Fully π - *P* – Stable Rings

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Received on:17/6/2009
Accepted on:5/11/2009

Abstract

M.S.Abbas [1] introduced and studied the concept of fully stable *R*-modules and called a ring *R* is fully stable (pseudo - stable) if it is fully stable (pseudo - stable) *R*-module . And A.M.Abdul-Daim [2] introduced and studied the concept of fully π – stable rings as a generalization of fully stable rings .

The purpose of this paper is to generalize the concept of fully pseudo – stable rings to fully π – pseudo - stable rings . Some properties and characterizations of fully π – pseudo – stable rings are obtained . A condition is given such that a fully π – pseudo – stable ring is fully π – stable .

الاستقرارية التامة من النمط - *p* - π

الخلاصة:

في [1] قدم ودرس لأول مرة مفهوم الموديولات التامة الاستقرارية (الاستقرارية الكاذبة) وسميت الحلقة *R* بأنها تامة الاستقرارية (الاستقرارية الكاذبة) اذا كانت تامة الاستقرارية (الاستقرارية الكاذبة) كموديول على نفسها . و في [2] قدم ودرس لأول مرة مفهوم الاستقرارية التامة من النمط – π كتعميم للاستقرارية التامة .

ان الغرض من هذا البحث هو تعميم مفهوم الاستقرارية الكاذبة التامة الى الاستقرارية الكاذبة التامة من النمط - π . درست بعض الصفات والخصائص للاستقرارية الكاذبة التامة من النمط – π . واعطي الشرط بحيث تكون الاستقرارية الكاذبة التامة من النمط – π استقرارية تامة من النمط – π .

Introduction

In this paper , *R* represents a commutative ring with identity and all modules are left unitary.

M.S.Abbas [1] was introduced the concept of a fully stable *R*-module and then introduced the concept of a fully pseudo-stable (fully *p*-stable) module as a generalization of a fully stable module.

(1) Definition

An *R*-module *M* is said to be fully stable module, if for each *R*-homomorphism $\alpha:N\to M$ of any submodule *N* of *M* into *M*, we have $\alpha(N)\subseteq N$. A ring *R* is fully stable if it is a fully stable *R*- module .

(2)Definition

An *R*-module *M* is said to be fully *p* - stable if for each *R*-

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https://doi.org/10.30684/etj.28.3.17
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monomophism $\alpha:N\to M$ of any submodule N of M into M, we have $\alpha(N) \subseteq N$. A ring R is fully pseudo stable (fully p-stable) if and only if it is a fully p-stable R-module [1].

In [2] the concept of a π – stable rings is investigated which includes the class of fully stable rings and that of π – regular rings.

(3)Definition

A ring R is called fully π –stable if and only if for each element x in R, there exists a positive integer n such that for every R-homomorphism $\alpha :Rx^n \to R$ we have $\alpha(Rx^n) \subseteq Rx^n$.

In an analogous manner, we introduce now a class of rings larger than the class *of* fully π –stable rings.

(4)Definition

Let R be any ring. An element x in R is called π - pseudo – stable (abbreviated π -p stable) if there exists a positive integer n such that for every R-monomorphism $\alpha :Rx^n \to R$ we have $\alpha(Rx^n) \subseteq Rx^n$.

A ring R is called fully π – pseudo – stable if and only if every element in it is π – pseudo – stable.

It is clear that every π – stable element of an arbitrary ring is π -p-stable. Hence every fully π –stable rings is fully π -p-stable, we conjecture the converse is not true , but we recall that a non zero R-module M is said to be uniform if each non zero submodules of M has non zero intersection with every non zero submodule of M . A ring R is uniform if it is uniform R-module , then we have the following:

(5) Proposition

Every fully π - p-stable uniform ring is fully π –stable ring.

Proof

Let R be a fully π - p-stable uniform ring. For any element x in R there exists a positive integer n and for every R – homomorphism $f :Rx^n \to R$. If ker (f) = (0), there is nothing to prove . Otherwise, let $y \in \ker(f) \cap \ker(I_{Rx}{}^n + f)$ then f(y) = 0 and $(I_{Rx}{}^n + f)(y) = 0$. Now, $y = y + f(y) = (I_{Rx}{}^n + f)(y) = 0$. Thus $\ker(f) \cap \ker(I_{Rx}{}^n + f) = (0)$, but R is uniform, hence $\ker(I_{Rx}{}^n + f) = (0)$, that is , $(I_{Rx}{}^n + f) :Rx^n \to R$ is an R-monomphism.Since R fully π – p – stable , then $(I_{Rx}{}^n + f)(Rx^n) \subseteq Rx^n$, hence $f(Rx^n) \subseteq Rx^n$.

W. D. Weakly [3] was introduced the concept of terse module. An R-module is said to be terse iff distinct submodules are not isomorphic. He proved that for an R-module to be terse, it is enough to have the property that distinct cyclic submodules are not isomorphic.

A ring R is terse if and only if it is terse R – module. The following is a generalization for terse rings.

(6) Definition

A ring R is called π-terse iff for any two elements x and y in R there exists a positive integer n such that if $Rx^n \neq Ry$ implies $Rx^n \cong Ry$.

In the following proposition we show that the concepts of a π – tersencess and full π - p – stability are coincide.

(7) Proposition

A ring R is π -terse if and only if it is fully π – p – stable ring.

Proof

Suppose that R is π –terse ring and there exists an element x in R and R -monomorphism $f : Rx^n \to R$ such that $f(Rx^n) \not\subset Rx^n$ for each positive integer n , then Rx^n and $f\,(R_x{}^n) = R_{f(x^n)}$ are two distinct ideals of R . Since R is π -terse ring , then $R_{f(x^n)} = f\ (Rx^n)$ is not isomorphic to Rx^n which is not true . Hence R is fully π – p stable ring.

Conversely , suppose that R is a fully π - p - stable ring and R has two elements x and y such that $Rx^t \cong Ry$ but $Rx^t \neq Ry$ for each positive integer t . We can assume that $Rx^t \not\subset Ry$. Then there exists a non – zero element z in Rx^t which is not in Ry . Let $f : Rx^t \to Ry$ be an isomorphism , consider the following two R-monomorphisms, $I_{Ry} \circ f : Rx^t \to R$ and $I_{Rx^t} \circ f^{-1}: Ry \to R$, since R is fully π - p - stable ring , then $(I_{Ry} \circ f)(Rx^t) \subseteq Rx^t$ and $(I_{Rx^t} \circ f^{-1})(Ry) \subseteq Ry$.
Now , $z = (I_{Rx^t} \circ f^{-1} \circ I_{Ry} \circ f)_{(Z)} \in Ry$ which is a contradiction .
Proposition (7) together with proposition (5) give:

(8)Corollary

Let R be a uniform ring and π -terse ring , then R is fully π-stable ring.
From proposition (7) we see that every fully π-stable ring , is π – terse , hence we have the following proposition :

(9) Proposition

Let R be a fully π – stable ring and let x and y be any two elements in R with Ry a direct summand of R then there exists a positive integer n such that if Rx^n is isomorphic to Ry , then Rx^n is direct summand of R

Proof

Since R is fully π – stable ring , then R is π -terse , so if $Rx^n \cong Ry$, then $Rx^n = Ry$, which implies that Rx^n is a direct summand of R .

Next, we will characterize fully π – stable rings among fully π-p-stable rings. However, we shall need the following lemma (for its proof , see[1]).

(10) Lemma

Let M be an R- module and I an ideal of R. Then $ann_M(I) \cong Hom_R(R/I, M)$.

(11)Theorem

Let R be a ring. Then the following statements are equivalent:-

(1) R is a fully π - stable ring .
(2) R is a π –terse ring and for every element x in R there exists a positive integer n such that $Rx^n \cong Hom_R(Rx^n, R)$.

Proof

(1) implies (2). Assume that R is a fully π-stable ring , then R is π-terse .Since R is fully π - stable ring ,then for every element x in R there exists a positive integer n such that $Rx^n = ann(ann(Rx^n))$ [2] . By Lemma (10) $ann(ann(Rx^n)) \cong Hom(R/ann(Rx^n),R) = Hom(Rx^n,R)$ which implies that $Rx^n \cong Hom(Rx^n, R)$.

(2) implies (1). Suppose that R is π –terse and for every element x in R there exists a positive integer n such that $Rx^n \cong Hom(Rx^n, R)$.By Lemma (10) $ann(ann(Rx^n)) \cong Hom(R/ann(Rx^n), R) \cong Hom(Rx^n, R)$, *then* $Rx^n \cong ann(ann(Rx^n))$ π -*tersenss* of R implies that $Rx^n = ann(ann(Rx^n))$. Hence R is fully π -stable ring.

The following corollary follows from proposition (7) which gives a characterization of fully π -stable rings among fully π-p- stable rings.

(12) Corollary

The following statements are equivalent for a ring R .

1) R is a fully π -stable ring.
2) R is a fully $\pi - p -$ stable ring and for every element x in R there exists a positive integer n such that $Rx^n \cong \text{Hom}(Rx^n,R)$.

Discussion

From all the above we have the following :-

(1) Every fully π – stable ring is fully $\pi - p$ – stable .

(2) Every fully $\pi - p -$ stable uniform ring is fully π – stable ring .

(3) A ring R is π - terse if and only if it is fully $\pi - p$ – stable .

(4) A ring R is fully π – stable ring if and only if R is fully $\pi - p$ – stable ring and for every element x in R there exists a positive integer n such that $Rx^n \cong Hom_R(Rx^n, R)$.

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