# A Moment Method for the Second Order Two-point Boundary Value Problems

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**Abstract**

In this paper a Moment method based on the second, third and fourth kind Chebyshev polynomials is proposed to approximate the solution of a linear two-point boundary value problem of the second order. The proposed method is flexible, easy to program and efficient. Two numerical examples are given for conciliating the results of this method, all the computation results are obtained using Matlab.

**Keywords:-** Moment method, Chebyshev polynomial, linear two-points boundary value problem (LTPBVP), Second order.

## الطريقه اللحظيه لمسائل القيم الحدوديه بين نقطتين من الرتبه الثانيه

**الخلاصة**

قدم في هذا البحث طريقه اللحظيه والمبنية على متعددات حدود شبشف للانواع الثاني, الثالث, الرابع لتقريب الحل لمسألة القيم الحدودية الخطية بين نقطتين من الرتبه الثانيه.الطريقة المقترحة مرنة, من السهل برمجتها وكفؤة. اعطي مثالين عددين لحساب النتائج بهذه الطريقة, كل النتائج تم حسابها باستخدام Matlab

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## 1. Introduction

There are large problems in engineering and physics that can be described through the use of linear two-point boundary value problem. Such that diffusion occurring in the presence of exothermic chemical reaction, heat conductions associated with radiation effect [10].

Consider the second order linear two-point boundary value

$$M[Z(x)] = z'' + p(x)z' + q(x)z = f(x) \quad x \in [a,b] \qquad ...(1)$$

$$z(a) = g_0 \quad , \; z(b) = g_1$$

problem [6]where M is the 2th order differential operator, the given function p,q and f are continuous on [a,b]. Many researchers used TPBVP of the second order in many subjects such as Bently,T.G. who used Spectral integration to present the solution of linear Two-point boundary value problem[2], while Jin & Wei studied wavelet functions applied for two-point boundary value problem.[8]

In this paper the moment method with the second, third and fourth kind of Chebyshev polynomials will be used to approximate the solution of the two-point boundary value problem of the second order.

## 2. Chebyshev Polynomials

Chebyshev polynomials are extremely important in approximating theory and they also arise in many other areas of applied mathematics [9].

The definitions and certain basic properties of the Chebyshev polynomials are presented. These properties are needed to prove our main results.

### 2.1 The first- kind Chebyshev Polynomials $T_n$: - [11]

The Chebyshev polynomials of the first kind $T_n(x)$ may be defined by the following recurrence relation. Set $T_0(x) = 1$ and $T_1(x) = x$, then

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \qquad n = 2,3,... \qquad ...(2)$$

Important properties of Chebyshev polynomials can be formulated as:-[7]

1-They are orthogonal with weighted function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ on the interval [-1, 1] that is:-

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x)\, dx = \begin{cases} 0 & m \neq n \\ p & m = n = 0 \\ p/2 & m = n = 1,2,.. \end{cases}$$

2-A Chebyshev polynomial at one point can be expressed by neighboring Chebyshev polynomial at the same point, that is:-

$$T_0(x) = 1 \;,\; T_1(x) = x \;,\; T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \quad ...(3)$$

3- $T_m(x)$ has m-distinct zeros $x_i$ that lie in the interval [-1, 1], given by:-

$$x_i = \cos(\frac{(2i+1)p}{2m}) \quad , \quad i = 0,1,\ldots,m-1$$

moreover, $T_m$ assumes it absolute extreme at:-

$$x_i = \cos(\frac{ip}{m}) \quad , \quad i = 0,1,\ldots,m$$

4-The explicit expression of general formula of $T_m$ is

$$T_m(x) = \frac{m}{2}\sum_{r=0}^{[m/2]}(-1)^r \frac{(m-r-1)!}{r!(m-2r)!}(2x)^{(m-2r)} \qquad for\,all\,m \geq 1$$

where

$$\left[m/2\right] = \begin{cases} \frac{m}{2} & if\;\; m\;\; is\;\; even \\ \frac{m-1}{2} & if\;\; m\;\; is\;\; odd \end{cases}$$

**Proposition: -** [9]

The n[th] derivatives of Chebyshev polynomials $T_m$ are formulated as:-

$$\frac{d^n T_m(x)}{dx^n} = \begin{cases} 0 & if\, m<n \\ 2^{m-1}m! & if\, m=n \\ \frac{m}{2}\sum_{r=0}^{[\frac{m-n}{2}]}(-1)^r \frac{(m-r-1)!}{r!(m-2r)!}2^n (2x)^{(m-2r-n)} & if\, m>n \end{cases} \qquad \ldots(4)$$

$where\; n=1,2,3,\ldots and\; m \geq 0$

## 2.2 The second- kind Chebyshev Polynomials $U_n$ : - [7]

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x and range same as for $T_n(x)$ defined by the recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad with \quad U_0(x)=1, U_1(x)=2x \qquad \ldots(5)$$

so we deduce that

$$U_0(x)=1, U_1(x)=2x, U_2(x)=4x^2-1, U_3(x)=8x^3-4x, \ldots \qquad \ldots(6)$$

## 2.3 The Third and Fourth kind Chebyshev Polynomials $V_n$ and $W_n$ :-[7]

Two other families of polynomials $V_n(x)$ and $W_n(x)$ may be constructed, which are related to $T_n(x)$ and $U_n(x)$, but which have trigonometric definitions involving the half angle $q/2$ (where $x = \cos q$ as before).These polynomials are also referred to as the 'airfoil polynomials', but Gantschi(1992) rather appropriately named them third and fourth- kind Chebyshev polynomials.

**Definition**

The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are polynomials of degree n in x:-

These polynomials too may be efficiently generated by the use of a recurrence relation.

$$V_n(x) = 2x\,V_{n-1}(x) - V_{n-2}(x) \quad with\, V_0(x)=1, V_1(x)=2x-1 \qquad \ldots(7)$$

$$W_n(x) = 2x\,W_{n-1}(x) - W_{n-2}(x) \quad with\, W_0(x)=1, W_1(x)=2x+1 \qquad \ldots(8)$$

where n=2,3,4,…

we may readily show that

$$V_0(x)=1,\; V_1(x)=2x-1,\; V_2(x)=4x^2-2x-1,\; V_3(x)=8x^3-4x^2-4x+1 \qquad \ldots.(10)$$

and

$$W_0(x)=1,\; W_1(x)=2x+1,\; W_2(x)=4x^2+2x-1, W_3(x) = 8x^3+4x^2-4x-1, \ldots \qquad \ldots..(11)$$

Thus $V_n(x)$ and $W_n(x)$ share precisely the same recurrence relation as $T_n(x)$ ,$U_n(x)$ and their generation differs only in the prescriptions of the initial condition n=1.

Delves and Mohmad used Chebyshev polynomials in Galerkin approximation method to present a solution of Fredholm integral equation [3], moreover Guglieme and Mario suggest a simple method based on Chebyshev approximation at Chebyshev nodes to approximate partial differential equations.[5]

### 3. Moment Method

Moment method is one of the weighted residual methods, Sarker and Su used this method for solving Fredholm integral equation of the first kind [12], while Abdolerza, Farhad and Jafar improved second moment method for solution of pure advection problem [1].

We consider the following second order linear two-point boundary value problem:-

$$M[Z(x)] = z'' + p(x)z' + q(x)z = f(x) \quad \ldots(12)$$

with the boundary conditions

$$z(a) = g_0 \quad , \quad z(b) = g_1 \quad \ldots(13)$$

where $x \in [a,b]$ and the function p,q and f are continuous on [a,b].

The problem of finding an approximation solution to the boundary value problem (12) is often obtained by assuming the solution Z(x) as:-

$$Z_N(x) = \sum_{i=0}^{N}{}' d_i C_i(x) = \frac{1}{2} d_0 C_0(x) + d_1 C_1(x) + \ldots + d_N C_N(x) \quad \ldots(14)$$

*for all* $a \le x \le b$ [7].

where $C_i$'s are the second , third and fourth kind Cheyshev polynomial defined in (6), (10) and (11). This approximation must satisfy the boundary conditions (13), substituting (14) in (12) we get the residue in the differential equation:-

$$E(x) = M[z_N(x)] - f(x) \quad \ldots(15)$$

the residue E(x) depends on x as well as on the way that the parameters $d_i$ are chosen.

We hope the residue E(x) will become smaller; the exact solution is obtained when the residue is identically zero. It is difficult to make E(x) =0; we shall try to make it as small as possible in some sense.

In weighted residual methods the unknown parameters $d_i$ are chosen to minimize the residual E(x) by setting its weighted integral equal to zero

i.e

$$\int w_j E(x) dx = 0 \qquad j = 0,1,\ldots,N \quad \ldots(16)$$

...(13)

where $w_j$ is prescribed weighting function.

In moment method, we put the weighting function:-

$$w_j = x^j$$

inserting (17) in (16) by yields:-

$$\int x^j E(x) dx = 0, \quad j = 0,1,2,\ldots,N$$

now, we have (N+1) linear equations for determent (N+1) coefficient $d_0, d_1, ..., d_N$.

## 4. The Solution of Second Order TPBVP Using Moment Method

Consider the following second order TPBVP equations:-[6]

$$z'' + p(x)z' + q(x)z = f(x) \quad ...(18)$$

Subject to the boundary conditions:-

$$z(a) = g_0 \qquad , \qquad z(b) = g_1$$

the unknown function u(x) is approximated using:-

$$Z_N(x) = \sum_{i=0}^{N}{}' d_i C_i(x) \quad ...(19)$$

where $C_i$'s are the second , third and fourth kind Cheyshev polynomials.

since these approximations must satisfy the boundary condition, we get:-

$$Z(a) = \frac{d_0}{2} C_0(a) + d_1 C_1(a) + d_2 C_2(a) + ... + d_N C_N(a) = g_0$$

*hence*

$$d_0 = 2\left(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}\right) \quad ...(20)$$

$$Z(b) = \frac{d_0}{2} C_0(b) + d_1 C_1(b) + d_2 C_2(b) + ... + d_N C_N(b) = g_1$$

*hence*

$$d_1 = \frac{1}{C_1(b)}[((g_1 - 2(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}))\frac{C_0(b)}{2}) - \sum_{i=2}^{N} d_i C_i(b)] \quad ...(21)$$

by substitute equation (20) and (21) into (19) we get

$$Z_N(x) = (2(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}))C_0(x) + \frac{1}{C_1(b)}[((g_1 - 2(\frac{g_0 - \sum_{i=1}^{N} d_i C_i(a)}{C_0(a)}))\frac{C_0(b)}{2}) - \sum_{i=2}^{N} d_i C_i(b)]C_1(x) + \sum_{i=2}^{N} d_i C_i(x)$$

using operator form to get:-

$$M[z] = f(x)$$

where the operator M is defined as:-

$$M[z] = \frac{d^2}{dx^2} z_N + p(x)\frac{d}{dx} z_N + q(x) z_N$$

the residue equation becomes:-

$$E(x) = M[z] - f(x)$$

then given linearly independent $w_0, w_1, ..., w_N$ on interval [a,b] that is:-

$$\int_b^a w_j(x) E(x) dx = 0, \; j = 0,1,...,N \quad ...(19)$$

where $w_j = x^j$.

Gaussian elimination procedure is used to solve system s and 'of N-1 equations to find $d_m$ substitute in eq.(18) to obtain the approximate solution of z(x).

## 5. Numerical Examples

Consider the following linear two-point value problem:-

$$y'' = y - 4xe^x$$

with boundary conditions:-

y(0)=y(1)=0

while the exact solution is :-

$$y(x) = x(1-x)e^x$$

This problem with N=4 using moment by assuming the approximated solution

$$y_4 = \sum_{i=0}^{4}{}' d_i C_i(x)$$

Table (1) presents a comparison between the exact and approximated solution which depends on the least square error.

Figure (1) shows a comparison between the exact solution against the approximate solution of the problem which is presented in example (1) using Moment method Cheyshev polynomials second, third and fourth kind of y(x).

**Example (2)**

Consider the following linear two-point boundary value problem:-

$$y'' = y - x^2 - x$$

with boundary conditions:-

y(0)=2 , y(1)=4

while the analytical solution is :-

$$y(x) = x^2 + x + 2$$

This problem with N=5 using moment by assuming the approximated solution

$$y_5 = \sum_{i=0}^{5}{}' d_i C_i(x)$$

Table (2) presents a comparison between the exact and approximated solution which depends on the least square error.

Figure (2) shows a comparison between the exact solution against the approximate solution of the problem which is presented in example (2) using Moment method Cheyshev polynomials second, third and fourth kind of y(x).

## 6. Conclusions

The method, was described using Chepyshev polynomials of second, third and fourth kind provides convenient and efficient way for solving two-point boundary value problem of the second order. The results show a marked improvement in the least square errors form which we conclude that:-

The Chepyshev polynomial of the second kind gives better accuracy and stability than the other kinds of Chepyshev polynomials depending on the least square error.

As "N the number of D notes" is increased, the term is decreased.

## 7. Reference


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**Table (1) solution of example (1)**

| x | The Exact solution | CH Pol.2 | CH Pol.3 | CH Pol.4 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.0995 | 0.1001 | 0.1012 | 0.1012 |
| 0.2 | 0.1954 | 0.1946 | 0.1961 | 0.1962 |
| 0.3 | 0.2835 | 0.2826 | 0.2826 | 0.2826 |
| 0.4 | 0.3580 | 0.3581 | 0.3564 | 0.3564 |
| 0.5 | 0.4122 | 0.4123 | 0.4108 | 0.4109 |
| 0.6 | 0.4373 | 0.4369 | 0.4369 | 0.4369 |
| 0.7 | 0.4229 | 0.4232 | 0.4235 | 0.4234 |
| 0.8 | 0.3561 | 0.3557 | 0.3570 | 0.3571 |
| 0.9 | 0.2214 | 0.2213 | 0.2219 | 0.2219 |
| 1 | 0 | 0 | 0 | 0 |
| L.S.E | | 0.00001 | 0.00001 | 0.00001 |

Table (2) solution of example (2)

| x | The Exact solution | CH Pol.2 | CH Pol.3 | CH Pol.4 |
|---|---|---|---|---|
| 0 | 2 | 2 | 2 | 2 |
| 0.1 | 2.1100 | 2.1100 | 2.1101 | 2.1101 |
| 0.2 | 2.2400 | 2.2400 | 2.2401 | 2.2401 |
| 0.3 | 2.3900 | 2.3900 | 2.3902 | 2.3902 |
| 0.4 | 2.5600 | 2.5600 | 2.5601 | 2.5601 |
| 0.5 | 2.7500 | 2.7500 | 2.7501 | 2.7501 |
| 0.6 | 2.9600 | 2.9600 | 2.9601 | 2.9601 |
| 0.7 | 3.1900 | 3.1900 | 3.1902 | 3.1902 |
| 0.8 | 3.4400 | 3.4400 | 3.4401 | 3.4401 |
| 0.9 | 3.7100 | 3.7100 | 3.7101 | 3.7101 |
| 1 | 4 | 4 | 4 | 4 |
| L.S.E | | 0 | 0.0000002 | 0.0000002 |

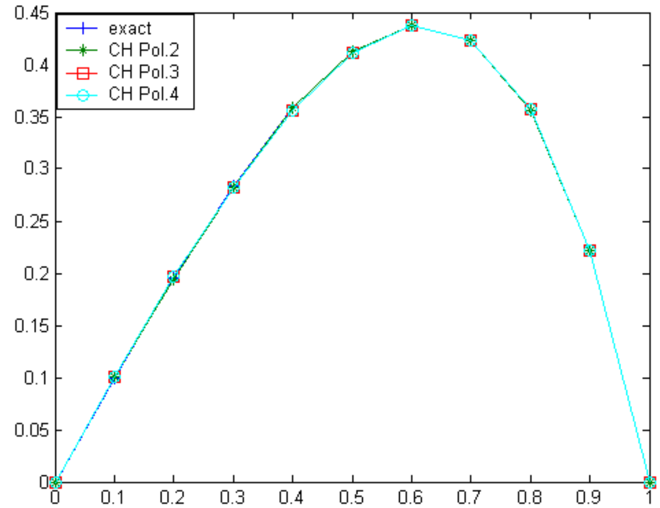


**Figure (1)**

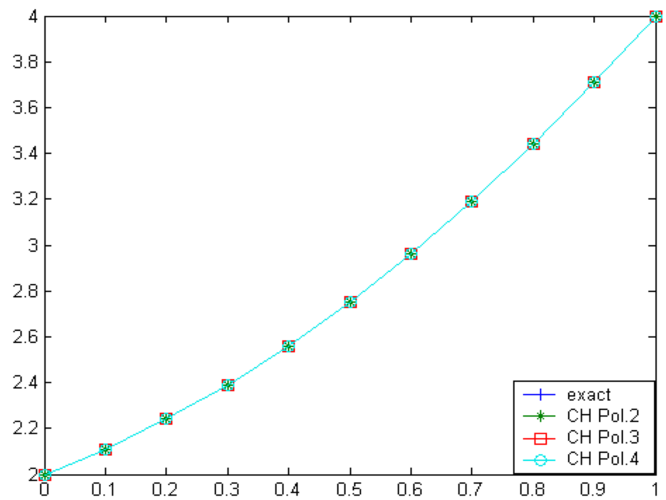


**Figure (2)**