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The Artin's Exponent of A Special Linear Group $SL(2,2^k)$

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Abstract

The set of all n×n non singular matrices over the field F form a group under the operation of matrix multiplication, This group is called the general linear group of dimension n over the field F, denoted by GL(n,F) .
The subgroup from this group is called the special linear group denoted by SL(n,F).

We take n=2 and $F=2^k$ where k natural, k>1. Thus we have SL $(2,2^k)$.
Our work in this thesis is to find the Artin's exponent from the cyclic subgroups of these groups and the character table of it's.
Then we have that: a $SL(2,2^k)$ is equal to 2^{k-1} .

Keywords: Linear Group, Special Group, Exponent.

"اس ارتن للزمر الخطية الخاصة $SL(2,2^K)$"

الخلاصة

أن مجموعة كل المصفوفات الشاذة على الحقل F تشكل زمرة تحت العملية الثنائية ضرب المصفوفات، هذه الزمرة تسمى الزمرة الخطية العامة ذات البعد n على الحقل F ويرمز لها GL(n,F).
الزمرة الجزئية من هذه الزمرة تسمى الزمرة الخطية الخاصة ويرمز لها بالرمز SL(n,F)
في بحثنا هذااخترنا n=2، $F=2^k$، k عدد طبيعي اكبر من الواحد أي سنأخذ الزمرة الجزئية الخاصة $SL(2,2^k)$.
في هذا العمل حاولنا إيجاد أس ارتن لهذه الزمرة من الزمر الجزئية الدائرية لها ، كما وقمنا بإيجاد جداول الكاركتر (Character Table) لمجموعة من الزمر الجزئية الخاصة $SL(2,2^k)$
ولقد حصلنا على النتيجة التالية :

$$a\,(SL(2,2^k))=2^{k-1}$$

1-Introduction

In this work our focus will lie on the representation and character theory of finite groups. R(G) is the group of all rational valued characters of G under point – wise addition, and T(G) is the group generated by the induced characters from the principal characters of certain subgroups of G satisfying the three conditions of Solomon theorem .Solomon theorem states that the factor group R(G)/T(G) has a finite exponent dividing |G|. E-Artin in (1927) proved that every rationally valued character of G is a rational sum of representation character of G, or in other words, the exponent of R(G)/T(G) is finite.

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In (1968) Lam proved a sharp form of Artin's theorem, he determined that the least positive integer A(G) such that A(G)χ is an integral linear combination of the induced principal characters of cyclic subgroups, for any rational valued character χ of G, A(G) is called the Artin exponent of G. In his paper, he studied A(G) extensively for many groups. He has shown that A(G) can be evaluated by knowing the Artin characters of G and that A(G) is equal to one if and only if G is cyclic.

Now, This thesis concentrates on the constructing of the character table of the irreducible rational representation and Artin's characters induced from all cyclic subgroups of SL$(2,2^k)$ where k natural number, k>1.We have found in this work that: a(SL$(2,2^k)$) = 2^{k-1} .

2- Representation Theory

Definition (2.2), [1, 9]

The set of all n×n non- singular matrices over the field ¢ of complex number under the operation of matrix multiplication is called the **general linear group** of dimension **n** over the field ¢ denoted by GL(n,¢).

Definition (2.3), [11]

Let R:G→GL(n,¢) be a matrix representation of G, then R is said to be **reducible** if for any x∈G, R(x) is equivalent to a matrix of the form

$$M^{-1}\, R(x)\, M = \begin{pmatrix} R_1(x) & E(x) \\ 0 & R_2(x) \end{pmatrix},\ \forall x \in G$$

where $R_1(x)$, $R_2(x)$ are two representations of G. $R_1(x)$,$R_2(x)$, and E(x) are matrices over ¢ of dimensions r×r, s×s and (n-r)(n-s) respectively, such that 0 < r < n and r+s = n Otherwise then the representation is called **irreducible**

Remark (2.4):

Any one dimensional representation is irreducible.

3. Character Theory

Definition (3.1), [8]:

Let R:G→GL(n,¢) a representation of G. the complex valued function X:G→G defined by χ (x) = Trace(R(x)) is called the **character of x** afforded by the representation R.

Definition (3.2), [5,7] :

Let x, y be two elements of a group G, then we said that x, y be conjugate if ∃ g∈G such that $g^{-1}xg = y$

Definition (3.3), [10,16]*:*

Let G be a finite group, F:G→¢ which is constant on the conjugate classes of G is called **class function** .

4. Induced characters [5]*:*

Definition (4.1), [8]:

Let H≤G and X:H→¢ is a character (or any class function). Then the induced character

$$Ind_H^G \ X(g) \sum_{h\in G} X(hgh^{-1}) =$$

(1/|H|)

where X(g) = 0 if g∉H

Definition (4.2), [10]:

The least integer A(G) such that A(G) Φ is an integral linear combination of the induced principal characters of the cyclic subgroup of G, for all rational valued characters Φ of G, A(G) is called the Artin exponent of G.

Definition (4.3), [4]:

The integer linear combination of arbitrary character induced from the cyclic subgroups of G, a(G) is determined as the least integer such that a(G) X is an integral linear combination of characters induced from cyclic subgroups of G, for all character X of G.

Notation:

The character induced from the characters of its cyclic subgroups of G is called Artin's exponent

5. Artin exponent a(G) of finite groups*:*

Definition (5.1), [6]:

If <t>is a cyclic subgroup of G we define n(t)=n(<t>) to be the number of subgroup <s>of <t> such that $N_G<s>/C_G<s>$ is non trivial .

Theorem (5.2): (Main Theorem)

Let G be a non cyclic group of order P^n. Let $k\geq 0$. The following conditions are necessary and sufficient that $a(G)\leq P^k$.

1) For each element χ of order P in G, $a(N_G(<\chi>)/<\chi>)\leq P^k$

2) For each element χ of order P in G, there exists a cyclic subgroup <t> containing $<\chi>$ such that $n(t)\geq m-k-1$, where $|N_G<\chi>|=P^m$.

Proof:

See [6].

Definition (5.3), [6]:

Let G be a finite group, the integral linear combination of arbitrary characters induced from the cyclic subgroups of G is called Artin's exponent of G and denoted by a(G).

Definition (5.4), [6]:

Let G be a finite group, the least integer such that a(G)X is an integral linear combination of characters induced from cyclic subgroup of G, for all characters X of G.

6. The Special Linear Group:

Definition (6.1), [1, 5, 9,]:

The general linear group of degree n in the set of n×n invertible (non singular) matrices, together with the operation of ordinary matrix multiplication. These form a group, because the product of two invertible matrices is again invertible, and the inverse of an invertible matrix is invertible.

Definition (6.2), [2] :

The general linear group over the field F is the group of n×n invertible matrices denoted by GL(n,F). the determination of these matrices is a homomorphism from GL(n,F) into F^*. Thus SL(n,F) is the subgroup of GL(n,F) which contains all matrices of determinate one and it is called special linear group .

Theorem (6.3):

Let $G=SL(2,2^k)$ has exactly (2^k+1) conjugacy classes C_g for $g\in G$as the table (1).

Proof:

See [5].

7.The Artin Exponent a(G) of $SL(2,2^k)$:

Theorem (7.1):

Let $G= SL(2,2^k)$, k=natural, k>1. Then $a(G)=2^{k-1}$ and the table of characters induced from the characters of all its cyclic subgroups see table (2).

Proof :

$| SL(2,2^k) |= 2^k (2^{2k}-1)$(by lemma (3.2.6))

From theorem (3.4.5.), $G= SL(2,2^k)$ has exactly (2^k+1) conjugacy classes C_g for $g \in G$ see table (3).

where:-

$1\le \ell \le (2^k-2)/2$ and $1\le m \le 2^k/2$

By the definition of inducing we obtained the induced characters Φ_1, Φ_2, Φ_3 and Φ_4 of $SL(2,2^k)$ from the characters of all cyclic subgroups see table(4):-

Then we have the following table see table(5) :

By multiply Φ_4 by -1 we get:

$-\ell (2^k (2^k+1))$

By multiply Φ_3 by -1 we get:

$-m (2^k (2^k-1))$

By multiply Φ_2 by $-(1/2^{k-1})$ we get:

$-1/2^{k-1} \Phi_2 =-(2^{k-1}(2^{2k}-1)/ 2^{k-1}) = -(2^{2k}-1)$

And then adding the result to $\Phi1 =2^k (2^{2k}-1)$ we get:

$-m2^k (2^k-1) -\ell 2^k (2^k+1) - (2^{2k}-1) +2^k(2^{2k}-1)$

$=-2^k/2(2^k (2^k-1)) - ((2^k-2)/2) 2^k (2^k+1) - 2^{2k}+1+2^{3k}-2^k$

$= -2^{2k-1}(2^k-1) - 2^{2k-1}(2^k+1) + 2^k (2^k+1) + 2^{3k}-2^{2k}-2^k+1$

$=-2^{3k-1}+2^{2k-1}-2^{3k-1}-2^{2k-1}+2^{2k}+2^k+2^{3k}-2^{2k}-2^{1k}+1$

$=2^{3k} (-\frac{1}{2}-\frac{1}{2}+1) +1=1$

Thus $a (SL (2,2^k)) = 2^{k-1}$.

Example (7.2):

If $k=2 \Rightarrow G = SL(2,2^2)$:-

$| SL(2,2^2) |= 2^k (2^{2k}-1) = 2^2(2^4-1) =60$

The conjugacy classes of $SL (2,2^2)$ is $2^k+1 =2^2+1 = 5$, for $g \in G$

$1, c, a^\ell, b^m$ where $1 \le \ell \le (2^k-2)/2 \Rightarrow 1 \le \ell \le 1$.

$1 \le m \le 2^k/2 \Rightarrow 1 \le m \le 2$.

$\Rightarrow 1, c, a^1, b^1, b^2$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$

$a = \begin{pmatrix} v^1 & 0 \\ 0 & v^{-1} \end{pmatrix}$

$|<v>|=|F^*|=2^k-1$

where the order of a is $2^k-1 =2^2-1 =3$.

Then $e,e' \le 3/2 \Rightarrow e = e' =1$

Also the order of b is $2^k+1=2^2+1=5$.

Then $f,f' \le 5/2 \Rightarrow f =f' =1$.

See table (6) and table (7)

Example (7.3):

If $k=3 \Rightarrow G = SL(2,2^3)$:-

The order of $SL (2,2^3) = 2^k(2^{2k}-1)=2^3(2^6-1)=8*63=504$

Have exactly $+1=2^3+1=9$ conjugacy classes.

Where $1\le \ell \le 2^3-2/2 \Rightarrow 1 \le \ell \le 3$

$1\le m \le 2^3/2 \Rightarrow 1 \le m \le 4$

$\Rightarrow 1, c, a^1, a^2, a^3, b^1, b^2, b^3, b^4$

Order of a is $7 \Rightarrow \ell =1$

Order of b is 9, the divisors of 9 is 1, 3 $\Rightarrow$f=1, 3 see table (8) and table(9).

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Table (1) The table of conjugacy classes of SL(2,2^k)

G	I	C	a^{ℓ}	b^m
| Cg |	1	$(2^{2k}-1)$	$2^k(2^k+1)$	$2^k(2^k-1)$
$\|C_G(g)\|$	$2^k(2^{2k}+1)$	2^k	2^k-1	2^k+1

where:- $1 \le \ell \le (2^k-2)/2$ and $1 \le m \le 2^k/2$.

$$1 \le \mathbf{l} \le \frac{2^k-2}{2} \quad and \quad 1 \le m \le \frac{2^k}{2} \ .$$

$SL(2,2^k)$	1	C	a^ℓ	b^m
$\lvert C_{(g)}\rvert$	1	$2^{2k}-1$	$2^k(2^k+1)$	$2^k(2^k-1)$
$\lvert C_G(g)\rvert$	$2^k(2^{2k}-1)$	2^k	2^k-1	2^k+1
Φ_1	$2^k(2^{2k}-1)$	0	0	0
Φ_2	$2^k(2^{2k}-1)/2$	$-2^k/2$	0	0
Φ_3	$\ell[2^k(2^{2k}-1)/(2^k-1)]$	0	$-(2^k-1)/(2^k1)$	0
Φ_4	$m[2^k(2^{2k}-1)/(2^k+1)]$	0	0	$-(2^k+1)/(2^k+1)$

where:- $1 \le \ell \le (2^k-2)/2$ and $1 \le m \le 2^k/2$.

G	1	C	a^ℓ	b^m
$\lvert C(g)\rvert$	1	$(2^{2k}-1)$	$2^k(2^k+1)$	$2^k(2^k-1)$
$\lvert G_G(g)\rvert$	$2^k(2^{2k}-1)$	2^k	(2^k-1)	(2^k+1)

SL(2,2^k)	I	C	a^ℓ	b^m
Φ_1	$2^k(2^{2k}-1)$		0	0
Φ_2	$2^k(2^{2k}-1)/2$	$2^k/2$	0	0
Φ_3	$\ell[2^k(2^{2k}-1)/(2^k-1)]$	0	$-(2^k-1)/(2^k-1)$	0
Φ_4	$m[2^k(2^{2k}-1)/(2^k+1)]$	0		$-(2^k+1)/(2^k+1)$

Table (2) The character table of rational representations of SL $(2,2^2)$

	I	C	a	b
1_G	1	1	1	1
Ψ	4	0	1	-1
χ	5	1	-1	0
θ	6	-2	0	1

Table (3)The table of artin's character of SL $(2,2^2)$

$SL(2,2^2)$	I	c	a^1	b^1	b^2
$\|Cg\|$	1	15	20	12	12
$\|C_G(g)\|$	60	4	3	5	5
Φ_1	60	0	0	0	0
Φ_2	30	-2	0	0	0
Φ_3	20	0	-1	0	0
Φ_4	24	0	0	-1	-1

We can see that:-

Þ **a $(SL(2,2^2)) = 2$**

Table (4) The character table of rational representations of SL $(2,2^3)$

$SL(2,2^3)$	I	C	a	b^1	b^3
1_G	1	1	1	1	1
Ψ	8	0	1	-1	-1
X	27	3	-1	0	0
θ_1	21	-3	0	0	3
θ_3	7	-1	0	1	-2

Table (5)The table of artin's characters of $SL(2,2^3)$

$SL(2,2^3)$	I	C	a^1	a^2	a^3	b^1	b^2	b^3	b^4
$\|C_{(g)}\|$	63	72	72	72	56	56	56	56	63
$\|C_G(g)\|$	8	7	7	7	9	9	9	9	8
Φ_1	0	0	0	0	0	0	0	0	0
Φ_2	-4	0	0	0	0	0	0	0	-4
Φ_3	0	-1	-1	-1	0	0	0	0	0
Φ_4	4(56)=224	0	0	0	0	-1	-1	-1	-1

We can see that:

$-\Phi_4$:	-224	0	0	0	0	1	1	1	1
$-\Phi_3$:	-216	0	1	1	1	0	0	0	0
-1/4 Φ_2:	-63	1	0	0	0	0	0	0	0
	-503	1	1	1	1	1	1	1	1
Φ_1:	+504	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1

$$\Rightarrow a(SL(2,2^3))=4=2^{3-1}$$