional two-sided fractional Diffusion equation with order $O(\Delta t) + O(\Delta x) + O(\Delta y) + O(\Delta z)$.

We show above that explicit Euler method is consistent and conditionally stable, then by Laxs equivalence theorem,[12], it converges at the rate $O(\Delta x + \Delta y + \Delta z + \Delta t)$.

NUMERICAL SIMULATION AND COMPARISON

In this section, we implement the proposed method to solve 3-dimensional two-sided fractional diffusion equation (1). Also, a comparison with numerical solution and exact solution, which is based on the explicit finite difference approximation of fractional derivative, is given.

Example: In this example, we consider (1) with $a = 1.6$, $b = 1.8$, $g = 1.5$, and d=0.4, 1-d=0.6, e=0.7, 1-e=0.3, v =0.8, 1-v =0.2, of the form:

$$\frac{\partial u(x,y,z,t)}{\partial t} = a(x,y,z)\left[0.4\frac{\partial^{1.6}u(x,y,z,t)}{\partial_+ x^{1.6}} + 0.6\frac{\partial^{1.6}u(x,y,z,t)}{\partial_- x^{1.6}}\right] +$$

$$b(x,y,z)\left[0.7\frac{\partial^{1.8}u(x,y,z,t)}{\partial_+ y^{1.8}} + 0.3\frac{\partial^{1.8}u(x,y,z,t)}{\partial_- y^{1.8}}\right] +$$

$$c(x,y,z)\left[0.8\frac{\partial^{1..5}u(x,y,z,t)}{\partial_+ z^{1..5}} + 0.2\frac{\partial^{1.5}u(x,y,z,t)}{\partial_- z^{1..5}}\right] +$$

$$q(x,y,z,t)$$

With the coefficient function:

$$a(x,y,z) = \frac{\Gamma(1.4)x^8(1-x)^7}{2},$$

$$b(x,y,z) = \Gamma(0.2)y^8(1-y)^7,$$

$$c(x,y,z) = \Gamma(2.5)z^7(1-z)^7,$$

and the source function:

$$q(x,y,z,t) = -0.4x^{8.4}(1-x)^7 yz^3 e^{-3t} - 0.6(1-$$

$$x)^{7.4}x^8 yz^3 e^{-3t} - 0.7x^2 y^{7.2}(1-y)^7 ze^{-3t} -$$

$$0.3x^2 y^8(1-y)^{6.2} z^3 e^{-3t} - 0.8x^2 yz^{8.5}(1-z)^7 e^{-3t}$$

$$-0.2x^2 yz^7(1-z)^{8.5} e^{-3t} - 3x^2 yz^3 e^{-3t}$$

subject to the initial condition

$$u(x,y,z,0) = x^2yz^3,\ 0< x<1,\ 0< y <1,\ 0< z <1$$

and the boundary conditions

$$u(0,y,z,t)=0,\ 0<y<1,\ 0<z<1,\ 0\leq t \leq 0.025$$
$$u(x,0,z,t)=0,\ 0<x<1,\ 0<z<1,\ 0\leq t \leq 0.025$$
$$u(x,,y,0,t)=0,0<x<1,0<y<1,\ 0\leq t \leq 0.025$$
$$u(1,y,z,t)=yz^3e^{-3t},\ 0< y < 1,\ 0 < z < 1,\ 0\leq t\leq 0.025$$
$$u(x,1,z,t)=x^2z^3e^{-3t},\ 0 < x < 1 ,\ 0 < z <1,\ 0\leq t\leq 0.025$$
$$u(x,,y,1,t)=x^2ye^{-3t},\ 0< x <1,\ 0< y < 1,\ 0\leq t\leq 0.025$$

Note that the exact solution to this problem is : $u(x,y,z,t) = x^2yz^3e^{-3t}$.

Tables 1 and 2 show the numerical solution using the explicit finite difference approximation. From tables 1 and 2, it can be seen that there is a good agreement between the numerical solution and exact solution.

CONCLUSIONS

In this paper

1- Numerical method for solving the 3-dimensional two-sided fractional Diffusion equation has been described and demonstrated.

2- The explicit difference method is proved to be conditionally stable and convergent.

3- Numerical example is given indicating the convergence of the solution with exact results.

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Table (1) The numerical solution of the example using the explicit difference method $(\Delta x = 0.2, \Delta y = 0.2, \Delta z = 0.2, \Delta t = 0.0125)$

x = y =z	**t**	**Numerical Solution**	**Exact Solution**	**$\|u_{ex} - u_{approx.}\|$**
0.2	0.0125	6.160E-5	0.61644E -4	4.44427 E -8
0.4	0.0125	3.943 E-3	0.39452E -2	2.24433 E -6
0.6	0.0125	0.045	0.44939 E -1	6.12012 E -5
0.8	0.0125	0.252	0.25296	4.95637 E -4
0.2	0.0250	5.929 E-5	0.59376E-4	8.55831 E -8
0.4	0.0250	3.796 E-3	0.38000E -2	4.03732 E -6
0.6	0.0250	0.043	0.43285 E -1	2.84800 E -4
0.8	0.0250	0.243	0.24320	2.02388 E -4

Table (2) The numerical solution of the example using the explicit difference method. $(\Delta x = 0.25, \Delta y = 0.25, \Delta z = 0.25, \Delta t = 0.0125)$

x =y =z	**t**	**Numerical Solution**	**Exact Solution**	**$\|u_{ex} - u_{approx.}\|$**
0.25	0.0125	2.350 E – 4	0.23515 E -3	1.54887E -7
0.50	0.0125	0.015	0.15050 E -1	4.99128 E -5
0.75	0.0125	0.171	0.17143	4.27913 E -4
0.25	0.0250	2.262 E – 4	0.22650E -3	2.99875 E -7
0.50	0.0250	0.014	0.14496 E -1	4.95992 E -4
0.75	0.0250	0.165	0.16512	1.18409 E -4

Table (3) Maximum error for the numerical solution of the example using the explicit difference method.

$\Delta x = \Delta y = \Delta z$	Δt	**Maximum Error**
0.20	0.0125	0.000202388
0.25	0.0125	0.000118409