Use of Bernstein Polynomial in Numerical Solution of Nonlinear Fred Holm Integral Equation

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Abstract

In this paper, Bernstein polynomials with different degree has been used to approximate the solution of nonlinear Fredholm integral equations. A comparison between the different degree of Bernstein polynomials has been made depending on absolute error and least squares errors.

keywords: Nonlinear Fredholm Integral equation, Bernstein polynomial.

استخدام متعددة حدود برنشتاين لايجاد الحل العددي لمعادلة فريدهولم التكاملية الغير خطية

الخلاصة

في هذا البحث متعددة حدود برنشتاين بدرجات مختلفة استخدمت لتقريب الحل لمعادلة فريدهولم التكاملية الغير خطية. كما تم اجراء مقارنة بين الدرجات المختلفة لمتعددة حدود برنشتاين باستخدام الخطأ المطلق والاخطاء التربيعية.

Introduction

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics, potential theory, electrostatic and radiative heat transfer problems. Therefore, many different methods are used to obtain the solution of the linear and non linear integral equations [1].

Several methods have been proposed for numerical solution of these equations. Discrete Galerkin method were used by Atkinson and F. Potra [2], to solve nonlinear integral equations , a survey of numerical methods for solving nonlinear integral equations were applied in [3]. The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations have been published by Brunner, A Pedas and G. Vainikko

[4], Numerical solution of two-dimensional nonlinear Fredholm integral equations by Spline functions were applied in [5], numerical algorithm based on a decomposition technique, were presented for solving a class of nonlinear integral equations in[6]. Also, Numerical method for solving nonlinear Fredholm integral equations based on the Haar Wavelet approach have been used in,[7]. Bhattacharya and R. N. Maandal in [8] use Bernstien polynomials in numerical solution of Volterra integral equations. A. Tahmasbi and O.S. Fard in [9] use power series to fined numerical solution of nonlinear Volterra integral equations of the second kind, in,[1] an iterative scheme based on the homotopy analysis methods introduced to solve nonlinear integral equations Vineet K.Singh, Rajesh K.Poundey and

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OMP. Singh in [10] introduced new stable numerical solutions for singular integral equations of Abel type based on the normalized Bernstein polynomials. New methods are always needed to solve integral equations because no single method work well for all such equations. In this paper the Bernstein polynomials are used to approximate the solution of non linear Fredholm Integral equations of the second kind.

Bernstein polynomials:

Bernstein polynomials of degree n can be defined by,[5]:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} , \; i = 0,1,\ldots,n$$

…(2.1)

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} , \; i = 0,1,\ldots,n$$

where

These polynomials form a partition of unity, that is

$$\sum_{i=o}^{n} B_{i,n}(x) = 1 ,$$ and can be used for approximating any function continuous in [a ,b] [6].

(2.1) properties of Bernstein polynomials:

Bernstein polynomials properties are: [6],[11]

1. $B_{i,n}(0) = B_{i,n}(1) = 0 , \; for \; i = 0,1,\ldots,n-1$
2. $B_{0,n}(0) = B_{n,n}(1) = 1$
3. $B_{i,n}(t) = 0 , \; if \; i < 0 \; or \; i > n$
4. $B_{i,n}(t) \geq 0 , \quad t \; n \; [\,0\,,1\,]$
5. $B_{i,n}(1-t) = B_{n-i,n}(t)$
6. $\sum_{i=o}^{n} B_{i,n}(t) = 1$
7. $B_{i,n-1}(t) = \left(\frac{n-t}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t)$
8. $\frac{d}{dt} B_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t))$

(2.2) A Matrix Representation for Bernstein polynomials :[10]

In many applications a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if only looking at a linear combination in terms of dot products. Given a polynomial written as a linear combination of the Bernstein basis function:

$$B(t) = C_0 B_{0,n} + C_1 B_{1,n}(t) + \cdots + C_n B_{n,n}(t)$$

It is easy to write this as a dot product of two vectors :

$$B(t) = \begin{bmatrix} B_{0,n}(t) & B_{1,n}(t) & \ldots & B_{n,n}(t) \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_n \end{bmatrix}$$

Which can be converted to the following form :

$$B(t) = \begin{bmatrix} 1 & t & t^2 & \ldots & t^n \end{bmatrix} \begin{bmatrix} b_{00} & 0 & 0 & \ldots & 0 \\ b_{01} & b_{11} & 0 & \ldots & 0 \\ b_{0n} & b_{n1} & \ldots & \ldots & b_{nn} \end{bmatrix}$$

Where b_{nn} are the coefficients of the power basis that are used to determine the respective Bernstein polynomials.

We note that the matrix in this case is lower triangular. In the quadratic case (i.e n=2) the matrix representation is :

$$B(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

The cubic case n=3 , the matrix representation is:

$$B(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & 6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

(3) Solution of Nonlinear Fredholm integral equations :

We consider the nonlinear Freadholm integral equation of the second kind given by :

$$u(x) = g(x) + \int_a^b k(x, t, u(t))dt \ldots (3.1)$$

Where u(x) is the unknown functions to be determined, k(x, t, u(t)), the kernel is a continuous function, g(x) being the known function.

To determine an approximate solution of (3.1), u(x) is approximated in the Bernstein polynomial basis on [a ,b] as :

$$u(x) = \sum_{i=o}^{n} a_i \, B_{i,n}(x) \ldots (3.2)$$

Where a_i $i = 0, \ldots \ldots, n$ are unknown constants to be determined using Newton-Raphson method. Substituting (3.2) in (3.1), we obtain:

$$\sum_{i=0}^{n} a_i \, B_{i,n}(x) = g(x) + \int_a^b k(x,t, \sum_{i=0}^{n} a_i \, B_{i,n}(t)) \, dt \ldots \ldots (3.3)$$

Now we put x = x_j , j = 0,1,…,n in (3,3), x_j's being chosen as suitable distinct points in [a, b], such that x_0 =a, x_n =b ,and x_j=x_0 +jh, where h=(b-a)/n. Putting x=x_j, we obtain the nonlinear system :

$$\sum_{i=0}^{n} a_i \, B_{i,n}(x_j) - g(x_j) + \int_a^b k(x_j, t, \sum_{i=0}^{n} a_i \, B_{i,n}(t)) \, dt \ldots \ldots (3.4)$$

The nonlinear system (3.4) can be solved by standard methods for the unknown constant a_i's   . These a_i $i = 0, \ldots \ldots, n$ are then used in (3.2) to obtain the unknown function u(x) approximately.

The following algorithm summarizes the steps for finding the approximate solution for the second kind of nonlinear Fredholm integral equation.

(3.1) Algorithm (BPNFIE):
Step (1):
Assume X_0 = a, X_n = b and X_j = X_0 + j h, where
h = (b- a)
Step (2):-
Putting X = X_j in (3.3) to obtain nonlinear System (3.4)
Step (3):
Solved the nonlinear system (3.4) to calculate the unknown a_i, *i*= 1,…, n
Step (4):
Used a_i's in (3.2) to obtain the function U(x) approximately.

(4) Numerical Examples:
(4.1) Example (1):
Consider the following nonlinear fredholm integral equation:

$$u(x) = e^x - x^2 - x + \frac{1}{4}e^2 + 4e^1 - \frac{139}{12} + \int_0^1 (x - tu(t)^2) dt$$

Table (1) presents the error between the results from a computer program that solves this problem

and the analytical solution which is

$$u(x) = e^x - x^2$$ over   the interval [0,1] using Bernstein polynomials of degree 1, 2, 3, 4, 5. With n = 10, *i.e* h = 0.1, and err_i=the absolute error when we used Bernstein polynomial of degree i to approximate the solution i=1,2,3,4,5 and L. S. E= least square error.

(4.2) Example (2):
Consider the following nonlinear fredholm integral equation:

$$u(x) = \sin(x) - x^2 + \sin(1) - \cos(1) + \int_0^1 (x^2 - tu(t)) dt$$

Table (2) presents the error between the results from a computer program that solves this problem and the analytical solution which is $u(x) = \sin x$ over the interval [0,1] using Bernstein polynomials of degree 1, 2, 3, 4, 5. With n = 10, *i.e* h = 0.1, and err_i=the absolute error when we used Bernstein polynomial of degree i to approximate the solution i=1,2,3,4,5 and L. S. E= least square error.

(4.3) Example (3):
Consider the following nonlinear fredholm integral equation:

$$u(x) = \cos(x) + \frac{5}{2} - x + \frac{1}{2}\cos(1)\sin(1) + 2\sin(1) + \int_0^1 (x - u(t)^2) dt$$

Table (3) presents the error between the results from a computer program that solves this problem and the analytical solution which is $u(x) =$cosx+1 over the interval [0,1] using Bernstein polynomials of degree 1, 2, 3, 4, 5. With n = 10, *i.e* h = 0.1, and err_i=the absolute error when we used Bernstein polynomial of degree i to approximate the solution i=1,2,3,4,5 and L. S. E= least square error.

(5) Conclusions
Bernstein polymials methods are   constructed   to   compute

numerical solution to a nonlinear Fredholm integral equation of the second kind.

For each degree a computer programs was written in MATLAB and several examples were solved using proposed method.

We conclude the following points:

1- This method can be used to approximate the solution of nonlinear fredholm integral equations.

2- It is clear that when the degree of Bernstein polynomial is increases the error is decreases.

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Table (1):The result of Example (1)

X	err_1	err_2	err_3	err_4	$err5$
0	0.01176	0.0035772	0.00017961	7.6691e-006	3.3872e-007
0.1	0.011582	0.012671	0.0005838	5.3197e-005	1.7205e-006
0.2	0.025986	0.015992	0.00047134	2.5066e-005	3.3872e-007
0.3	0.032614	0.014704	8.5166e-006	4.9884e-006	1.0444e-006
0.4	0.032752	0.010092	0.00046964	9.4857e-006	3.3872e-007
0.5	0.02782	0.0035772	0.00066111	7.6691e-006	1.7973e-007
0.6	0.019389	0.0032702	0.00048144	2.5404e-005	3.3872e-007
0.7	0.0091951	0.0087149	5.6963e-006	2.1198e-005	1.0858e-006
0.8	0.00084485	0.010839	0.00055413	1.1555e-005	3.3872e-007
0.9	0.0086108	0.0075223	0.00071596	4.4343e-005	1.9695e-006
1	0.01176	0.0035772	0.00017961	7.6691e-006	3.3872e-007
L.S.E	0.067318	0.032147	0.0015286	8.3331e-005	3.1352e-006

Table (2):The result of Example (2)

X	err_1	err_2	err_3	err_4	err_5
0	0.013786	0.00074333	2.7481e-005	1.7106e-006	5.0913e-008
0.1	0.0019008	0.0061854	0.00017234	2.8019e-005	5.0727e-007
0.2	0.01659	0.0079298	0.00014731	1.1534e-005	5.0913e-008
0.3	0.029293	0.0069641	1.9525e-005	5.2676e-006	2.483e-007
0.4	0.039044	0.0042558	0.0001088	7.5169e-006	5.0913e-008
0.5	0.044904	0.00074333	0.00016492	1.7106e-006	9.7974e-008
0.6	0.045974	0.0026741	0.000115	1.0771e-005	5.0913e-008
0.7	0.041402	0.005145	2.6977e-005	8.4388e-006	2.7012e-007
0.8	0.030394	0.0058743	0.00019062	7.5888e-006	5.0913e-008
0.9	0.012217	0.0041313	0.00024123	2.2744e-005	6.3798e-007
1	0.013786	0.00074333	2.7481e-005	1.7106e-006	5.0913e-008
L.S.E	0.099793	0.015956	0.00044812	4.211e-005	9.078e-007

Table (3):The result of Example (3)

X	err$_1$	err$_2$	err$_3$	err$_4$	err$_5$
0	0.056393	0.00010552	0.00010308	5.353e-008	1.9417e-007
0.1	0.015419	0.0024041	0.00034456	1.2223e-005	9.8291e-007
0.2	0.015613	0.0033555	0.00026834	4.7977e-006	1.9417e-007
0.3	0.036853	0.0031089	8.1801e-006	3.4615e—006	5.8141e-007
0.4	0.048547	0.0019114	0.0002593	4.7892e-006	1.9417e-007
0.5	0.051038	0.00010552	0.00035467	5.353e-008	7.8367e-008
0.6	0.044761	0.0018744	0.00025591	5.2017e-006	1.9417e-007
0.7	0.030238	0.0035063	1.2251e-005	4.0264e-006	5.6948e-007
0.8	0.0080719	0.0041857	0.00024469	5.649e-006	1.9417e-007
0.9	0.021055	0.0032319	0.00030692	1.551e-005	9.1151e-007
1	0.056393	0.00010552	0.00010308	5.353e-008	1.9417e-007
L.S.E	0.12879	0.0086119	0.00079043	2.2871e-005	1.6406e-006