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One Parameter Composite Semigroups of Linear Bounded Operators in Strong Operator Topology of Schatten Class C_p

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Abstract

For semigroups of linear bounded operators on Hilbert spaces, the problem of being in C_p, $0 < P < \infty$ is more interesting than of begin compact , so we developed some new results of a composite semigroups , T(t), $0 < t < \infty$ which defined on a Banach space of linear bounded operators on Hilbert space to be in the Schatten class C_p in strong operator topology.

زمر مركبة لمؤثرات خطية مقيدة ذات معلمة واحدة لصف شاتين C_p, في فضاء تبولوجي مؤثر قوي

الخلاصة

ان دراسة الزمرة شبه الاولية لمؤثرات خطية مقيدة على فضاء هلبرت في الصف $0 < P$ C_p, $< \infty$ أكثر أهمية من دراستها في أن تكون مرصوصة ولذلك طورت بعض النتائج الجديدة لزمرة شبه أولية مركبة $\mathbb{T}(t)$, $0 < t < \infty$ معرفة على فضاء بناخ لمؤثرات خطية مقيدة ومعرفة على فضاء هلبرت حتى تكون في صف شاتين C_p, في فضاء تبولوجي مؤثر قوي.

1. Introduction

Let L(H) be a Banach space, a one-parameter family$\{T(t)\}_{t\geq 0} \subset L(L((H))$, $t\in[0,\infty)$ of bounded linear operators defined by:

$T(t)X = T_1(t)XT_2(t)$, for any $X\in L(H)$ and $t \in[0,\infty)$. (1)

with generator $\mathbb{A}$ is called composite semi group if:

(i) $T(0)X = IX$, (I the identity operator of L(H)).

(ii) $T(t + s)X = T(t)T(s)X= T(s)T(t)X$, for every t, $s \geq 0$.

Where $T_1(t),T_2(t)$ are two semigroups defined from H into H for A_1 and A_2 generaters respectively, [1].

The infinitesimal generator $\mathbb{A}$ of $\mathbb{T}(t)$ a strong operator topology defined as the limit:

$$A X = \tau - lim_{t\downarrow 0} \left\{\frac{\mathbb{T}(t)Xh - Xh}{t}\right\} \in D(A),$$

Where $D(\mathbb{A}) \subset L(H)$ is the domain of $\mathbb{A}$ defined as follows:

$$D(A) = \left\{ X \in L(H):\ \tau - lim_{t\downarrow 0} \left\{\frac{\mathbb{T}(t)Xh - Xh}{t}\right\}\ exist\ in\ \{L(H), \tau\} \right\}.$$

where {L(H), τ} stands for L(H) equipped with the strong operator topology τ, i.e., topology induced by family of seminorms $\rho = \{\rho_h\}_{h\in H}$, where siminorms $\rho_h(X) = ||Xh||_H$, X

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$\in$ L(H). T(t) $\in$ L(L(H)) is a strong-operator and continuous at the origin, i.e.,

$\tau\text{-}\lim_{t\downarrow 0}\|(T(t)X)h - (T(0)X)h\|_H = 0$, h $\in$ H, X $\in$ L(H).

Remarks(1.1):

a- The different between the usual strongly continuous semigroups and the composite semigroups (1) follows from the fact that in general for X $\in$ L(H), the function $[0,\infty) \supseteq t \mapsto T(t)X \in L(H)$ is continuous in {L(H), τ}, and which cannot be continuous in {L(H), ‖.‖} unless the semigroups $\{T_1(t)\}_{t\geq 0}$, $\{T_2(t)\}_{t\geq 0} \subset L(H)$ are uniformly continuous. However, this takes place case only if their generators A_1, A_2 are bounded operators on H.

b- The generator A is densely defined only in {L(H), τ} and does not in {L(H), ‖.‖}. This implies that the closure of D(A) in L(H) is only a proper set and not the whole L(H).

The problem of being in C_p is more interesting than of begin compact. This is due to the fact that for a C_o-semigroup $T(t)\in C_p$, $0< t <\infty$, $\{\|T(t)\|, 0<t\leq a\}$ is a bounded in H for every finite a. But if $T(t)\in C_p, 0<t<\infty$, then $\|T(t)\|_p$ need not to be bounded in any interval (0,a) for any finite a. For the basic theory of semigroups we refer to Pazy [5].

For $1\leq p<\infty$, set:

$$C_p = \left\{ T\in L(H) : \sup \sum_{n=1}^{\infty} \left|\langle Te_n, f_n\rangle\right|^p < \infty \right\}$$

Where the supremum is taken over all orthonormal bases (e_n) and (f_n) of H. For $T\in C_p$, one defines $\|T\|_p = \sup(\sum |\langle Te_n, f_n\rangle|^p)^{1/p}$. This defines a norm on C_p. With this norm, , C_p is a two sided Banach ideal in L(H). For more on C_p we refer to KhaliL and Deeb[3].

Lemma(1.2),[3]:

Let $T_n(t)\in C_p$ such that $\sup_n \|T_n\|_p < \infty$ if $T_n \to T (n\to\infty)$ in the operator norm, then $T(t)\in C_p$

2. **Main results**:

The following results of compsite semigroups of Schatten Class C_p have been presented as follows:

Lemma (2.1):

Let $\mathbb{T}(t)X = T_1(t)XT_2(t)$, $t\geq 0$ be a C_o- composite semigroup in {L(H), τ}. If for some $t_0>0$, $T_1(t)\in C_p$ and $T_2(t)\in C_p$, for all $t>t_0$. Further there exists $M_1, M_2 \in [0,\infty)$ and $w_1, w_2 \in (0,\infty)$ such that $\|\mathbb{T}(t)\|_p$

$$\leq \|T_1(t)\|_p M_1 M_2 e^{(w_1+w_2)(t-t_0)} \|T_2(t)\|_p$$

Proof:

For the semigroup property, we have $\mathbb{T}(t).$ = $T_1(t_0)\,T_1(t-t_0)\,.\,T_2(t-t_0)\,T_2(t_0)$, for t, $t_0 \geq 0$

Since C_p is a two sided ideal and $T_1(t) \in C_p$, it follows that $\mathbb{T}(t) \in C_p$. Further Banach ideal property of C_p gives $\|\mathbb{T}(t)\,.\,\|_p$

=

$$\left\|T_1(t_0)T_1(t-t_0)\,.\,T_2(t-t_0)\,T_2(t_0)\right\|_p$$

$\leq$

$$\left\|T_1(t_0)\right\|_p \left\|T_1(t-t_0)\,.\,T_2(t-t_0)\right\|_{L(H)} \left\|T_2(t_0)\right\|_p \left\|.\right\|_{L(H)}$$

$$\leq \left\|T_1(t)\right\|_p M_1M_2e^{(w_1+w_2)(t-t_0)} \left\|T_2(t)\right\|_p \left\|.\right\|_{L(H)}$$

Thus

$\|\mathbb{T}(t)\,.\,\|_p$

$$\leq \left\|T_1(t)\right\|_p M_1M_2e^{(w_1+w_2)(t-t_0)} \left\|T_2(t)\right\|_p \left\|.\right\|_{L(H)}$$

for

$M_1, M_2 \geq 1, w_1, w_2 \geq 0$ and for t ≥ 0.

Theorem (2.1):

Let $T_{1,n}(t), T_{2,n}(t) \in C_p$ such that $\sup_n \left\|T_{1,n}(t)\right\| < \infty$, for and $\sup_n \left\|T_{2,n}(t)\right\| < \infty$ and $\sup_n \left\|T_{2,n}(t)\right\| < \infty$.If $\mathbb{T}_n(t)X$ = $T_{1,n}(t)XT_{2,n}(t) \rightarrow T_1(t)XT_2(t)$ in $\{L(H), \tau\}$ as $n \rightarrow \infty$,then $T_1(t)XT_2(t) \in C_p$.

Proof:

Since $T_{1,n}(t), T_{2,n}(t) \in C_p$, each one of $T_{1,n}(t), T_{2,n}(t)$ is compact ,see [2]. Hence

$\mathbb{T}_n(t)X$=

$$T_{1,n}(t)XT_{2,n}(t) = \sum_{k=1}^{\infty}\sigma_{nk}(t)e_{nk} \otimes f_{nk} X \sum_{k=1}^{\infty}\delta_{nk}(t)e_{nk} \otimes f_{nk}$$

, for $t \geq 0$

where

$$\sum_{k=1}^{\infty}\left|\sigma_{nk}(t)\right|^p \leq \lambda_1 < \infty, \sum_{k=1}^{\infty}\left|\delta_{nk}(t)\right|^p \leq \lambda_2 < \infty$$

, $t \geq 0$

for all n, (e_n) and (f_n) are orthonormal sequences for each n. Since

$$\left\|T_{1,n}(t) - T_1(t)\right\| \rightarrow 0, \left\|T_{2,n}(t) - T_2(t)\right\| \rightarrow 0$$

, as $n \rightarrow \infty$, it follows that

$T_1(t), T_2(t)$ are compact.

Let $T_1(t) = \sum_{k=1}^{\infty}\sigma_k(t)e_k \otimes f_k$

$T_2(t) = \sum_{k=1}^{\infty}\delta_k(t)e_k \otimes f_k$.

Using Theorem 1.20[6], we get

$\sigma_{nk}(t) \rightarrow \sigma_k(t)$ and $\delta_{nk}(t) \rightarrow \delta_k(t)$ as $n \rightarrow \infty$ for all k.

Since

$$\sum_{k=1}^{r}\left|\sigma_k(t)\right|^p = \sum_{k=1}^{r}\lim_n\left|\sigma_{nk}(t)\right|^p = \lim_n \sum_{k=1}^{r}\left|\sigma_{nk}\right|^p \leq \lambda_1$$

, for $t \geq 0$,

Also, we obtain

$\sum_{k=1}^{r}\left|\delta_k(t)\right|^p \leq \lambda_2$ is true for every r ,it follows that,

$\|\mathbb{T}(t)\|_p$ =

$$(\sum_{k=1}^{\infty}|\sigma_k(t)|^p)^{\frac{1}{p}}(\sum_{k=1}^{\infty}|\delta_k|^p)^{\frac{1}{p}} \le \lambda_1\lambda_2$$

, for $t \ge 0$.

Hence

$\mathbb{T}(t)X = T_1(t)XT_2(t) \in C_p$.

Definition (2.2):

Let $T(t)Z= T_1(t)ZT_2(t)$, $t \ge 0$ be a C_o- composite semigroup in {L(H), τ}. We say T(t) is of type P if :

(i) $T(t) \in C_p$ for all $t \ge 0$.

(ii) There exists an $\in > 0$ and $\alpha > 0$ such that $||\mathbb{T}(t)||_p \le \alpha$ for all $t \in (0, \varepsilon)$.

Let $T(t)X= T_1(t)XT_2(t)$, $t \ge 0$ be a C_o- composite semigroup of operators with generator $A=A_1 .+. A_2$.

Let $\lambda \in \rho$(A) such that the real part Re(λ) $> w_1 + w_2$, where

$$||\mathbb{T}(t)|| \le M_1M_2e^{(w_1+w_2)t}.$$

We define a family of operators $\{R_t(\lambda, \mathbb{A})\}$, where

$$R_t(\lambda, A)X = \int_t^{\infty} e^{-\lambda s}\ \mathbb{T}(t)X\, ds.$$

$$= \int_t^{\infty} e^{-\lambda s}T_1(s)XT_2(s)ds \cdot$$

We say $\{R_t(\lambda, A)\}$ is of type p if

(i) $R_t(\lambda, A) \in C_p$ for all $t \ge 0$ and $\lambda \in \rho(\mathbb{A})$ such that Re(λ) $> w_1 + w_2$.

(ii) There exists an $\beta > 0$ such that $\|\lambda R_t(\lambda, A)\|_p \le \beta$ for all $t \in (0, \infty)$ and $\lambda \in \rho$(A), Re (λ) $> a > w_1 + w_2$, where a is positive constant.

Now we prove the following results.

Theorem (2.2):

Let $T(t)Z= T_1(t)ZT_2(t)$, $t \ge 0$ be a C_o- composite semigroup of operators in {L(H), τ} with generator $A=A_1 .+. A_2$ .Then the following are equivalent

(i) $T_1(t)$ is of type p.

(ii) $\{R_t(\lambda, A)\}$ is of type p and T(t) is uniformly continuous on $(0, \infty)$.

Proof:

$(i) \to (ii)$, we have

$R\ (\lambda, A)X$

$$= \int_0^{\infty} e^{-\lambda s}T_1(s)XT_2(s)ds$$

$$=\tau\text{-}\lim_{t\to 0}\int_t^{\infty} e^{-\lambda s}T_1(s)XT_2(s)ds$$

$$=\tau\text{-}\lim_{t\to 0} R_t(\lambda, A).$$

where the above limit is uniform limit.

Now,

$R_t(\lambda, A)X$

$$= \int_t^{\infty} e^{-\lambda s}T_1(s)XT_2(s)ds$$

=

$$T_1(t)[\int_t^{\infty} e^{-\lambda s} T_1(s-t) X T_2(s-t) ds \ ] T_2(t)$$

Since $T_1(t) \in C_p$, it follows that $R_t(\lambda, A) \in C_p$.

Furthermore ,

$$\|\lambda R_t(\lambda, A)\|_p =$$

$$\left\|\lambda T_1(t) \int_t^{\infty} e^{-\lambda s} T_1(s-t) X T_2(s-t) ds T_2(t)\right\|_p$$

$$\leq |\lambda| \|T_1(t)\|_p \left|\int_t^{\infty} e^{-\lambda s} M_1 M_2 e^{(w_1+w_2)(s-t)} ds\right| \|T_2(t)\|_{L(H)} \|X\|_{L(H)}$$

Since $\|T_2(t)\|_{L(H)} \leq M_1 e^{w_1 t}$, So

$$\|\lambda R_t(\lambda, A)\|_p \leq |\lambda| \|T_1(t)\|_p \frac{1}{|(w_1+w_2)-\lambda|} \xi \|X\|_{L(H)}$$

.

if $t \in (0, \delta), \delta \leq \varepsilon$, we get $\|\lambda R_t(\lambda, A)\|_p \leq \beta$.

Conversely , $(ii) \rightarrow (i)$

Since $T_1(t)$, $T_2(t)$ are uniformly continuous , it follows that $R_t(\lambda, A) \rightarrow R(\lambda, A) =$ $\int_0^{\infty} e^{-\lambda s} T_1(s).T_2(s) ds$ convergent uniformly, as $t \rightarrow 0$. By the assumption, $\| R_t(\lambda, A) \|_p \leq \beta$. It follows that from convergence theorem in C_p, [7], that $R(\lambda, A) \in C_p$.

Also

$$\|\lambda R(\lambda, A)\|_p \leq \lim_{t \to 0} \|\lambda R_t(\lambda, A)\|_p \leq \beta \quad \forall t \in (0, \delta)$$

Further it follows from [4] that $\lambda R(\lambda, A) T_1(t) \rightarrow T_1(t)$ as $\lambda \rightarrow \infty$ uniformly,

And $\|\lambda R(\lambda, A) T_1(t)\|_p \leq \beta \|T_1(t)\|$, from the semigroup property,[4],

Since $\|T_1(t)\| \leq M_1 e^{w_1 t}$, so as $t \in (0, \delta)$, we obtain $\|T_1(t)\| \leq \eta$, for some $\eta > 0$. Thus $\lambda R(\lambda, A) T_1(t)$ is uniformly bounded in C_p.

$$\|\lambda R_t(\lambda, A) T_1(t)\|_p \leq \beta \|T_1(t)\| \leq \beta\eta.$$

Consequently,[7] , $T_1(t) \in C_p$ for all $t \in (0, \varepsilon]$.

from the semigroup property that $T_1(t) \in C_p$ for all t>0. Further;

$$\|T_1(t)\| \le \left\| \lim_{\lambda\to\infty} \lambda R(\lambda, A) \right.$$

$$T_1(t)\|_p \le \left\| \lim_{\lambda\to\infty} \lambda R(\lambda, A) \right.$$

$$\|_p \|T_1(t)\|$$

$\le \beta\eta$, for $t \in (0, \varepsilon]$.

Remark(2.3):

(i) $T_1(t) \in C_p$ and $T_2(t) \in C_p$ then $T(t) \in C_p$ for all p and $t \in (0,\infty)$.

(ii) There exts a C_0- semigroup of operators $T(t)$ such that $T(t) \in C_p$ for all $t \in (0,\infty)$, but $\|\mathbb{T}(t)\|_p \le \infty$ as $t \to 0$ as the following example.

Example(2.4):

Let A_1 and A_2 are a positive compact operators which are not of finite ranks and $\|A_1\|, \|A_2\| \le 1$. So

$$A_1 = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n, A_2 = \sum_{n=1}^{\infty} \sigma_n e_n \otimes e_n$$

for some $0 < \lambda_n, \sigma_n < 1$ and decreasing, and (e_n) is some orthonormal basis .Define a one parameter family of composite operators as follows:

$T(t)X = T_1(t)XT_2(t) =$

$$(\sum_{n=1}^{\infty} \lambda_n^t e_n \otimes e_n) X (\sum_{n=1}^{\infty} \sigma_n^t e_n \otimes e_n)$$

It is easily seen that $T_1(t)$,$T_2(t)$ are C_0- semigroups of operators on H, see [4].

Choose $(\lambda_n), (\sigma_n) \in \bigcap_{p>0} \ell^p$,

where ℓ^p is the space of p-summable sequences. Then $T_1(t), T_2(t) \in C_p$ for all p and all t and also from Remark(2.3)(i), we have that $T(t) \in C_p$.

Now,

$$\|T_1(t)\|_p = (\sum_{n=1}^{\infty} \lambda_n^{tp})^{1/p}, \text{and} \|T_2(t)\|_p = (\sum_{n=1}^{\infty} \sigma_n^{tp})^{1/p}$$

Further

$$\|T_1(t)\|_p \le \|T_1(s)\|_p, \|T_2(t)\|_p \le \|T_2(s)\|_p \text{ for } \quad t > s.$$

The Monotone Convergence Theorem implies that $\|T_1(t)\|_p \to \infty$ and $\|T_2(t)\|_p \to \infty$,as $t \to 0$

Since

$\|\mathbb{T}(t)\|_p \le \|T_1(t)\|_p \|T_2(t)\|_p$, we have that

$$\lim_{t\to 0} \|\mathbb{T}(t)\|_p \le \lim_{t\to 0} \|T_1(t)\|_p \|T_2(t)\|_p$$

Thus

$\lim_{t\to 0} \|\mathbb{T}(t)\|_p \le \infty$ and this completes the proof.

Theorem(2.3):

Let $T(t)Z = T_1(t)ZT_2(t)$, $t \ge 0$ be a C_o- composite semigroup of operators in $\{L(H), \tau\}$ with generator A then the following are equivalent $=A_1$.+. A_2 . If $w \in (0,\infty)$ such that $T(t) \le e^{(w_1+w_2)t}$,

(i)

$(t) \in C_p$ for $t \in (0,\infty)$ and if

$\|T(\frac{1}{n})\|_p \le \|T_1(\frac{1}{n})\|_p \|T_2(\frac{1}{n})\|_p \le \gamma_1\gamma_2$

for $n \ge n_0$, for some n_0 and some $\gamma_1, \gamma_2 > 0$.

(ii) $R(\lambda, A) \in C_p$ and $\|\lambda R(\lambda, A)\|_p \le \frac{\gamma_1\gamma_2}{|(w_1+w_2)-\lambda|}$,

for some $\gamma_1, \gamma_2 > 0$, and all $\lambda > 0$.

Proof:

(i) $\to$ (ii), set $R_n(\lambda, A)$

$$X = \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T_1(s) X T_2(s) ds$$

$$= T_1(\tfrac{1}{n})\left[\int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T_1(s-\tfrac{1}{n}) X T_2(s-\tfrac{1}{n}) ds\right] T_2(\tfrac{1}{n}) ds$$

Since (i) is satisfied , we have that $T_1(\frac{1}{n})$, $T_2(\frac{1}{n}) \in C_p$ and from theorem(2.2) that $R_n(\lambda, A) \in C_p$, and also implies that

$\| R_n(\lambda, A) X\|_p =$

$$\|T_1(\tfrac{1}{n})\|_p \left|\int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T_1(s-\tfrac{1}{n}) X T_1(s-\tfrac{1}{n}) ds\right| \|T_2(\tfrac{1}{n})\|_p$$

$$\le \gamma_1\gamma_2 \frac{1}{|(w_1+w_2)-\lambda|},$$ for

$n > n_0$ for large value of n.

But

$R_n(\lambda, A)X \to R(\lambda, A)X$ for all $X \in L(H)$, as $n \to \infty$.

Consequently, lemma(1.2), implies that $R(\lambda, A) \in C_p$, and

$$\| R(\lambda, A) \|_p \le \gamma_1\gamma_2 \frac{1}{|(w_1+w_2)-\lambda|},$$ for $\lambda > (w_1+w_2)$.

(ii) $\to$ (i) by expansion formulation of any semigroup, see [4] we have that

T(t)X=

$$\lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^n t^n)}{n!} [\lambda R(\lambda, A)]^n X,$$ for $(w_1+w_2) > \lambda$

where w_1, w_2 is as given in the assumption. Then

$$\|T(t)\|_p \le \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^n t^n)}{n!} \lambda^n \|R(\lambda, A)\|^{n-1} \|R(\lambda, A)\|_p$$

But

$$\| R(\lambda, A)\|_{L(L(H))} \le \frac{M_1 M_2}{\lambda - (w_1+w_2)}.$$

Since

$(\lambda-(w_1+w_2)>0 \Rightarrow |\lambda-(w_1+w_2)|>0 \Rightarrow |(w_1+w_2)-\lambda|>0$

Thus

$\| R(\lambda, A)$

$$\|_{L(L(H))} \leq \frac{M_1 M_2}{|\lambda-(w_1+w_2)|}.$$

Hence

$$||\mathbb{T}(t)||_p$$

$$\leq \lim_{\lambda\to\infty} \sum_{n=0}^{\infty} \frac{(\lambda^n t^n)}{n!|(w_1+w_2)-\lambda|^{n-1}} \frac{\gamma_1\gamma_2}{|(w_1+w_2)-\lambda|}$$

$$\leq \lim_{\lambda\to\infty} e^{-\lambda t}\gamma_1\gamma_2 \sup_n \left|\frac{\lambda}{(w_1+w_2)-\lambda}\right|^n \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!}$$

$$\leq \lim_{\lambda\to\infty} \gamma_1\gamma_2 \left|\frac{\lambda}{(w_1+w_2)-\lambda}\right| \leq \gamma_1\gamma_2 K, \quad \text{for } K \geq 1.$$

Consequently $\mathbb{T}(t) \in C_p$.

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