On Pólya's random walk constants

On Pólya's random walk constants*

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Abstract

A celebrated result in probability theory is that a simple symmetric random walk on the d-dimensional lattice $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d \geq 3$. In this note, we derive a closed-form expression, in terms of the Lauricella function of type C, for the return probability for all $d \geq 3$. Previously, a closed-form formula had only been available for $d = 3$.

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1 Introduction

Let $p(d)$ be the probability that a simple symmetric random walk on the d-dimensional lattice $\mathbb{Z}^d$ returns to origin, for $d \geq 1$. A celebrated result of Pólya [10] states that $p(1) = p(2) = 1$ but $p(d) < 1$ for $d \geq 3$. An explicit formula is available in the three-dimensional case:

$$p(3) = 1 - 1/u(3) = 0.3405373296\ldots,$$

where

$$\begin{aligned} u(3) &= \frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x-\cos y-\cos z} && (1.1)\\ &= \frac{\sqrt{6}}{32\pi^3}\Gamma\Big(\frac{1}{12}\Big)\Gamma\Big(\frac{5}{12}\Big)\Gamma\Big(\frac{7}{12}\Big)\Gamma\Big(\frac{11}{12}\Big) && (1.2)\\ &= 1.5163860592\ldots \end{aligned}$$

(see [2, 5, 7, 13]). The integral in (1.1) is one of Watson's triple integrals [13] up to a multiplicative factor.

Closed-form expressions for the case $d \geq 4$ are not available to date in the literature, although numerical values are reported in [4, 8] and an integral representation was obtained by [8]: for $d \geq 3$,

$$p(d) = 1 - 1/u(d), \tag{1.3}$$

where

$$u(d) = \int_{(-\pi,\pi)^d}\Big(d - \sum_{k=1}^{d}\cos x_k\Big)^{-1}\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_d \tag{1.4}$$

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$$= \int_0^\infty \left[I_0 \left(\frac{x}{d} \right) \right]^d \mathrm{e}^{-x} \, \mathrm{d}x, \tag{1.5}$$

with $I_0(\cdot)$ denoting the modified Bessel function of the first kind of order zero, defined by

$$I_0(x) = \sum_{k \geq 0} \frac{1}{(k!)^2} \left(\frac{x}{2} \right)^{2k}. \tag{1.6}$$

The integral in (1.4) is a d-fold integral generalisation of the Watson triple integral (1.1) (again, up to a multiplicative factor). Note that the integral (1.5) is not convergent for $d = 1, 2$, which is easily seen from the limiting form $I_0(x) \sim \mathrm{e}^x / \sqrt{2\pi x}$, $x \to \infty$ (see [9]).

In this note, we derive a closed-form expression for the return probability $p(d)$ for any positive integer $d \geq 3$. The expression involves the Lauricella function of type C (see [3, 6]), defined by

$$F_C^{(d)}(a, b; c_1, \ldots, c_d; x_1, \ldots, x_d) = \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{(a)_{k_1 + \cdots + k_d} (b)_{k_1 + \cdots + k_d}}{(c_1)_{k_1} \cdots (c_d)_{k_d}} \frac{x_1^{k_1} \cdots x_d^{k_d}}{k_1! \cdots k_d!}, \tag{1.7}$$

where $(f)_k = f(f+1) \cdots (f+k-1) = \Gamma(f+k)/\Gamma(f)$ denotes the ascending factorial or the Pochhammer symbol. Numerical routines for the direct computation of (1.7) are available; see, for instance, the *Mathematica*-based routine presented in [1].

2 Closed-form expression for the return probability

Our main result is the following.

Theorem 2.1. *For any positive integer $d \geq 3$,*

$$u(d) = F_C^{(d)} \left(1, \frac{1}{2}; 1, \ldots, 1; \frac{1}{d^2}, \ldots, \frac{1}{d^2} \right). \tag{2.8}$$

Proof. Using (1.6), we can write (1.5) as

$$\begin{aligned}
u(d) &= \int_0^\infty \left[\sum_{k \geq 0} \frac{1}{(k!)^2} \left(\frac{x}{2d} \right)^{2k} \right]^d \mathrm{e}^{-x} \, \mathrm{d}x \\
&= \int_0^\infty \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{1}{(k_1! \cdots k_d!)^2} \left(\frac{x}{2d} \right)^{2k_1 + \cdots + 2k_d} \mathrm{e}^{-x} \, \mathrm{d}x \\
&= \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{1}{(k_1! \cdots k_d!)^2 (2d)^{2k_1 + \cdots + 2k_d}} \int_0^\infty x^{2k_1 + \cdots + 2k_d} \mathrm{e}^{-x} \, \mathrm{d}x \\
&= \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{1}{(k_1! \cdots k_d!)^2 (2d)^{2k_1 + \cdots + 2k_d}} \Gamma(2k_1 + \cdots + 2k_d + 1).
\end{aligned} \tag{2.9}$$

Using the duplication formula for the gamma function, (2.9) can be written as

$$\begin{aligned}
u(d) &= \frac{1}{\sqrt{\pi}} \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{1}{(k_1! \cdots k_d!)^2 d^{2k_1 + \cdots + 2k_d}} \Gamma\left(k_1 + \cdots + k_d + \frac{1}{2} \right) \Gamma(k_1 + \cdots + k_d + 1) \\
&= \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{(1)_{k_1 + \cdots + k_d} \left(\frac{1}{2} \right)_{k_1 + \cdots + k_d}}{(1)_{k_1} \cdots (1)_{k_d} k_1! \cdots k_d! d^{2k_1 + \cdots + 2k_d}}.
\end{aligned}$$

Now (2.8) follows from the definition in (1.7). □

Remark 2.2. *The return probability* (1.3) *becomes*

$$p(d) = 1 - \left[F_C^{(d)}\left(1, \frac{1}{2}; 1, \dots, 1; \frac{1}{d^2}, \dots, \frac{1}{d^2}\right)\right]^{-1},$$

for all positive integers $d \geq 3$.

Corollary 2.3. *The following reduction formula holds:*

$$F_C^{(3)}\left(1, \frac{1}{2}; 1, 1, 1; \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) = \frac{\sqrt{6}}{32\pi^3}\Gamma\Big(\frac{1}{12}\Big)\Gamma\Big(\frac{5}{12}\Big)\Gamma\Big(\frac{7}{12}\Big)\Gamma\Big(\frac{11}{12}\Big). \tag{2.10}$$

Proof. Combine (1.2) and (2.8). □

Remark 2.4. 1. *The reduction formula* (2.10) *appears to be new. We could not locate it in standard references such as* [12]*.*

2. *We were unable to obtain a further simplification of* (2.8) *for* $d \geq 4$ *from reduction formulas for Lauricella functions in standard references such as* [12]*. However, we cannot not rule out this possibility, especially in the light of the fact that we could not locate* (2.10) *in the existing literature.*

The direct Laplace transform [11, p. 346, Eq. 3.15.16.35] turns out to be erroneous. Here we give its corrected form. On specifying $\lambda = \nu_j = 0$, $a_j = d^{-1}$ and $p = 1$ in (2.12) below we arrive at (2.8). Recall that the modified Bessel function of the first kind of order $\nu \in \mathbb{R}$ is defined for $x \in \mathbb{R}$ by the power series

$$I_\nu(x) = \sum_{k\geq 0} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}. \tag{2.11}$$

Lemma 2.5. *Denote* $\nu = \sum_{j=1}^d \nu_j$*, where* d *is a positive integer. Let* $\Re(\lambda) + \nu > -1$ *and* $\nu_j > -1$*;* $j = 1, \dots, d$*. Let* $a_1, \dots, a_d > 0$*. Then, the Laplace transform*

$$\begin{aligned}\mathscr{L}_p\Big[x^\lambda \prod_{j=1}^d I_{\nu_j}(a_j x)\Big]\,\mathrm{d}x &= \frac{\Gamma(\lambda+\nu+1)}{2^\nu p^{\lambda+\nu+1}}\left\{\prod_{j=1}^d \frac{a_j^{\nu_j}}{\Gamma(\nu_j+1)}\right\} \\ &\quad \cdot F_C^{(d)}\Big(\frac{\lambda+\nu+1}{2}, \frac{\lambda+\nu}{2}+1; \nu_1+1, \dots, \nu_d+1; \frac{a_1^2}{p^2}, \dots, \frac{a_d^2}{p^2}\Big),\end{aligned} \tag{2.12}$$

provided $p > \sum_{j=1}^d a_j$*, or* $p = \sum_{j=1}^d a_j$ *and* $\Re(\lambda) < d/2 - 1$*.*

Proof. The conditions $p > \sum_{j=1}^d a_j$, or $p = \sum_{j=1}^d a_j$ and $\Re(\lambda) < d/2 - 1$ are required to ensure that the integral in the Laplace transform is convergent; this is easily seen from the limiting form $I_\nu(x) \sim \mathrm{e}^x/\sqrt{2\pi x}$, $x \to \infty$ (see [9]).

Applying the power series definition (2.11) of the function $I_\nu(x)$, denoting $\boldsymbol{n} = (n_1, \dots, n_d)$ and $n = \sum_{j=1}^d n_j$, we conclude by the Legendre duplication formula (twice) that

$$\begin{aligned}\mathscr{L}_p\Big[x^\lambda \prod_{j=1}^d I_{\nu_j}(a_j x)\Big]\,\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-px} x^\lambda \prod_{j=1}^d I_{\nu_j}(a_j x)\,\mathrm{d}x \\ &= \sum_{\boldsymbol{n}\geq 0}\prod_{j=1}^d \frac{\big(\frac{a_j}{2}\big)^{2n_j+\nu_j}}{\Gamma(n_j+\nu_j+1)\,n_j!}\int_0^\infty \mathrm{e}^{-px} x^{\lambda+2n+\nu+1}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{n}\geq 0}\prod_{j=1}^{d}\frac{\left(\frac{a_j}{2}\right)^{2n_j+\nu_j}}{\Gamma(n_j+\nu_j+1)\,n_j!}\frac{\Gamma(\lambda+2n+\nu+1)}{p^{\lambda+2n+\nu+1}}\\
&= \frac{2^{\lambda}}{\sqrt{\pi}p^{\lambda+\nu+1}}\prod_{j=1}^{d}\frac{a_j^{\nu_j}}{\Gamma(\nu_j+1)}\sum_{\boldsymbol{n}\geq 0}\Gamma\Big(\frac{\lambda+\nu+1}{2}+n\Big)\Gamma\Big(\frac{\lambda+\nu}{2}+1+n\Big)\prod_{j=1}^{d}\frac{(a_j^2/p^2)^{n_j}}{(\nu_j+1)_{n_j}\,n_j!}\\
&= \frac{\Gamma(\lambda+\nu+1)}{2^{\nu}p^{\lambda+\nu+1}}\prod_{j=1}^{d}\frac{a_j^{\nu_j}}{\Gamma(\nu_j+1)}\sum_{\boldsymbol{n}\geq 0}\left(\frac{\lambda+\nu+1}{2}\right)_n\left(\frac{\lambda+\nu}{2}+1\right)_n\prod_{j=1}^{d}\frac{(a_j^2/p^2)^{n_j}}{(\nu_j+1)_{n_j}\,n_j!},
\end{aligned}
$$

which is equivalent to the statement. □

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