Fine Analysis of Mean Curvature Flow through Singularities

by

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Contents

List of Figures

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.
The work presented was carried out by the author except in the cases outlined below:

- In Chapter 2, we summarise the classical results from the field relevant to the present work, with references to the original works where appropriate.
- In Section 3.1, we recall the results for 2-convex surgery of Haslhofer–Kleiner, [HK17b].
- In Section 4.1, we summarise the background material relevant to Chapter 4, with references to the original works where appropriate.

Additionally, extensive references are given throughout the text.
Parts of this thesis were achieved in collaboration:

- The new results obtained in Chapter 4 were achieved in collaboration with Felix Schulze and Otis Chodosh.

Parts of this thesis have been published by the author:

- *Approximation of mean curvature flow with generic singularities by smooth flows with surgery.* Advances in Mathematics, Volume 410, Part A, 2022.

Explicitly, Section 1.3.1 and Chapter 3 appeared in [DH22a].

Abstract

We provide a short survey on the history of the mean curvature flow and the theory of flow through singularities. We establish the existence of a smooth flow with surgery approximating weak mean curvature flows with only spherical and neck-pinch singularities, thereby dropping the standard global 2-convexity assumption. This makes use of the resolution of the mean-convex neighbourhood conjecture of Choi–Haslhofer–Hershkovits, and Choi–Haslhofer–Hershkovits–White and a barrier argument for flows with surgery. We conclude our discussion of surgery by utilising the surgery flow, in combination with results of Choi–Chodosh–Mantoulidis–Schulze for generic flows, to increase the known entropy bound for the Schoenflies conjecture in $\mathbb{R}^4$. We then consider mean curvature flow of compact hypersurfaces through conical singularities. We demonstrate a uniqueness theorem for flows with tangent flows modelled on the flow generated by a smooth, stable expander with a linearly growing Jacobi field. Moreover, we demonstrate the forward tangent flow at the conical singularity of the outer-most Brakke flows are modelled on the outer-most expanders of the cone, when said expanders are smooth. Combined with work of Ilmanen–White, this demonstrates genus drop for the outer-most flows through such singularities, answering a conjecture of Ilmanen. Finally, we deduce the following dichotomy in dimensions $2 \leq n \leq 6$: The flow from a compact hypersurface with isolated conical singularity fattens if and only if the flow from the model cone fattens.

Chapter 1

Introduction

The field of Geometric Analysis encompasses the study of partial differential equations that govern the geometry of an object, where these equations are frequently derived from an associated variational problem. The analysis of variational problems originally arose as part of the description of physical phenomena through mathematics, indeed, the cornerstone of mathematical physics is the principle of least action, which states the dynamics of a physical system are determined by the critical points of the functional describing the energy of the system.

In the present work, we will be considering the mean curvature flow of hypersurfaces. We say a family of smooth hypersurfaces, $M_t^n \subset \mathbb{R}^{n+1}$, is a mean curvature flow if

$$\left(\frac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \mathbf{H}_{M_t}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{H}_{M_t}(\mathbf{x}) = -H(\mathbf{x})\nu_{M_t(\mathbf{x})}$ is the mean curvature vector of M_t at a point $\mathbf{x}$ in the hypersurface.

From the perspective of parameterised hypersurfaces, one may define the scalar mean curvature, H, as the sum of the principal curvatures, however, the definition through the first variation formula is perhaps more physically pertinent. By considering how the area of a hypersurface, $M^n \subset \mathbb{R}^{n+1}$, changes under the variation generated by a compactly supported smooth vector field, $X \in \mathrm{Vec}_c(\mathbb{R}^{n+1})$, one deduces the first variation of area, first studied by Allard:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\phi_t^X(M)} d\mu_t = -\int_M \langle X, \mathbf{H}\rangle \, d\mu,$$

where ϕ_t^X is the variation generated by the vector-field X. From this variational perspective, mean curvature flow is the L^2-gradient flow for the area functional. We note the above formula is valid for measure-theoretic generalisations of hypersurfaces known as varifolds, where the variational definition is the only available definition for the mean curvature vector. Certainly, the first variation formula motivates the interpretation of mean curvature as the surface tension, the 'desire' of fluid boundaries to reduce their area. In this manner, mean curvature appears in the equation of capillarity, attributed to Young and Laplace, governing the shape of the interface between two fluids in a tube, and as the elastic force in Germain's study of vibrating plates. Famously, soap films and bubbles are described by the constant mean curvature equation. The case of $H = 0$ is often distinguished as the minimal surface equation, so called as minimal surfaces are the critical points of the area functional, and non-zero constant mean curvature surfaces are critical points under constraints on the volume of the domain bounded by the surface. The mean curvature vector also appears as the normal velocity in models of the motion of grain boundaries in annealing metals, where motion is driven by the surface tension. Indeed, this was the physical motivation for Brakke's foundational text on mean curvature flow.

Mean curvature flow belongs to a family of parabolic equations known as geometric flows. It would be a disservice to the field to not mention the foundational work of Eells–Sampson, [ES64], who introduced the harmonic map heat flow. Their ideas propagated through the mathematical community, inspiring the definitions of other flows, such as Ricci flow, Yamabe flow and the Yang–Mills flow.

Examining equation 1.1, we see that mean curvature flow of hypersurfaces is a degenerate, 2nd order, scalar, parabolic equation. Moreover, by writing the mean curvature of a hypersurface as the surface Laplacian of the position vector, we have

$$\left(\frac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \Delta_{M_t}\mathbf{x}.$$

In doing so, it is apparent mean curvature flow may be viewed as the heat equation of the immersion. Indeed, mean curvature flow enjoys many of the properties of the heat equation, and parabolic equations in general, such as short time existence and uniqueness, and instantaneous improvement of regularity. The distance between two hypersurfaces (at least one of which is compact) satisfies a maximum principle, a staple of elliptic and parabolic equations, and is hence monotonic under the flow. This property is known as the *avoidance principle* and demonstrates mean curvature flow from a compact initial condition must form singularities in finite time, in stark contrast to the heat equation.

1.1 Singularities of Mean Curvature Flow

Rather unusually, weak solutions of the mean curvature flow were studied before smooth solutions. In [Bra78], Brakke considered families of varifolds moving by their mean curvature. These measure-theoretic solutions to mean curvature flow have become known as *Brakke flows*. A precise definition has been given in Definition 2.2.1. The flow is defined as a family of Radon measures, rather than smooth manifolds, readily allowing us to consider flow from singular initial conditions and, crucially, through singularities that may form during evolution.

Another formulation of a weak solution to the mean curvature flow is that of the *Level-set Flow*. Motivated by numerical simulations of Osher and Sethian, [Set90],[OS88], the level-set flow was introduced as a viscosity solution to the mean curvature flow independently by Evans–Spruck [ES91] and Chen–Giga–Goto [CGG91]. As the name suggests, one finds a function on the ambient space-time such that the level-sets move by their mean curvature. Later, an equivalent geometric definition was given by Ilmanen, [Ilm93, Ilm94], recasting the level-set flow purely in terms of the avoidance principle. Ilmanen redefines the level-set flow as the 'maximal motion' of weak set flows - families of closed sets that avoid smooth mean curvature flows.

The notion of level-set flow agrees with classical mean curvature flow from smooth hypersurfaces up to the first singular time. In [ES91], Evans–Spruck provide singular initial conditions from which the viscosity solutions develop an interior, a phenomenon known as *fattening*. They pose the question: can the flow from a smooth, compact hypersurface fatten? An example of such a surface was found numerically by Angenent–Ilmanen–Chopp, [AIC95], and a construction was outlined by Ilmanen and White, [Whi02]. By noting the support of any Brakke flow starting from a given smooth hypersurface, $M \subset \mathbb{R}^{n+1}$, defines a weak set flow, and is thus contained in the level-set flow from M by maximality, one may characterise fattening as capturing an essential non-uniqueness of the Brakke problem. We speak of 'essential non-uniqueness' as the original definition of a Brakke flow allows for instantaneous vanishing. To (partially) overcome this vanishing, one usually works with *unit-regular* flows, Definition 2.2.3, which can be thought of as a condition that forbids the flow to disappear at smooth points. Several conditions are known to ensure non-fattening of the level-set flow through singularities: The flow from star-shaped sets was shown not to fatten by Sonner, [Son93], mean-convex mean curvature flow was shown not to fatten by Evans–Spruck, [ES91], and later White, [Whi00], using a geometric argument, and more recently, Hershkovits–White [HW20] demonstrated non-fattening assuming mean-convexity only in some neighbourhood of the singular set. Hershkovits also demonstrated that Reifenberg initial conditions do not immediately fatten, [Her17],[Her18].

1.1.1 Singularity formation

A standard tool for studying singularities in geometric flows is the *tangent flow*. Colloquially speaking, a tangent flow is a parabolic 'zooming in' on a point to see the first order behaviour of the flow. More formally, a tangent flow is the limiting object attained by taking a (subsequential) limit of a sequence of increasing parabolic dilations of the flow centred at a point. Through the introduction of the monotonicity formula, Definition 2.1.3, Huisken, [Hui90], showed that the backwards tangent flow of any point is modelled on a self-similarly shrinking solution to mean curvature flow. A hypersurface Σ is said to be a self-shrinker if the family $\left\{\sqrt{t}\Sigma\right\}_{t\in(-\infty,0)}$ is a mean curvature flow, see Definition 2.3.2 for other equivalent definitions. In [Whi97], White further explored singularities of mean curvature flow, stratifying the structure singular set in terms of the parabolic Hausdorff dimension of the tangent flows. Moreover, White calculated the Hausdorff dimension of each of the strata ($\mathcal{L}^1$ a.e. in time).

The introduction of the monotonicity formula motivated the study and classification of self-shrinking solutions. Huisken was able to show the only mean-convex self-shrinkers with bounded second fundamental-form are the spheres, $\mathbb{S}^n$, and generalised cylinders, $\mathbb{S}^{n-k} \times \mathbb{R}^k, 1 \leq k \leq n-1$, [Hui90]. See also [CM12]. In [Wan16], L.Wang demonstrated that the ends of a non-compact shrinker in $\mathbb{R}^3$ are asymptotic to either a cylinder or a smooth cone. High genus examples of smooth, asymptotically conical self-shrinkers in $\mathbb{R}^3$ were constructed independently by Kapouleas–Kleene–Møller in [KKM18] and X.H. Nguyen [Ngu14], moreover, recent work of Buzano–H.T. Nguyen–Schulz [BNS21], constructed shrinkers with arbitrary genus, conjectured to have one asymptotically conical end. It is not yet known if it is possible to have a shrinker with 'mixed' ends, though such shrinkers are thought not to exist in $\mathbb{R}^3$. Indeed, the *No Cylinder Conjecture*, [Ilm03, #12], states that the only shrinker with a cylindrical end is a cylinder. In recent work of Chodosh–Choi–Schulze, [CCS23], it was shown that mixed-end shrinkers can be disposed of in $\mathbb{R}^3$ by perturbing the initial condition.

Spheres and generalised cylinders were conjectured by Huisken to be the only singularity models to occur generically, [Ilm03, #8]. Weak flows with only these singularities are known as *generic flows*, the study of which was pioneered by Colding–Minicozzi in their works [CM12, CM15, CM16]. Through their introduction of the entropy functional it was demonstrated that spheres and generalised cylinders are the only linearly stable singularity models. For a definition of entropy, see Definition 2.1.7. Recent work of Chodosh–Choi–Mantoulidis–Schulze, [CCMS20, CCMS21] resolved the genericity conjecture in $\mathbb{R}^3$ for surfaces with entropy $\lambda(M) \leq 2$ and in $\mathbb{R}^4$ for hypersurfaces with entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2) + \varepsilon_0$. Here ε_0 is a positive constant and $\lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ is the entropy of the *bubble-sheet* in $\mathbb{R}^4$. Generic flows are particularly nice to work with, as the structure of their singular set is well understood, see [Whi97, CM15, CM16].

A priori, tangent flows are non-unique. Examining the definition, Definition 2.3.1, one can see non-uniqueness enters the discussion in two ways. Firstly, one works with subsequential convergence. It is hence possible the limit may depend on the sub-sequence chosen. Secondly, one has the freedom to dictate the rate of dilation along the sequence. Indeed, the limiting flows attained by dilating at different rates could, hypothetically, extract different 'modes' of singularity formation. This 'mode selection' is certainly seen when performing an-isotropic blow-ups, as demonstrated in the work of Angenent–Velázquez, [AV97], on the degenerate neck-pinch.

A related notion to the tangent flow is the Type-II blow-up procedure of Hamilton, [Ham95], which can be used to identify the translating 'modes' in the formation of the degenerate neck-pinch. We note that the Type-II blow-up construction also requires suitable base-point selection, rather than just dilating around a fixed point, as in the tangent flow.

Uniqueness of (multiplicity one) tangent flows has been established for compact tangent flows in all dimensions and codimensions, [Sch14], generalised cylinders for (hyper)surface flows in all dimensions, [CM15], and smooth asymptotically conical shrinkers for (hyper)surface flows in all dimensions, [CS21], and high co-dimension, [LZ23]. Recall, this list of singularity models is believed to be the full range of singularities that occur in mean curvature flow from smooth, compact surface in $\mathbb{R}^3$, as 'mixed-end' shrinkers are conjectured not to exist. See the aforementioned *no-cylinder* conjecture. The above uniqueness results were proven using Łojasiewicz-type inequalities motivated by the work of Simon, [Sim83], where, what has become known as, the Łojasiewicz-Simon inequality was used to prove uniqueness of *tangent cones* for minimal surfaces. Tangent cones are the 'elliptic analogue' of tangent flows. It should be noted that Colding–Minicozzi do not directly apply the abstract methods of Simon, instead proving their Łojasiewicz-type inequality by exploiting the structure of the (generalised) cylinder.

1.1.2 Resolution of Singularities

The flow emerging from singularities that have formed is more elusive. The ground-breaking resolution of the mean-convex neighbourhood conjecture by Choi–Haslhofer–Hershkovits for surfaces ($n = 2$), [CHH22], and Choi–Haslhofer–Hershkovits–White for hypersurfaces ($n \geq 3$), [CHHW22], establishes a canonical neighbourhood theorem, hence classifying the behaviour of the flow nearby a neck-pinch singularity in space-time. Their canonical neighbourhood theorem states, at smooth points near a neck-pinch singularity, the flow will look like either a sphere, cylinder, ancient oval or bowl soliton at the scale of the mean curvature. Intuitively, this indicates singularity resolution of neck-pinches is governed by the bowl soliton. Recent classification results of B. Choi, K. Choi, Daskalopoulos Du, Haslhofer, Hershkovits and Šešum [CDD$^+$22, DH21a, DH21b, DH22b, DH23, CHH23a, CHH23b], for ancient 3-convex flows in $\mathbb{R}^4$ indicate there may be an analogous picture in a neighbourhood of a bubble-sheet singularity, $\mathbb{S}^1 \times \mathbb{R}^2$.

If the tangent flow is modelled on a shrinker with only conical ends, the work of Chodosh–Schulze, [CS21], shows that at the singular time, the limiting generalised surface has an isolated conical singularity. An elementary blow up argument demonstrates the forward tangent flow at isolated conical singularities will be modelled on flows from the cone. The details of this argument are contained in proof of Theorem 4.6.7. Moreover, by noting the scaling invariance of the cone $\mathcal{C}$ and the maximality of the level-set flow, it is readily seen that the level-set flow from a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ is inherently self-similar. It is not given, however, that the level-set flow is a hypersurface, as one must be concerned as to whether the flow fattens. We find solace in considering the (space-time) boundary of the level-set flow.

Provided one can show regularity, the space-time boundary will be an *asymptotically conical self-expander*. A hypersurface Σ is said to be a self-expander if the family $\left\{\sqrt{t}\Sigma\right\}_{t\in(0,\infty)}$ is a mean curvature flow, see Definition 2.3.4 for other equivalent definitions. Shown by Ilmanen, [Ilm95a] and expanded upon by Ding, [Din20], there exists at least one (weak) expander asymptotic to any given cone. The recent work of Chodosh–Choi–Schulze, [CCS23], confirms that when the boundary of the level-set flow of cone is smooth, each connected component of the boundary is a smooth expander, moreover, these expanders are one-sided minimizers of the expander energy. In particular, they show this is always true for the outermost flows for a cone in dimensions $2 \leq n \leq 6$. Similar ideas were discussed by Ilmanen [Ilm95a] for surfaces ($n = 2$).

Optimistically, one might hope the resolution of a conical singularity is governed entirely by self-expanders, just as formation is governed by self-shrinkers. Alas, there is no currently known forward monotonicity formula for compact flows. Whilst Bernstein–Wang were able to prove a monotonicity formula for flows from a cone proposed by Ilmanen, [BW22b], it does not rule out non-expanding flows. Furthermore, recent work of L.Chen, [Che22], constructs rescaled mean curvature flows flowing from an (unstable) expander towards another. Such flows are known in the study of dynamical systems as *heteroclinic orbits*. In addition to establishing existence, Chen shows these flows can occur as mean curvature flows from the asymptotic cone. It is hence possible, though currently unknown, if these flows can appear as forward tangent flows at conical singularities that form in the flow from a smooth, compact initial condition.

Some of the more complicated behaviour seen during singularity resolution is well illustrated by work on flows forming singularities modelled on the Simons cone. The Simons cone of dimension $n \geq 7$ is a singular, area minimising minimal surface, moreover, it is a critical point of the expander energy, trivially asymptotic to itself. In [Vel94], Velázquez constructed $O(4) \times O(4)$ symmetric smooth solutions that form a Type-II singularity modelled on the Simons cone, moreover, a subset of these solutions were shown by Stolarski, [Sto23], to have bounded mean curvature up to the singular time. The work of Angenent–Daskalopoulos–Šešum, [ADS21], shows that the generalised hypersurface formed at the singular time of (at least one of) these flows can be used as initial data for mean curvature flow, rigorously proving the existence of the continuation of the flow with bounded mean curvature, formally proposed by Velázquez. It is believed that the compactification process of Stolarski, [Sto22] can be used to compactify the Velázquez solutions, and the Angenent–Daskalopoulos–Šešum continuation.

In Chapter 4, we establish the relationship between the outermost flows from a compact hypersurface with a conical singularity and the outermost flows from the model cone, provided they are smooth expanders. Furthermore, we establish a uniqueness result for the outermost flows from such a compact hypersurface. These results determine the evolution past a conical singularity in dimensions $2 \leq n \leq 6$, however, in higher dimensions, regularity of solutions introduces complications. Recall, the regularity theory for minimal surfaces of Allard, Almgren, DeGiorgi, Federer, Flemming, and Simons demonstrates that the singular set of minimal surface will have co-dimension $k \geq 7$. Since expanders are minimal surfaces in the Gaussian metric on $\mathbb{R}^{n+1}$, in dimensions $n \geq 7$ it is possible for expanders to have singular points, see the previously discussed Simons cone. Currently, our results do not apply to such expanders.

1.1.3 Flows with Surgery

Instead of using a weak flow to continue past a singularity, an alternate approach is to approximate the flow by a piece-wise smooth flow, known as a flow with surgery. As a general principle, such flows will have finitely many surgical modifications, making them desirable for topological applications. Ricci flow with surgery was introduced by Hamilton in [Ham97] for 4-manifolds with positive isotropic curvature. Hamilton also proposed a surgery procedure for 3 dimensional Ricci flow as a method to prove the Poincaré conjecture. This program culminated in the spectacular works of Perelman, [Per02, Per03], proving Thurston's geometrization conjecture.

The surgery procedure for mean curvature flow from a 2-convex hypersurface of dimension $n \geq 3$ was introduced by Huisken–Sinestrari in [HS09] and extended to $n = 2$ by Huisken–Brendle [BH18]. Independently, Haslhofer–Kleiner [HK17a, HK17b] established a surgery procedure that works for all dimensions $n \geq 2$. By classifying blow ups for a more general class of 2-convex flows, they showed regions of high curvature in such flows have a canonical structure.

In both methodologies, existence of 2-convex surgery boils down to the classification of regions of high curvature that develop: a canonical neighbourhood theorem for 2-convex flow. As mentioned above, canonical neighbourhoods of neck-pinch singularities for unit-regular cyclic (mod 2) Brakke flows of dimension $n = 2$ were established in [CHH22] and for $n \geq 3$ in [CHHW22], as a corollary to their resolution of the mean-convex neighbourhood conjecture for neck-pinch singularities. It is from this result that we can extend the smooth mean curvature flow with surgery to flows without a global curvature assumption.

Recall, a flow with surgery will have finitely many surgeries and hence allows for topological information to be tracked in an elementary fashion. See Section 3.5, where we prove the low-entropy Schoenflies conjecture [CCMS21, Conjecture 1.9] in such a manner. Indeed finiteness is desirable, as despite the groundbreaking results concerning the structure and size of the singular set, see White [Whi97] and Colding–Minicozzi [CM15], it is still unknown if there are finitely many singular times, or if spherical singularities can accumulate to a neck-pinch singularity. See the work of B.Choi–Haslhofer–Hershkovits [CHH21].

To highlight why existence of a surgical flow without global curvature assumptions is non-trivial, consider a hypersurface, M, whose weak mean curvature flow has only spherical and neck-pinch singularities. Suppose there is an isolated (non-degenerate) neck-pinch singularity at the first singular time, then, using the canonical neighbourhood theorems of [CHH22, CHHW22], one can follow the arguments of [HK17b] to pick surgery parameters suitable for surgical modifications to be made before the singular time. This process constructs a new hypersurface M'. One immediately runs into a problem: without assuming global 2-convexity, we do not have any knowledge of how the flow from M' will proceed. In the worst case, it may run into non-generic singularities. Moreover, the concatenation of these flows is no longer a weak flow, so passing to global limits along sequences of modified flows becomes impractical.

To overcome these difficulties, one needs to show that by choosing parameters carefully, the surgery flows stay sufficiently close to the weak flow. This is achieved by developing a technical framework that allows us to pass to limits locally. Further, we show the flows with surgical modification converge, in a smooth sense, to the original weak flow away from the singular set. This demonstrates that the surgery process is in some sense stable, allowing for one to perform subsequent surgeries. These stability results are dependent on showing that if the flow from a hypersurface encounters only (multiplicity one) spheres and neck-pinches, then it is 'well-posed'. Explicitly, a flow with only these singularities is unique, moreover, the unit-regular flows from hypersurfaces nearby to our initial condition also remain close to the weak flow we seek to approximate, see Theorem 3.3.1 and Lemma 3.3.2.

1.2 Well-posed problems

A central question in the study of differential equations is whether the problem is *well-posed*. Introduced by Hadamard in the context of modelling physical systems, a problem is said to be *well-posed* if:

1. a solution exists,

2. the solution is unique,

3. the solutions vary 'continuously' in the initial data.

Of course, continuity must be defined using some 'reasonable' topologies on the set of initial conditions and solutions.

Well-posedness is considered a desirable property as it lends credence to approximating solutions via some suitable scheme. Certainly, one must consider if the problem is well-posed when doing numerical simulations. Well-posedness also plays an important role when evolving families of geometric objects under a geometric flow, as one often requires a deformation that varies continuously along the entire family. A notable example is the work of Bamler–Kleiner, [BK23], proving the generalised Smale conjecture using Ricci flow. In [BK22], Bamler–Kleiner showed weak Ricci flow in 3-dimensions is well-posed, confirming a conjecture of Perelman. This enabled them to show that the diffeomorphism group of every 3-dimensional spherical space form deformation retracts to its isometry group. Existence of this notion of weak Ricci flow as the limit of surgery flows was shown in earlier work of Kleiner–Lott, [KL17].

Showing mean curvature flow from smooth, closed initial conditions is well-posed for some short-time follows readily from the avoidance principle and the Arzela–Ascoli theorem. Unfortunately, the essential non-uniqueness captured by the fattening of the level-set flow demonstrates the Brakke problem from smooth, compact hypersurfaces can be ill-posed past singularities.

As mentioned above, there are several known situations in which fattening is known not to occur. Recall, in [Whi00], White showed mean-convex mean curvature flow does not fatten, additionally showing mean-convex mean curvature flow does not develop higher multiplicity through singularities. As a consequence, we deduce the unit-regular Brakke problem from mean-convex hypersurfaces is well-posed. Without imposing curvature conditions on the initial condition, generic singularities offer reprieve to the notion of well-posedness in mean curvature flow. Combining the non-fattening result of Hershkovits–White, [HW20], with the resolution of the mean-convex neighbourhood conjecture for neck-pinch singularities by Choi–Haslhofer–Hershkovits–White, $n = 2$ [CHH22] and $n \geq 3$ [CHHW22], shows that the level-set flow from a hypersurface M does not fatten, provided that the flow from M encounters only spherical and neck-pinch singularities. In $\mathbb{R}^3$, Choi–Haslhofer–Hershkovits combined this observation with Brendle's classification of non-trivial, genus zero shrinkers, [Bre16], to deduce mean curvature flow of spheres in $\mathbb{R}^3$ is well-posed, provided the multiplicity one conjecture holds.

Similarly, Theorem 3.3.1 shows that if the only singularities of the flow are multiplicity one spheres and neck-pinches, then the unit-regular, cyclic mod 2 Brakke problem is well-posed. This is not merely an academic observation, recalling the recent work of Chodosh–Choi–Mantoulidis–Schulze, we see flows encountering only spherical and neck-pinch singularities are generic in $\mathbb{R}^3$ assuming either $\lambda(M) < 2$ or the multiplicity one conjecture, and in $\mathbb{R}^4$ assuming low entropy i.e. $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2) + \varepsilon_0$, [CCMS20, CCMS21].

The well-posed nature of flows with 'generic singularities' is one of the key observations required in Chapter 3 to show existence of the surgical solutions. In combination with an avoidance principle for surgeries, well-posedness is used to show unit-regular Brakke flows from nearby hypersurfaces are barriers to flows with surgical modifications, thereby showing surgical modification is 'stable' with respect to the weak flow.

1.3 Overview of Results

1.3.1 Results for Mean Curvature Flows with Surgery

In Chapter 3, we will be considering an n-dimensional unit-regular, cyclic (mod 2) integral Brakke flow $\mathcal{M}$ that encounters only spherical or neck-pinch singularities (with multiplicity one), evolving from the smoothly embedded, closed hypersurface $M^n \subset \mathbb{R}^{n+1}$. We recall the definition of such singularities.

Definition 1.3.1. A (multiplicity-one) singularity is said to be

(a) *spherical* if it has the shrinking sphere,

$$(-\infty, 0) \ni t \mapsto \mathbb{S}^n \left(\sqrt{-2nt}\right) \times \mathbb{R},$$

as a tangent flow,

(b) a *neck-pinch* if it has the shrinking cylinder,

$$(-\infty, 0) \ni t \mapsto \mathbb{S}^{n-1} \left(\sqrt{-2(n-1)t}\right) \times \mathbb{R},$$

as a tangent flow.

By the work of Hershkovits-White [HW20], and the resolution of the mean-convex neighbourhood conjecture, a level-set flow with only these singularities does not fatten. Moreover, these results, plus the recent work [CCMS21], provide the tools required to prove a uniqueness theorem for weak mean curvature flows with only spherical and neck-pinch singularities. In Theorem 3.3.1, we show that if the outer flow from a given hypersurface $M^n \subset \mathbb{R}^{n+1}$ encounters only spherical and neck-pinch singularities, then it is the unique, unit-regular, cyclic (mod 2), integral Brakke flow starting from M.

Our principal result concerns the existence of a smooth flow with surgery from a given hypersurface. We adapt the definitions of [HK17b] to construct a unit-regular Brakke flow with surgical modification. This gives one the freedom to localise the surgery procedure of Haslhofer–Kleiner.*

The existence of a surgery flow is dependent on two parameters, $H_{\min}$ and Θ. Recall, the parameters of surgery detailed in [HK17b] are: H_{th}, the scale at which

*Ultimately, one will use the maximum principle to show the existence arguments can be applied directly. There is no reason that the formalism of [HS09] and [BH18] could not be used, however, the formalism of [HK17b] makes it very clear what data one has to control on the boundary.

components are dropped, H_{neck}, the scale of the necks which we perform surgery on, and H_{trig}, the trigger scale, at which we pause the flow and perform surgery. The parameter Θ governs the ratios between these quantities. We say $\mathbb{H} \geq \Theta$ if $H_{\text{trig}}/H_{\text{neck}} \geq \Theta$ and $H_{\text{neck}}/H_{\text{th}} \geq \Theta$. We also require $H_{\text{th}} > H_{\min}$.

Theorem 1.3.2 (Existence of a smooth flow with surgery). *Let $M^n \subset \mathbb{R}^{n+1}$ be a smoothly embedded hypersurface, and $\mathcal{M}$ be a unit-regular, cyclic mod 2 integral Brakke flow, emerging from M with only spherical and neck-pinch singularities. Then, the parameters $H_{\min}(M) < \infty$ and $\Theta(M) < \infty$ can be chosen (depending only on the initial hypersurface) such that every weak $(\alpha, \delta, \mathbb{H})$-flow, $\mathcal{M}_{\mathbb{H}}$, with $H_{\text{th}} > H_{\min}$, $\mathbb{H} > \Theta$ satisfies:*

- *$|H| \leq H_{\text{trig}} < \infty$ everywhere,*
- *$\mathcal{M}_{\mathbb{H}}$ vanishes in finite time.*

i.e. $\mathcal{M}_{\mathbb{H}}$ is a smooth mean curvature flow with surgery.

For the precise definition of a weak $(\alpha, \delta, \mathbb{H})$-flow, see Definition 3.2.17. Our proof relies on two key ideas. The first is the construction of barriers to flow with surgery, Theorem 3.3.6, to establish Hausdorff convergence of surgical flows to the level-set flow. Such an idea was first explored by Lauer [Lau13] for 2-convex flows. Their idea is not directly applicable, as they take advantage of the set monotonicity of such flows. Instead, we consider flows from nearby initial conditions and show they act as barriers to surgery flows.

Before detailing the second tool, we make the following observations. Let $\{\mathcal{N}^i\}_{i\in\mathbb{N}}$ be a sequence of integral unit-regular Brakke flows, and presume each flow has a singular set of small Hausdorff dimension. Suppose the sequence converges in the Hausdorff sense to a Brakke flow $\mathcal{M}$. By further assuming $\mathcal{N}^i$ converge smoothly to $\mathcal{M}$ at the initial time, the result of [CCMS20] allows for Hausdorff convergence to be improved to Brakke convergence. Turning our attention back to weak flows with surgery, we observe in regions where no surgical modifications take place, a surgical flow is a smooth mean curvature flow. It is hence desirable to understand where surgical modifications take place. This is the purpose of our second tool, Proposition 3.3.15, which shows surgeries accumulate in the singular set. Moreover, we actually show the smooth convergence of the flows with surgery by probing their behaviour in neighbourhoods of regular points of $\mathcal{M}$ with a careful combination of pseudolocality for mean curvature flow [INS19], graphical estimates [EH91] and the curvature estimates of Haslhofer–Kleiner, [HK17b]. This second tool requires us to only permit surgery in a set with somewhat technical restrictions on the behaviour of the flow along the boundary. These requirements ensure that the hypotheses of the curvature estimates are satisfied.

We consider $\Omega_{(\alpha,\beta)}$ - an open neighbourhood of the singular set with finitely many connected components, along the boundary of which the flow $\mathcal{M}$ behaves in a fashion suitable for surgery in the interior. We examine the class of weak flows with surgery, derived from M. Surgeries are performed only in the set $\Omega_{(\alpha,\beta)}$.

As previously noted, a priori little can be said about the long time behaviour of modified flows due to the parabolic nature of mean curvature flow. Using the above tools we demonstrate the parameters can be chosen suitably such that flows modified by surgery can be written as a small graph over $\mathcal{M}$ along the boundary of $\Omega_{(\alpha,\beta)}$. The existence of suitable parameters is shown by a convergence result, Proposition 3.3.17. It then follows that the weak surgery flows are smooth flows with surgery in the sense of Haslhofer–Kleiner inside $\Omega_{(\alpha,\beta)}$, the canonical neighbourhoods of the flow $\mathcal{M}$, via the maximum principle, and hence the arguments of Haslhofer–Kleiner can be applied to show the existence of a smooth flow with surgery.

In addition, we show that such mean curvature flows with surgery approximate the weak flow, compare [Lau13, Hea13] in the 2-convex case.

Theorem 1.3.3. *Taking the limit as $H_{\text{th}} \to \infty$, the weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$ surgical flows converge in the Hausdorff sense to $\mathcal{M}$. In particular, away from the singular set of $\mathcal{M}$, the convergence is smooth.*

Finally, we combine our proof of the existence of a mean curvature flow with surgery with the existence of generic low entropy flows established by Chodosh–Choi–Mantoulidis–Schulze to get a new bound on entropy for the low-entropy Schoenflies conjecture, as conjectured in [CCMS21, Conjecture 1.9].

Theorem 1.3.4 (Low-entropy Schoenflies for $\mathbb{R}^4$)**.** *Let $\Sigma^3 \subset \mathbb{R}^4$ be a hypersurface homeomorphic to $\mathbb{S}^3$ with entropy $\lambda(\Sigma) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. Then M is smoothly isotopic to the round $\mathbb{S}^3$.*

Surgery is used to decompose the surface into spheres and tori, at which point the topological properties of the flow are exploited to rule out tori. The previous best bound was established independently by Bernstein–Wang [BW22a] and Chodosh–Choi–Mantoulidis–Schulze [CCMS20].

1.3.2 Results for flow through Conical Singularities

Joint work with F.Schulze and O.Chodosh. We study unit-regular, cyclic mod 2 Brakke flows emerging from conical singularities that form in the flow from a smooth compact hypersurface M. If a smooth, conical self-shrinker appears as the backward (multiplicity one) tangent flow of a singular point X, the work of Chodosh–Schulze, [CS21], demonstrates said shrinker is the unique backwards tangent flow at X, moreover, the limiting (generalised) hypersurface at the singular time has an isolated conical singularity. It is therefore sufficient to consider flow from smooth, compact hypersurfaces with isolated conical singularities. To simplify the discussion further, we assume there is a single, isolated, conical singularity, as the methods described may be generalised to finitely many such singularities without much complication. We recall the definition of a hypersurface with isolated conical singularity:

Definition 1.3.5. We say a closed set $M \subset \mathbb{R}^{n+1}$ is a smooth hypersurface with a conical singularity at 0 modelled on the smooth cone $\mathcal{C}$ if:

1. $M\backslash\{0\}$ is a smooth hypersurface,
2. $\lim_{\rho\to\infty} \rho M = \mathcal{C}$,

where the convergence is taken in $C^{\infty}_{loc}(\mathbb{R}^{n+1}\backslash\{0\})$.

From this definition, it is immediate any forward tangent flow centred at the origin of any unit-regular, cyclic mod 2 Brakke flow from M can be extended to include the cone, $\mathcal{C}$, at time 0. It follows any forward tangent flow must be some flow from the cone, hence demonstrating the link between flows from M and flows from $\mathcal{C}$. Since there are known examples of cones with fattening and non-fattening level-set flows, it is natural to ask 'How does the evolution from the model cone govern the uniqueness (or lack thereof) of flows from hypersurfaces with conical singularities?'. In its strongest form, this question may be restated as the following 'folklore' conjecture.

Conjecture 1.3.1 (Fattening Dichotomy)**.** *The level-set flow from a smooth, compact hypersurface with a conical singularity fattens instantaneously if and only if the level-set flow from the model cone fattens.*

We confirm this conjecture, provided the outermost flows from the cone are smooth expanders. In particular, the conjecture is true in dimensions $2 \leq n \leq 6$, by a result of Chodosh–Choi–Mantoulidis–Schulze. Our resolution demonstrates that, near the cone point, the outermost flows from compact hypersurfaces with conical singularities are modelled on the outermost expanders of the model cone. We additionally establish a uniqueness theorem for smooth flows satisfying a Type-I curvature bound and a blow-up assumption near the cone point, a condition we demonstrate is satisfied by the outermost flows. Recall, the outermost expanders necessarily have zero genus, by work of Ilmanen and White [Ilm95b]. Consequently, our work demonstrates genus drop through conical singularities for compact flows. This answers a conjecture of Ilmanen, [Ilm03, #13], for conical singularities occurring in the outermost flows. In aggregate, the tangent flow classification and uniqueness results may be considered a canonical neighbourhood theorem for the outermost flows around conical singularities.

Answering Conjecture 1.3.1 provides a crucial stepping stone towards fully understanding the flow from smooth, compact hypersurfaces. We hope that this work may be combined with that of Hershkovits–White, [HW20], to show non-fattening when the cones that appear as singularity models have unique evolution and other singularities are of mean-convex type. If we believe the multiplicity-one and no-cylinder conjectures hold, combining this work with [HW20], and the resolution of the mean convex neighbourhood conjecture, [CHH22], would complete the understanding of mean curvature flow from smooth, compact surfaces through singularities in $\mathbb{R}^3$.

We now provide an overview of our proof of Conjecture 1.3.1. Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone. We consider a stable, smooth expander Σ, asymptotic to $\mathcal{C}$, and suppose that there is a positive Jacobi field on Σ with linear growth at infinity. These assumptions are satisfied by the outermost expanders from the cone $\mathcal{C}$ (in low dimensions): If the outermost flows from a cone $\mathcal{C}$ are modelled on smooth expanders $\Sigma^{\pm}$, the work Chodosh–Choi–Mantoulidis–Schulze,[CCMS20], shows the expanders $\Sigma^{\pm}$ are outwards minimising, furthermore, a small modification to the construction of Deruelle–Schulze, [DS20] shows the outermost expanders admit a Jacobi field with linear growth. It is important to note the construction of Deruelle–Schulze should also show stable expanders between the outermost should also possess such a linearly growing Jacobi field, and so our work applies to a more general class of expanders.

Let $M_0^n \subset \mathbb{R}^{n+1}$ be a smooth, compact hypersurface with conical singularity modelled on $\mathcal{C}$ and let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0. We make the assumption that every forward tangent flow of $\mathcal{M}$ at the origin is modelled on $\mathcal{M}_\Sigma := \{\sqrt{t}\Sigma\}_{t\in(0,\infty)}$, Assumption 4.2.2, the flow from $\mathcal{C}$ generated by the expander Σ. This assumption is obviously satisfied when there is a unique smooth expander regularising the cone, but we must prove it for the outermost flows from M_0 when there is non-unique evolution from the cone. We show, provided the orientations of M_0 and $\mathcal{C}$ have been chosen in agreement, the following.

Theorem 1.3.6. *Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone, with fattening level-set flow. Suppose the outermost flows from $\mathcal{C}$ are modelled on smooth expanders. Let M_0 be a smooth hypersurface with a conical singularity modelled on $\mathcal{C}$. Then, every tangent flow of any unit-regular, cyclic mod 2 Brakke flow supported on the inner (resp. outer) flow from M_0 is modelled on the inner (resp. outer) expander.*

This theorem immediately demonstrates the level-set flow from M_0 fattens if the flow from $\mathcal{C}$ fattens, as not only do the outermost flows from M_0 disagree, but the level-set flow must have an interior.

Corollary 1.3.7. *Let $\mathcal{C}$ be a smooth cone with fattening level-set flow, and suppose the outermost flows from $\mathcal{C}$ are modelled on smooth expanders. Then, if M_0 has an isolated conical singularity modelled on $\mathcal{C}$, the level-set flow from M_0 fattens instantaneously.*

Under the above blow-up assumption, we demonstrate that a unit-regular, cyclic mod 2 Brakke flow $\mathcal{M}$ satisfies a forward Type-I estimate on the curvature, and is hence smooth on some time interval $(0, T)$, Lemma 4.3.13. Indeed, this shows that assumption $\mathcal{M}$ is a unit-regular, cyclic mod 2 Brakke flow could instead be replaced with the assumption that the flow $\mathcal{M}$ is smooth and satisfies a forward Type-I curvature estimate, i.e. Assumption 4.2.2 B.

Using a barrier argument, we show $\mathcal{M}$ is the unique flow that satisfies this assumption on the tangent flows.

Theorem 1.3.8. *Let $\mathcal{M}^1$ and $\mathcal{M}^2$ be two mean curvature flows from M_0, smooth on some time interval $(0,T)$, and suppose every tangent flow at the origin is given by $\mathcal{M}_\Sigma$, where Σ is a stable, smooth expander possessing a linearly growing Jacobi field. Then, for as long as the flows are smooth,*

$$\mathcal{M}^1 \equiv \mathcal{M}^2.$$

As an immediate consequence, we deduce non-fattening flow from the cone implies the level-set flow from M_0 does not fatten.

Theorem 1.3.9. *Let $\mathcal{C}$ be a smooth cone. Suppose the level-set flow from $\mathcal{C}$ is non-fattening and is given by $\mathcal{M}_\Sigma$, where Σ is smooth expander asymptotic to $\mathcal{C}$. If M_0 is a compact hypersurface with a conical singularity modelled on $\mathcal{C}$, the level-set flow from M_0 does not fatten instantaneously. Moreover, there is a unique mean curvature flow from M_0, smooth until the next singular time.*

For dimensions $2 \leq n \leq 6$, we immediate deduce the following,

Corollary 1.3.10. *For $n \in [2,6]$, let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone. Suppose $M_0^n \subset \mathbb{R}^{n+1}$ satisfies Assumption 4.2.1. If the level-set flow from $\mathcal{C}$ does not fatten, then the level-set flow from M_0 does not fatten instantaneously. Moreover, there is a unique mean curvature flow from M_0, smooth until the next singular time.*

Finally, in the case when the level-set flow from $\mathcal{C}$ fattens, we may combine Theorem 1.3.6 with Theorem 1.3.8 to further show,

Corollary 1.3.11. *The unit-regular, cyclic mod 2 Brakke flows supported on the outermost flows from M_0 are smooth until the next singular time and are the unique flows from M_0 with tangent flows modelled on the outermost expanders from $\mathcal{C}$.*

Both the tangent flow claim for outermost flows, Theorem 1.3.6, and uniqueness, Theorem 1.3.8, are proven using closely related barrier arguments. Said barriers are constructed by using the first eigen-function for some compact region and the Jacobi field in conjunction with the behaviour of the linearised dynamics along the ends of the expander.

It is well known that the geometric quantities of a hypersurface written as a graph over another can be calculated using the linearised operator with a quadratic error, provided the height of the graph is sufficiently small, i.e. the graph function u, satisfies $|A||u| << 1$. These methods are typically employed when the C^1 norm of u is small enough that a Taylor series expansion can be used to calculate the error. One may think of this as the 'low-energy regime'. Indeed, in a compact region, constructing the barriers via linearised dynamics is a standard exercise. In order to construct barriers along the end, we note the curvature term of the linearisation decays quadratically, and so constant functions can be viewed as 'asymptotic eigen-functions' with 'eigen-value' $\frac{1}{2}$. Furthermore, by studying the dependence of the error on the curvature, we see the error term (for a constant function) also decays at least quadratically in the radial parameter along the end. We conclude the rescaled mean curvature vector of graphs of constants points towards the expander. This permits us to view the construction of barriers as a compact problem, comparable to the use of the (negative) Gaussian weight in the study of shrinkers.

Conjoining these two barrier regimes exploits the asymptotic linear growth of the Jacobi field. By choosing parameters carefully, we can ensure that the barrier over the compact region 'eventually' lies above a given constant, ensuring the two barriers intersect appropriately. A related idea has been explored by Chodosh–Choi–Schulze, [CCS23], in the setting of mixed end shrinkers. They baptised the conjoining process 'welding', terminology we will adopt.

Our explicit construction is as follows. Recall, the expander Σ is assumed to be stable, and thus on every compact subset the first Dirichlet eigen-value is positive, as is the first eigen-function. We pick R such that the normal graph of a constant defines a barrier over the end $E_R := \Sigma \backslash B(0, R)$. For $\mathfrak{R} \geq C_{\text{len}} R$, let $\phi_1^{\mathfrak{R}}, \mu_1^{\mathfrak{R}}$ be the first Dirichlet eigen-function and eigen-value (respectively) for the stability operator on $\Sigma_{\mathfrak{R}} = \Sigma \cap B(0, \mathfrak{R})$. Let ϕ_0 denote the (positive) Jacobi field with linear growth at infinity. For $\alpha > 0$, we define the family of functions

$$\begin{aligned} f_\alpha^{\mathfrak{R}} &: \Sigma_{\mathfrak{R}} \to \mathbb{R} \\ f_\alpha^{\mathfrak{R}} &: = \phi_0 + \alpha \phi_1^{\mathfrak{R}}. \end{aligned}$$

The parameter α is tuned such that $f_\alpha^{\mathfrak{R}}$ has linear growth in the region $\Sigma_{C_{\text{len}} R} \cap E_R$. We set $h = \max_{\mathbf{x},\ |\mathbf{x}| = C_{\text{len}} R} f_\alpha^{\mathfrak{R}}(\mathbf{x})$ and consider, for $s \in [0, 1]$, the following 'welded function':

$$u_s = \begin{cases} s f_\alpha^{\mathfrak{R}} & \mathbf{x} \in \overline{\Sigma_R} \\ s \min\{f_\alpha^{\mathfrak{R}}, h\} & \mathbf{x} \in E_R \backslash \Sigma_{C_{\text{len}} R} \\ sh & \mathbf{x} \in E_{C_{\text{len}} R}. \end{cases}$$

We demonstrate there exists an s_0 such that for $s \in [0, s_0]$, the normal graph of u_s over Σ yields a global barrier lying strictly to one side of Σ. Similarly, $-u_s$ is a barrier to the other side. In Section 4.7, we use these barriers with an interior approximation argument to demonstrate the tangent flows of the outermost flows at the cone point are modelled on the outermost expanders. This argument is motivated by the construction of unit-regular, cyclic mod 2 Brakke flows supported on the outermost flows by Hershkovits–White. Whilst this technique appears to be restricted to co-dimension 1, we believe these globally defined barriers could instead be used in conjunction with a gluing procedure. Such a gluing procedure could be used to show the existence of smooth flows from M_0 with tangent flows modelled on stable expanders in between the outermost flows from $\mathcal{C}$. This construction should generalise to high co-dimension mean curvature flow and Ricci flow.

Tackling uniqueness requires a more sophisticated barrier. Moreover, we no longer wish to 'weld' the barrier over an end of the expander, rather, we need a barrier over the complement of the expander region of radius R. An expander region is the subset of space-time where the flow looks like a graph of the expander, see Definition 4.3.11.

We desire to show equality of two smooth flows, $\mathcal{M}^1, \mathcal{M}^2$, starting from M_0, both satisfying the assumption that every forward tangent flow at the space-time origin is $\mathcal{M}_\Sigma$. We thus require barriers that govern the separation of these two solutions, rather than the closeness to the flow $\mathcal{M}_\Sigma$. In the expander region of radius $C_{\text{len}}R$, our barrier is defined using the function

$$\begin{aligned} &u^{\pm}_{s,\alpha,R,\mathfrak{R}} : \Sigma_{C_{\text{len}}R} \times (-\infty, \tau_0] \to \mathbb{R}, \\ &u^{\pm}_{s,\alpha,R,\mathfrak{R}}(\mathbf{x}, \tau) := u_1(\mathbf{x}, \tau) \pm s f^{\mathfrak{R}}_\alpha(\mathbf{x}), \end{aligned}$$

where u_1 is the function parameterising $\mathcal{R}\mathcal{M}^1$ as a normal graph over Σ_R. By viewing the error as a homogenous degree 2 polynomial, we use a binomial decomposition to deduce one may choose $s > 0$ sufficiently small to yield barriers relative to the flow $\mathcal{R}\mathcal{M}^1$, rather than Σ, on $\Sigma_{C_{\text{len}}R}$. Of course, this requires the function u_1 is not too large, which is achieved by noting u_1 converges smoothly to 0 as $\tau \to -\infty$.

Outside the expander region of radius R, our barrier takes the form of a separation estimate for mean curvature flows that initially agree on a large, smooth region, Theorem 4.5.1. Our separation estimate states if two (non-rescaled) mean curvature flows separate at rate $h\sqrt{t}$ ($h > 0$) along the boundary of a ball of radius $R\sqrt{t}$, then the rate of separation of the flow contained in the complement of the ball is at most $h\sqrt{t}$.

Uniqueness follows by demonstrating we can weld these two barrier regimes together to construct a global barrier to other flows starting from M_0 satisfying the blow-up assumption. To understand the welding step, we turn our attention to Figure 1.3.2. We note that $\text{graph}_{M^1_t} h\sqrt{t}$ corresponds to $\text{graph}_{RM^1_\tau} h$ after transformation to the rescaled flow and radius $r = R\sqrt{t}$ corresponds to radius $r = R$ after transformation to the rescaled flow. We sketch the welding process. Considering the hypersurfaces, $\Gamma^{\pm}_s(\tau)$, defined over the expander region, $r \leq C_{\text{len}}R$, we deduce a constant bound $h > 0$ on the separation between two rescaled solutions at radius $r = R$. Explicitly, the constant h will depend on s and the maximum value of $f^{\mathfrak{R}}_\alpha$ at radius R. This separation is then propagated over the rescaled flow outside of radius R via our separation estimate, Theorem 4.5.1. Considering now the flow at

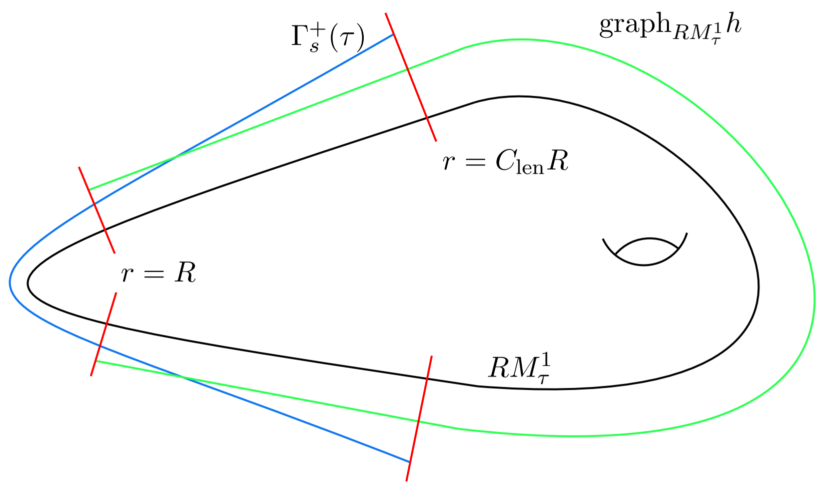

Figure 1.1: Sketch of the barriers at rescaled time τ.

radius $r = C_{\mathrm{len}}R$, we see that the separation estimate yields an improvement on the previously known separation at this greater radius, as the barriers $\Gamma_s^\pm$ have linear growth between the two radii. It is thus clear that another flow from M_0 satisfying the tangent flow assumption can never touch the 'boundary' of the hypersurfaces $\Gamma_s^\pm(\tau)$, and so our welding is well defined. In particular, the $\Gamma_s^\pm(\tau)$ are barriers on the time interval they are defined.

1.4 Outline

The thesis is structured in three parts with four chapters. Chapters 1 and 2 cover the history and background of mean curvature flow in codimension 1. Chapter 3 discusses surgery for flows with only spherical and neck-pinch singularities, and Chapter 4 discusses mean curvature flow through conical singularities.

Chapter 3 comprises of the following sections. In Section 3.1, we recap the structure of Haslhofer–Kleiner surgery. In Section 3.2, we discuss the necessary adaptations to the definitions of [HK17b] for our more general setting. In Section 3.3, we construct barriers and detail the structure and stability of weak surgery flows. In Section 3.4, we prove the existence of a smooth mean curvature flow with surgery approximating the unit-regular Brakke flow. Finally, in Section 3.5 we apply the results to the low-entropy Schoenflies conjecture.

Chapter 4 comprises of the following sections. Section 4.1 covers the some background results not previously covered. In Section 4.2, the assumptions are stated and justified. Section 4.3 establishes regularity and graphicality results for flows from compact hypersurfaces with conical singularities. In Section 4.4, the barriers are constructed using the linearised dynamics. In Section 3.1.7, we establish a separation estimate. In Section 4.6 the ideas are brought together to demonstrate uniqueness of flows satisfying our assumptions. Finally, in Section 4.7, we demonstrate the relation between the outermost flows from our compact initial condition and the cone, and conclude our fattening dichotomy.

Chapter 2

Preliminaries

We recap standard definitions, notation, and results from the field that will be used throughout this work.

Definition 2.0.1. The parabolic cylinder of radius $r > 0$ centred at the space-time point $X = (\mathbf{x}, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ is defined as

$$P(X, r) = B(\mathbf{x}, r) \times (t - r^2, t + r^2)$$

We use the terminology 'backwards (resp. forwards) parabolic cylinder' for a parabolic cylinder with a time interval of the form $(t - r^2, t]$, (resp. $[t, t + r^2)$).

Definition 2.0.2 (Mean Curvature Flow)**.** Let $M^n \subset \mathbb{R}^{n+1}$ be a smoothly embedded hypersurface. A mean curvature flow $\mathcal{M} = \{M_t \subset U\}_{t \in [0, t_0)}$ in an open subset $U \subset \mathbb{R}^{n+1}$ is a smooth family of hypersurfaces such that

$$\begin{aligned} M_0 &= M, \\ \left(\frac{\partial}{\partial t}\mathbf{x}\right)^{\perp} &= \mathbf{H}_{M_t}(\mathbf{x}), \end{aligned}$$

where $\mathbf{H}_{M_t}(\mathbf{x})$ is the mean curvature vector.

Definition 2.0.3. Given a choice of unit normal, ν, we fix an orientation, and thus can write

$$\mathbf{H} = -H\nu$$

We refer to $H = H(\mathbf{x})$ as the (scalar) mean curvature.

2.1 Gaussian Area and The Monotonicity formula

Definition 2.1.1 (Gaussian density ratio)**.** For $X_0 := (\mathbf{x}_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$, consider the backward heat kernel based at $(\mathbf{x}_0, t_0)$:

$$\rho_{X_0}(\mathbf{x}, t) = (4\pi(t_0 - t))^{-n/2} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4(t_0 - t)}\right),$$

for $\mathbf{x} \in \mathbb{R}^{n+1}, t < t_0$. For a Mean curvature flow (or Brakke flow) $\mathcal{M}$ and $r > 0$ we define

$$\Theta_{\mathcal{M}}(X_0, r) := \int_{\mathbb{R}^{n+1}} \rho_{X_0}(\mathbf{x}, t_0 - r^2)\, d\mu_{t_0 - r^2}.$$

$\Theta_{\mathcal{M}}(X_0, r)$ is known as the *Gaussian density ratio*, of $\mathcal{M}$ at X_0 at scale $r > 0$.

Definition 2.1.2 (Area ratios)**.** We say a hypersurface (or varifold) $M^n \subset \mathbb{R}^{n+1}$ has bounded area ratios if

$$\sup_{\mathbf{x} \in \mathbb{R}^{n+1}} \sup_{R > 0} \frac{\mathcal{H}^n(M \cap B_R(\mathbf{x}))}{\omega_k R^k} \leq D, \tag{2.1}$$

for some $D < \infty$, where ω_k is the volume of the unit ball in $\mathbb{R}^k$.

The Gaussian density ratios were shown by Huisken, [Hui90], to be monotonic under the flow.

Theorem 2.1.3 (Huisken's monotonicity formula [Hui90], [Ilm95b])**.** *Suppose $\mathcal{M} := \{M_t\}_{t \in [0,T)}$ is a smooth mean curvature flow (resp. Brakke flow) with bounded area ratios. Then,*

$$\frac{d}{dt} \int_{M_t} \rho_{X_0}(\mathbf{x}, t)\, d\mu_t = (\textit{resp.} \leq) - \int_{M_t} \left|\mathbf{H} - \frac{(\mathbf{x} - \mathbf{x}_0)^\perp}{2(t - t_0)}\right|^2 \rho_{X_0}(\mathbf{x}, t)\, d\mu_t\,.$$

Definition 2.1.4 (Gaussian Density)**.** The Gaussian density of $\mathcal{M}$ at X_0 is defined by

$$\Theta_{\mathcal{M}}(X_0) := \lim_{r \searrow 0} \Theta_{\mathcal{M}}(X_0, r)\,.$$

This limit is well defined by the monotonicity formula.

The monotonicity formula has been suitably localised by work of Ecker [Eck04] and White [Whi97]. Such localisations allow for us to drop the hypothesis of bounded area ratios on the entire flow, requiring it only on the initial condition. One commonly used localisation is the *spherically shrinking localisation*: one uses the cut-off function

$$\varphi^R_{X_0}(\mathbf{x}, t) := \left(1 - \frac{|\mathbf{x} - \mathbf{x}_0|^2 + 2n(t - t_0)}{R^2}\right)^3_+,$$

where + denotes the positive portion of the function, and 0 elsewhere.

We hence define

$$\Theta^R(\mathcal{M}, X_0, r) := \int_{\mathbb{R}^{n+1}} \rho_{X_0}(\mathbf{x}, t_0 - r^2)\, \varphi^R_{X_0}(\mathbf{x}, t)\, d\mu_{t_0 - r^2}.$$

Definition 2.1.5 (Parabolic Dilation)**.** Let $\mathcal{M} = \{M_t\}_{t\in[0,T)}$ be a mean curvature flow, or $\mathcal{M} = \{\mu_t\}_{t\in[0,T)}$ be a Brakke flow. For any $\lambda > 0$, we denote the parabolic rescaling of space-time by λ as $\mathcal{D}_\lambda : (\mathbf{x}, t) \mapsto (\lambda\mathbf{x}, \lambda^2 t)$. We denote by $\mathcal{D}_\lambda(\mathcal{M} - X_0)$ the mean curvature flow, resp. Brakke flow, obtained from $\mathcal{M}$ by parabolic dilation around X_0 by λ. That is,

$$\begin{aligned} \mathcal{D}_\lambda(\mathcal{M} - X_0) =& \{M^\lambda_t\}_{t'\in[-\lambda^2 t_0, \lambda^2(T-t_0))}, \\ \text{resp. } =& \{\mu^\lambda_t\}_{t'\in[-\lambda^2 t_0, \lambda^2(T-t_0))}. \end{aligned}$$

Where $M^\lambda_t = \lambda M_{t_0+\lambda^{-2}t}$ and $\mu^\lambda_t(A) = \lambda^n \mu_{t_0+\lambda^{-2}t}(\lambda^{-1}A + \mathbf{x}_0)$, for $A \subset \mathbb{R}^{n+1}$.

In the following theorem, P denotes the backwards parabolic cylinder $P(X_0, r) := B(\mathbf{x}_0, r) \times (t_0 - r^2, t_0]$.

Theorem 2.1.6 (White Regularity [Whi05])**.** *There exists constants $\varepsilon > 0$, $C < \infty$ depending only on the dimension n such that, if $\mathcal{M}$ is a smooth mean curvature flow in $P(X_0, 4nR)$ with*

$$\sup_{X\in P(X_0,r)} \Theta^R(\mathcal{M}, X, r) < 1 + \varepsilon$$

for some $r \in (0, R)$, then

$$\sup_{X\in P(X_0,r/2)} |A| \le C r^{-1}$$

Definition 2.1.7 (Entropy [CM15])**.** The Entropy of a hypersurface Σ is

$$\lambda(\Sigma) = \sup_{\mathbf{x}_0, t_0} \left(\frac{1}{4\pi t_0}\right)^{\frac{n}{2}} \int_\Sigma \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4t_0}\right) d\mu,$$

i.e. the supremum of the Gaussian densities over all scales and base-points. It can be considered a measure of the complexity of an embedding.

2.2 Weak formulations of Mean Curvature Flow

The flow is quasi-linear and develops singularities. A rich theory has been developed to continue the flow past such singularities. We recap the fundamentals of Brakke and Level-set flow.

2.2.1 Brakke flow

Definition 2.2.1 (Integral Brakke Flow [Bra78, Ilm94])**.** We follow the formalism of [Whi21]. An (n-dimensional) *integral Brakke flow* in $\mathbb{R}^{n+1}$ is a 1-parameter family of Radon measures $\{\mu_t\}_{t\in I}$ over an interval $I \subset \mathbb{R}$ such that:

(i) For almost every t there exists and integral n-dimensional varifold $V(t)$ with $\mu_t = \mu_{V(t)}$ so that $V(t)$ has locally bounded first variation and has mean curvature $\mathbf{H}$ orthogonal to $\mathrm{Tan}(V(t), \cdot)$ almost everywhere.

(ii) For a bounded interval $[t_1, t_2] \subset I$ and any compact set K

$$\int_{t_1}^{t_2} \int_K (1 + |\mathbf{H}|^2)\, d\mu_t\, dt < \infty\,.$$

(iii) If $[t_1, t_2] \subset I$ and $f \in C^1_c(\mathbb{R}^{n+1} \times [t_1, t_2])$ has $f \geq 0$ then

$$\begin{aligned} &\int f(\cdot, t_2)\, d\mu_{t_2} - \int f(\cdot, t_1)\, d\mu_{t_1} \\ &\qquad \leq \int_{t_1}^{t_2} \int_K \Big(-|\mathbf{H}|^2 f + \mathbf{H} \cdot \nabla f + \frac{\partial}{\partial t} f \Big)\, d\mu_t\, dt \end{aligned} \tag{2.2}$$

We write $\mathcal{M}$ for a Brakke flow $\{\mu_t\}_{t\in I}$ to refer to the family of measures $I \ni t \mapsto \mu_t$ satisfying Brakke's inequality (2.2).

Remark 2.2.2. The left hand side of Brakke's inequality, 2.2, depends only on the masses of the (generalised) hypersurfaces at times t_1 and t_2. This permits gratuitous vanishing, e.g. a smooth hypersurface at time $t = 0$, followed by the 'empty flow' would satisfy the above definition. To get a more 'canonical' notion of a weak flow, we work with *unit-regular* and *cyclic mod 2* flows, introduced by White in [Whi09]. These properties are defined below.

Definition 2.2.3 (Unit-regular and cyclic Brakke Flows [Whi09])**.** An integral Brakke flow $\mathcal{M} = \{\mu_t\}_{t\in I}$ is said to be

- *unit-regular* if $\mathcal{M}$ is smooth in some space-time neighbourhood of any space-time point X with $\Theta_{\mathcal{M}}(X) = 1$;
- *cyclic (mod 2)* if, for a.e. $t \in I, \mu_t = \mu_{V(t)}$ for an integral varifold $V(t)$ whose unique associated rectifiable mod-2 flat chain $[V(t)]$ has $\partial[V(t)] = 0$.

Theorem 2.2.4 (Compactness for Integral Brakke Motions [Ilm94, Bra78])**.** *Let M be complete. Let $\{\mu^i_t\}_{t\geq 0}$, $i \in \mathbb{N}$, be a sequence of integral Brakke motions in M. Suppose*

$$\sup_{i,t} \mu^i_t(U) \leq C_1(U) < \infty$$

for each $U \subset\subset M$. Then,

(i) There is a subsequence $\{\mu_t^{i_j}\}_{t\geq 0}$, $j \in \mathbb{N}$ and an integral Brakke motion $\{\mu_t^\infty\}_{t\geq 0}$ such that

$$\mu_t^{i_j} \to \mu_t^\infty \text{ as Radon measures for each } t \geq 0.$$

(ii) For almost every $t \geq 0$, there is a subsequence $\{i'_j\}_{j\geq 1}$ of $\{i_j\}_{j\geq 1}$ (depending on t) such that

$$\lim_{j\to\infty} V(\mu_t^{i'_j}) = V(\mu_t^\infty) \text{ as varifolds.}$$

Finally, we state the following theorem from [CCMS20]. The ideas will be used in Section 3.3 to show convergence properties of the ε-barriers and flows with surgery.

Definition 2.2.5. For a Brakke flow $\mathcal{M}$, we define $\widehat{\operatorname{reg}}\,\mathcal{M}$ to be the set of points $X = (\mathbf{x}, t)$ such that there is an $\varepsilon > 0$ with

$$\mathcal{M}\lfloor(B_\varepsilon(\mathbf{x}) \times (t - \varepsilon^2, t] = k\mathcal{H}^n\lfloor M(t),$$

where k is a positive integer and $M(t)$ is a smooth mean curvature flow. We write $\operatorname{reg}\mathcal{M}$ as the above set with $k = 1$; thus, $\operatorname{reg}\mathcal{M} \subset \widehat{\operatorname{reg}}\,\mathcal{M}$.

Theorem 2.2.6 ([CCMS20, Corollary F.4])**.** *Suppose that $\mathcal{M}$ is a unit-regular integral n-dimensional Brakke flow in $\mathbb{R}^{n+k}$ with $\mu(t) = \mathcal{H}^n\lfloor M(t)$ for $t \in [0, \delta)$, where $M(t)$ is a mean curvature flow of connected, properly embedded submanifolds of $\mathbb{R}^{n+k}$ and $\delta > 0$. If*

$$\mathcal{H}^n_P(\operatorname{supp}(\mathcal{M})\backslash\widehat{\operatorname{reg}}\mathcal{M}) = 0$$

Then $\widehat{\operatorname{reg}}\,\mathcal{M} = \operatorname{reg}\mathcal{M}$ is connected.

Here $\mathcal{H}^n_P$ denotes n-dimensional parabolic Hausdorff measure. This theorem provides vital information on the behaviour of unit-regular Brakke flows with small singular set.

2.2.2 Level set flow

We recall the definition of level-set flow outlined by Ilmanen. The interested reader is directed to Evans–Spruck, [ES91], and Chen–Giga–Goto, [CGG91], for the origin definition in terms of viscosity solutions.

Definition 2.2.7 (Weak and Level set flow, [Ilm94])**.** Let $K \subset \mathbb{R}^{n+1}$ be closed. A one-parameter family of closed sets, $\{K_t\}_{t\geq 0}$, with initial condition $K_0 = K$ is said to be a *weak set flow* for K if for every smooth mean curvature flow M_t of compact hypersurfaces defined on $[t_0, t_1]$, we have

$$K_{t_0} \cap M_{t_0} = \emptyset \implies K_t \cap M_t = \emptyset$$

for all $t \in [t_0, t_1]$.

The *level set flow* is defined as the maximal weak set flow, i.e. the union of all weak set flows from K. We also write $F_t(K) = K_t$ to denote the t time-slice of the level set flow of the set K.

2.2.3 Inner and Outer Flows

We recall the definitions of Hershkovits–White, [HW20], relevant to the fattening phenomenon.

Definition 2.2.8. Let M be a compact, smoothly embedded hypersurface. The *fattening time* of the level set flow of M is defined as

$$T_{\text{fat}} := \inf\{t > 0 : F_t(M) \text{ has non-empty interior}\}.$$

Definition 2.2.9. Let $M_0^n \subset \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, and let U be the compact region bounded by M_0. Let $U' = \overline{U^c}$. Using the set theoretic formulation of the level set flow, we define space-time tracks of the evolution of U, U' under level-set flow by

$$\begin{aligned} \mathcal{U} &:= \{(\mathbf{x}, t) \subset \mathbb{R}^{n+1,1} |\ \mathbf{x} \in F_t(U)\}, \\ \mathcal{U}' &:= \{(\mathbf{x}, t) \subset \mathbb{R}^{n+1,1} |\ \mathbf{x} \in F_t(U')\}. \end{aligned}$$

We hence define

$$\begin{aligned} M(t) &:= \{\mathbf{x} \in \mathbb{R}^{n+1} |\ (\mathbf{x}, t) \in \partial \mathcal{U}\}, \\ M'(t) &:= \{\mathbf{x} \in \mathbb{R}^{n+1} |\ (\mathbf{x}, t) \in \partial \mathcal{U}'\}. \end{aligned}$$

We call $t \mapsto M(t)$ the *outer flow* from M_0 and $t \mapsto M'(t)$ the *inner flow* from M_0. One can see $M(t), M'(t) \subset F_t(M)$.

Definition 2.2.10. The *discrepancy time* is defined as

$$T_{\text{disc}} = \inf\{t > 0 : M(t), M'(t), \text{ and } F_t(M) \text{ are not all equal}\}.$$

Theorem 2.2.11 ([HW20])**.** *There exists a unit-regular Brakke flow $\mathcal{M}$ on the time interval $[0, \infty)$ starting from M_0, such that the space-time support of $\mathcal{M}$ is given by the outer flow, $t \mapsto M(t)$. The same holds true for the inner flow.*

2.3 Tangent flows and Self-Similar solutions

Definition 2.3.1 (Tangent flow)**.** Let $\{\lambda_i\}$ be a sequence s.t. $\lambda_i \to \infty$. We define a tangent flow at the space-time point $X_0 \in \mathcal{M}$ as a subsequential limiting Brakke flow of the sequence parabolic rescalings of $\mathcal{M}$ around X_0 by λ_i.

The *backwards* tangent flow is defined by restricting the tangent flow to the time interval $(-\infty, 0)$. The *forwards* tangent flow is defined by restricting to the time interval $(0, \infty)$.

The monotonicity formula implies that all (backward) tangent flows are self-similar, i.e. their time -1 slice is given by a (weak) self-shrinker.

Definition 2.3.2 (Self-shrinker)**.** A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is called a *self-shrinking soliton* if any of the following, equivalent, definitions hold.

- Σ satisfies $\mathbf{H}_\Sigma(\mathbf{x}) + \frac{\mathbf{x}^\perp}{2} = 0$.
- The family $\left\{\sqrt{t}\Sigma\right\}_{t\in(-\infty,0)}$ is a mean curvature flow.
- Σ is a critical point of the (negative) Gaussian area functional,

$$F(\Sigma) := \frac{1}{4\pi}\int_\Sigma \exp\left(\frac{-|x|^2}{4}\right) d\mu.$$

Definition 2.3.3 (Translator)**.** A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is called a *translating soliton* if there exists $V \in \mathbb{R}^{n+1}$ such that any of the following, equivalent, definitions hold.

- Σ satisfies $\mathbf{H}_\Sigma(\mathbf{x}) = \langle V, \nu\rangle\, \nu$.
- The family $\{\Sigma + tV\}_{t\in(-\infty,\infty)}$ is a mean curvature flow.
- Σ is a critical point of the area functional

$$F(\Sigma) := |V| \int_\Sigma \exp\left(\mathbf{x}\cdot V\right) d\mu.$$

Definition 2.3.4 (Self-expander)**.** A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is called a *self-expanding soliton* if any of the following, equivalent, definitions hold.

- Σ satisfies $\mathbf{H}_\Sigma(\mathbf{x}) - \frac{\mathbf{x}^\perp}{2} = 0$.
- The family $\left\{\sqrt{t}\Sigma\right\}_{t\in(0,\infty)}$ is a mean curvature flow.
- Σ is a critical point of the (positive) Gaussian area functional

$$F(\Sigma) := \frac{1}{4\pi}\int_\Sigma \exp\left(\frac{|x|^2}{4}\right) d\mu.$$

The quantity $\mathbf{H}_{M(t)}(\mathbf{x}) + \frac{\mathbf{x}^\perp}{2}$, is known as the *backward rescaled mean curvature vector*, or *shrinker mean curvature vector* of Σ. Similarly, $\mathbf{H}_{M(t)}(\mathbf{x}) - \frac{\mathbf{x}^\perp}{2}$ is known as the *forward rescaled mean curvature vector*, or *expander mean curvature vector*, of Σ. These quantities are the mean curvatures of a hypersurface in $\mathbb{R}^{n+1}$ considered with the Gaussian metric of the related sign.

A closely related concept to tangent flows is that of the *rescaled mean curvature flow*. We recall the definition of the forward rescaled flow.

Definition 2.3.5 (Forward Rescaled Mean Curvature Flow)**.** A smooth family of hypersurfaces $\mathcal{RM} = \{M(t)\}_{t\in I}, I \subset \mathbb{R}$ is said to evolve by the forward rescaled mean curvature flow if

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^\perp = \mathbf{H}_{M(t)}(\mathbf{x}) - \frac{\mathbf{x}^\perp}{2}.$$

Remark 2.3.6. One may obtain the equation for the backward rescaled flow by replacing the expander mean curvature with the shrinker mean curvature. That is, changing the sign in front of $\frac{\mathbf{x}^\perp}{2}$.

Note, shrinkers and expanders define static eternal solutions to the respective rescaled flow, i.e. solutions defined on the time interval $(-\infty, \infty)$.

Ancient solutions of the forward rescaled flow are of interest, as they can be constructed from smooth mean curvature flows starting from a singular initial condition via the following transformation.

$$\tau = \log(t),\ \tilde{\mathbf{x}}(\tau) = \frac{\mathbf{x}(\exp(\tau))}{\sqrt{\exp(\tau)}}.$$

Stated precisely,

Lemma 2.3.7 (Transformation to Rescaled Mean Curvature Flow)**.** *Suppose*

$$\mathcal{M} = \{M(t)\}_{t\in(0,t_0)}$$

is a mean curvature flow. Then,

$$\mathcal{RM} := \left\{e^{-\frac{\tau}{2}} M(e^\tau)\right\}_{\tau\in(-\infty,\log(t_0))}$$

defines a forward rescaled mean curvature flow.

Definition 2.3.8. If $\mathcal{M}$ is a mean curvature flow, we write $\mathcal{RM}$ to denote the related rescaled flow.

2.4 Pseudolocality and Interior Estimates

Finally, we record the well known pseudolocality result for mean curvature flow, and the graphical estimates of Ecker–Huisken. Throughout the following chapters, we combine these results to demonstrate interior regularity.

Definition 2.4.1. Let $\mathbf{x} \in \mathbb{R}^{n+1}$ and let $\Pi_{\mathrm{x}} \subset \mathbb{R}^{n+1}$ be an n-plane passing through $\mathbf{x}$. Denote by ν the normal to Π_x. We define the n-ball

$$B^n(\mathbf{x}, r) := B(\mathbf{x}, r) \cap \Pi_{\mathbf{x}}$$

and n-cylinder

$$C(\mathbf{x}, r) := \{\mathbf{x} \in \mathbb{R}^{n+1} \text{ s.t. } \mathbf{x} = \mathbf{y} + \alpha\nu,\ \mathbf{y} \in B^n(\mathbf{x}, r),\ |\alpha| < r\}.$$

We state the pseudolocality result of Ilmanen–Neves–Schulze, in the co-dimension 1 case for smooth mean curvature flows. See also the pseudolocality result stated in [CY07]. In the formulation of their result presented below, no bounds are assumed on the area ratios of the flow. As noted in their remark, [INS19, Remark 1.6], this is a made possible by the local version of the monotonicity formula.

Theorem 2.4.2 (Pseudolocality, [INS19])**.** *Let $\{M_t\}_{t\in[0,T)}$ be a smooth mean curvature flow of embedded hypersurfaces in $\mathbb{R}^{n+1}$. Then, for any $\eta > 0$ there exists $\varepsilon, \vartheta > 0$ depending only on n, η such that if $\mathbf{x}_0 \in M_0$ and $M_0 \cap C(\mathbf{x}_0, 1)$ can be written as* graph(u)*, where $u : B^n(\mathbf{x}_0, 1) \to \mathbb{R}$ with Lipschitz constant less than ε, then*

$$M_t \cap C(\mathbf{x}_0, \vartheta), t \in [0, \vartheta^2) \cap [0, T)$$

is a graph over $B^n(\mathbf{x}_0, \vartheta)$ with Lipschitz constant less than η and height bounded by $\eta\vartheta$.

Remark 2.4.3. As commented in [INS19, Remarks 1.6 iii)], the above result holds for Integral Brakke flows if we presume there is no sudden mass loss in $C(\mathbf{x}_0, 1)$. To satisfy this hypothesis, we will always work with Brakke flow that are unit-regular. The proof is identical, except one must use Brakke's local regularity theorem, [Bra78], in place of White's local regularity, [Whi05].

Theorem 2.4.4 (Interior estimates for Graphs [EH91])**.** *Let $M^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface. Let $R > 0$ be such that M_t can be written as a graph of some function u over $B^n(\mathbf{y}_0, R)$, an n-ball of radius R centred at a point $\mathbf{y}_0 \in \mathbb{R}^{n+1}$ in some hyperplane, for $t \in [0, T]$. Suppose further that the gradient is bounded, i.e. for each $t \in [0, T]$ we have*

$$\sqrt{1 + |Du_t|^2} \leq 1 + \eta,$$

where $\eta > 0$ depends only on the dimension. Then, for any $t \in [0, T]$ and $\theta \in (0, 1)$, we have

$$\sup_{B^n(y_0,\theta R)\times[0,T]} |A|^2 \leq C(n, \theta, R) \sup_{B^n(y_0,R)\times\{0\}} |A|^2$$

This is immediate from the Theorem 3.1 of [EH91] under the assumption of bounded initial curvature. The interested reader is directed to [BM17, Chapter 8], where the estimates are established for high co-dimension.

Chapter 3

Mean Curvature flow with Surgery

3.1 Overview of 2-Convex Surgery

We begin by recapping the results of [HK17b] that we will use.

Definition 3.1.1 (α-noncollapsed, [And12], [HK17a])**.** Let $\alpha > 0$. A mean convex hypersurface M^n bounding an open region Ω in $\mathbb{R}^{n+1}$ is α–noncollapsed (on the scale of the mean curvature) if for every $x \in M$ there are closed balls $B_{\text{int}} \subset \overline{\Omega}$ and $B_{\text{ext}} \subset \mathbb{R}^{n+1} \backslash \Omega$ of radius at least $\alpha / H(x)$ tangential to M at x, from the interior and exterior of M respectively. A smooth mean curvature flow is said to be α-noncollapsed if every time slice is α-noncollapsed.

This definition may be suitably localised. See Definition 3.2.3.

Definition 3.1.2 (β-uniformly 2-convex)**.** A mean convex hypersurface M is said to be β-uniformly 2-convex, for $\beta > 0$, if

$$\lambda_1 + \lambda_2 > \beta H.$$

Where λ_i are the ordered principal curvatures with $\lambda_1 \leq \ldots \leq \lambda_n$, and H is the mean curvature.

Recall, 'α-noncollapsed'-ness is preserved under the mean curvature flow by the maximum principle, [And12]. β-uniform 2-convexity is preserved by the Hamilton tensor maximum principle.

Definition 3.1.3 (Strong δ-neck, [HK17b, Definition 2.3])**.** Let $\delta > 0$. We say a mean curvature flow $\mathcal{M} = \{M_t \subset U\}_{t \in I}$ has a strong δ-neck with centre p and radius s at time $t_0 \in I$ if $\mathcal{M}_{(p,t_0),s^{-1}} = \mathcal{D}_{s^{-1}}(\mathcal{M} - (p, t_0))$ is δ-close in $C^{\lfloor 1/\delta \rfloor}$ in $(B^U_{1/\delta} \times (-1, 0])$ to the evolution of a solid round cylinder of radius 1 at $t = 0$. Here $B^U_{1/\delta} = s^{-1}((B(p, s/\delta) \cap U) - p) \subseteq B(0, 1/\delta) \subset \mathbb{R}^{n+1}$ and $\mathcal{D}_\lambda$ denotes the parabolic dilation by λ.

Definition 3.1.4 (Standard cap, [HK17b, Definition 2.2])**.** A standard cap is a smooth convex domain $K^{st} \subset \mathbb{R}^{n+1}$ that coincides with a solid round half-cylinder of radius 1 outside a ball of radius 10.

The evolution from such a cap is unique, β-uniformly 2-convex and α-noncollapsed for some $\alpha, \beta > 0$, [HK17b, Proposition 3.8]. This is a key component of the canonical neighbourhood theorem for mean curvature flows with surgery.

A surgery algorithm seeks to replace δ-necks with standard caps, the following is the gluing algorithm used.

Definition 3.1.5 (δ-neck replacement, [HK17b, Definition 2.4])**.** We say that the final time slice of a strong δ-neck with centre p and radius s is replaced by a pair of standard caps if the pre-surgery domain $K^- \subset U$ is replaced by a post-surgery domain $K^\# \subset K^-$ such that the following statements hold.

1. The modification takes place inside a ball $B = B(p, 5\Gamma s)$

2. There are bounds for the second fundamental form and its derivatives

$$\sup_{\partial K^\# \cap B} |\nabla^\ell A| \leq C_\ell s^{-1-\ell}$$

3. If B from point (1) satisfies $B \subset U$ then for every point $p_\# \in \partial K^\# \cap B$ with $\lambda_1(p_\#) < 0$ there is a point $p_- \in \partial K^- \cap B$ with $\frac{\lambda_1}{H}(p_-) \leq \frac{\lambda_1}{H}(p_\#)$

4. If $B(p, 10\Gamma s) \subset U$ then $s^{-1}(K^\# - p))$ is δ-close in $B(0, 10\Gamma)$ to a pair of disjoint standard caps which are at distance Γ from the origin.

Here, $\Gamma > 0$ denotes a cap separation parameter that is fixed later.

Haslhofer–Kleiner begin by defining a broader class of flows, of which mean curvature flow with surgery belongs. It is a class of piece-wise smooth, mean convex, α-noncollapsed, mean curvature flows with δ-necks replaced by caps. They fix a $\mu \in [1, \infty)$, used below.

Definition 3.1.6 ((α, δ)-flow, [HK17b, Definition 1.3])**.** An (α, δ)-flow $\mathcal{K}$ is a collection of finitely many smooth α-noncollapsed flows $\{K_t^i \subset U\}_{t \in [t_{i-1}, t_i]}$, $(i = 1, \ldots, k;\ t_0 < \cdots, t_k)$ in an open set $U \subset \mathbb{R}^{n+1}$ such that the following statements hold.

1. For each $i = 1, \ldots, k-1$, the final time slices of some collection of disjoint strong δ-necks are replaced by pairs of standard caps as described in definition 3.1.5, giving a domain $K_{t_i}^\# \subseteq K_{t_i}^i =: K_{t_i}^-$

2. The initial time slice of the next flow $K_{t_i}^{i+1} =: K_{t_i}^+$, is obtained from $K_{t_i}^\#$ by discarding some connected components.

3. There exists $s_\# = s_\#(\mathcal{K}) > 0$, which depends on $\mathcal{K}$, such that all necks in item (1) have radius $s \in [\mu^{-1/2}s_\#, \mu^{1/2}s_\#]$.

Proposition 3.1.7 (One-sided minimization, [HK17b, Proposition 2.9]). *There exists a $\overline{\delta} > 0$ and $\Gamma_0 < \infty$ with the following property. If $\mathcal{K}$ is an (α, δ)-flow ($\delta < \overline{\delta}$) in an open set U, with cap separation parameter $\Gamma \geq \Gamma_0$ and surgeries at scales between $\mu^{-1}s$ and s, and if $\overline{B} \subset U$ is a closed ball with $d(\overline{B}, \mathbb{R}^{n+1} \backslash U) \geq 20\Gamma s$, then*

$$|\partial K_{t_1} \cap \overline{B}| \leq |\partial K' \cap \overline{B}|$$

for every smooth comparison domain K' that agrees with K_1 outside $\overline{B}$ and satisfies $K_{t_1} \subset K' \subset K_{t_0}$ for some $t_0 < t_1$.

Theorem 3.1.8 (Global Curvature Estimate, [HK17b, Theorem 1.10]). *For all $\Lambda < \infty$, there exists $\overline{\delta}(\alpha) > 0, \xi = \xi(\alpha, \Lambda) < \infty$ and $C_0 = C_0(\alpha, \Lambda) < \infty$ with the following property. If $\mathcal{K}$ is an (α, δ)-flow ($\delta < \overline{\delta}$) in a parabolic ball $P(p, t, \xi r)$ centred at $p \in \partial \mathcal{K}_t$ with $H(p, t) \leq r^{-1}$, then*

$$\sup_{P(p,t,\Lambda r) \cap \partial \mathcal{K}'} |A| \leq C_0 r^{-1}$$

where $\mathcal{K}'$ denotes the connected component of the flow containing p.

Remark 3.1.9. Of course, this extends to higher derivatives, $|\nabla^l A|$, as is standard for parabolic equations.

Definition 3.1.10 ($\boldsymbol{\alpha}$-controlled initial condition, [HK17b, Definition 1.15]). Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma) \in (0, n-1) \times (0, \frac{1}{n-1}) \times (0, \infty)$. A hypersurface $M^n \subset \mathbb{R}^{n+1}$ is said to be $\boldsymbol{\alpha}$-controlled if it is α-noncollapsed, β-uniformly 2-convex: $\lambda_1 + \lambda_2 \geq \beta H$ and $\max_{x \in M}\{H(x)\} \leq \gamma$.

Definition 3.1.11. The surgery parameter $\mathbb{H}$ is defined as the triple

$$\mathbb{H} = \{H_{\text{th}}, H_{\text{neck}}, H_{\text{trig}}\} \in \mathbb{R}^3,$$
$$0 < H_{\text{th}} < H_{\text{neck}} < H_{\text{trig}} < \infty.$$

H_{trig} is the trigger curvature, once achieved the flow is stopped. H_{neck} is the mean curvature of neck points. H_{th} is the curvature that is used to determine high curvature regions of the flow. For $\Theta < \infty$ we say $\mathbb{H} > \Theta$ if the ratios satisfy

$$\frac{H_{\text{neck}}}{H_{\text{th}}}, \frac{H_{\text{trig}}}{H_{\text{neck}}} > \Theta$$

We say the ratios degenerate along a sequence if these ratios tend to infinity.

The definition of a mean curvature flow with surgery is made formal in the following definition.

Definition 3.1.12 (($\mathfrak{a}, \delta, \mathbb{H}$)-flow, [HK17b, Definition 1.17])**.** Let $M^n \subset \mathbb{R}^{n+1}$ be an $\mathfrak{a} = (\alpha, \beta, \gamma)$ controlled initial condition. An $(\mathfrak{a}, \delta, \mathbb{H})$-flow is an (α, δ) flow such that:

1. $H \leq H_{\text{trig}}$ everywhere. Surgery and/or discarding occurs precisely at times t when $H = H_{\text{trig}}$ somewhere.

2. The collection of necks in Definition 3.1.6 (1) is a minimal collection of necks with curvature $H = H_{\text{neck}}$ which separate the set $\{H = H_{\text{trig}}\}$ from $\{H \leq H_{\text{th}}\}$ in the domain K_t^-.

3. K^+ is obtained from $K_t^\#$ by discarding precisely those connected components with $H > H_{\text{th}}$ everywhere. In particular, of each pair of facing surgery caps, precisely one is discarded.

4. If a strong δ-neck from item (2) is also a strong $\hat{\delta}$-neck for $\hat{\delta} < \delta$ then definition 3.1.6 (4) also holds with $\hat{\delta}$ instead of δ.

The above theory is then used to prove existence of the flow, provided one is replacing strong enough necks (controlled by $\overline{\delta}$) that are sufficiently long (controlled by Θ and the curvature estimates).

Theorem 3.1.13 (Existence of surgery flow, [HK17b, Theorem 1.21])**.** *There are constants $\overline{\delta} = \overline{\delta}(\mathfrak{a}) > 0$ and $\Theta(\delta) = \Theta(\mathfrak{a}, \delta) < \infty$ ($\delta \leq \bar{\delta}$) with the following significance. If $\delta \leq \overline{\delta}$ and $\mathbb{H} = (H_{trig}, H_{neck}, H_{th})$ are positive numbers with $H_{trig}/H_{neck}, H_{neck}/H_{th} \geq \Theta(\delta)$, then there exists an $(\mathfrak{a}, \delta, \mathbb{H})$-flow $\{K_t\}_{t\in[0,\infty)}$ for every $\mathfrak{a}$-controlled initial condition K_0.*

Additionally, a canonical neighbourhood theorem is proved.

Theorem 3.1.14 (Canonical Neighbourhoods, [HK17b, Theorem 1.22])**.** *For all $\varepsilon > 0$, there exist $\overline{\delta} = \overline{\delta}(\mathfrak{a}) > 0$, $H_{can}(\varepsilon) = H_{can}(\mathfrak{a}, \varepsilon) < \infty$ and $\Theta_\varepsilon(\delta) = \Theta_\varepsilon(\mathfrak{a}, \delta) < \infty$ ($\delta \leq \bar{\delta}$) with the following significance. If $\delta \leq \overline{\delta}$ and $\mathcal{K}$ is an $(\mathfrak{a}, \delta, \mathbb{H})$-flow with $H_{trig}/H_{neck}, H_{neck}/H_{th} \geq \Theta_\varepsilon(\delta)$, then any $(p, t) \in \partial\mathcal{K}$ with $H(p, t) \geq H_{can}(\varepsilon)$ is ε-close to either (a) a β-uniformly 2-convex ancient α-noncollapsed flow, or (b) the evolution of a standard cap preceded by the evolution of a round cylinder.*

A consequence of the canonical neighbourhood theorem is the classification of discarded components. This result allows one to use surgery to decompose the topology of the original hypersurface.

Theorem 3.1.15 (Discarded components, [HK17b, Corollary 1.25])**.** *For $\varepsilon > 0$ small enough, for any $(\mathfrak{a}, \delta, \mathbb{H})$-flow with $H_{\text{neck}}/H_{\text{th}}, H_{\text{trig}}/H_{\text{neck}} > \Theta_\varepsilon(\delta)$, and $H_{\text{th}} > H_{\text{can}}(\varepsilon)$, all discarded components are diffeomorphic to $\overline{D}^{n+1}$ or $\overline{D}^n \times \mathbb{S}^1$.*

3.2 Definitions for Local Surgery

Let $\mathcal{M}$ be an n-dimensional unit-regular, cyclic (mod 2) integral Brakke flow that encounters only multiplicity one spherical or neck-pinch singularities, evolving from the smoothly embedded, closed hypersurface $M^n \subset \mathbb{R}^{n+1}$. We will always presume these singularities are multiplicity one. We fix a neck separation parameter Γ_0 that satisfies the conclusions of Proposition 3.1.7, and a $\bar{\delta} > 0$ that satisfies the conclusions of Theorem 3.1.13 and Theorem 3.1.14.

All of the above definitions for surgery make use of the 'fattened' flow, where at each time K_t is defined to be the set such that the boundary $\partial K_t = M_t$ is the motion by mean curvature from the initial hypersurface M. Since the flow is mean convex, the direction of flow is always into such a K.

With no assumption on the initial mean curvature, $\mathcal{M}$ can have 'outward' necks, where the mean curvature vector (direction of flow) is pointing exterior to the compact set the hypersurface bounds. Observe, however, that the mean convex neighbourhood conjecture gives a neighbourhood of the singularity in which the mean curvature vector always points in the same direction. Recall, we are considering Brakke flows that are cyclic (mod 2), so the ambient $\mathbb{R}^{n+1}$ is separated (at almost every time) into two components by the support of the Brakke flow. Let Ω be a set such that $\mathcal{M} \cap \Omega$ is 2-convex. Observe, this gives a 'local orientation' in the following sense. We say the set K_t, with $\partial K_t \backslash \partial \Omega = M_t \cap \Omega$ is the local interior if $\mathbf{H}$ points into K_t.

We use the same definition for the local interior of a surgery flow. Such a definition will be shown to be well defined in the definition of our flow with surgery.

Definition 3.2.1 (Neck replacement)**.** We localize definition 3.1.5 by using the above 'local interior' K_t as opposed to the interior of the entire flow.

Remark 3.2.2. In this local sense, we still have the chain of inclusions

$$K_{t_i}^{+} \subseteq K_{t_i}^{\#} \subseteq K_{t_i}^{-}$$

This is important for lemma 3.3.6 in order to replicate the argument of [Lau13].

Note, we will not have this sequence of inclusions for the interior of the surgery flow. Such a statement would not be true for outward necks: the caps are glued inside the solid neck, which equates to being exterior of the pre-surgery hypersurface.

Definition 3.2.3 (Locally α-noncollapsed)**.** Let $M^n \subset \mathbb{R}^{n+1}$ be a smooth, closed hypersurface bounding the region Ω. Suppose M is mean convex in the open balls $B(\mathbf{y}, 2r)$. We say M is *locally α-noncollapsed in* $B(\mathbf{y}, r)$ if

(a) $H(\mathbf{x}) > 1/r$ for $x \in M \cap B(\mathbf{y}, r)$, and

(b) There is an $\alpha > 0$ such that the balls $B_{\text{int}} \subset \overline{\Omega}$ and $B_{\text{ext}} \subset \mathbb{R}^{n+1} \backslash \Omega$ of radius

$\alpha/H(\mathbf{x})$ situated either side of the hypersurface, with $x \in \partial B_{\text{int}}, \partial B_{\text{ext}}$, are contained in $B(\mathbf{y}, 2r)$ and each ball has no intersection with $M \cap B(\mathbf{y}, 2r)$.

Examining the structure of the singular set of the flow $\mathcal{M}$, we can start to build the definitions for a more general surgery.

Definition 3.2.4. We denote the singular set of $\mathcal{M}$ as $\mathfrak{S}$.

We recall the canonical neighbourhood theorem of [CHH22, CHHW22].

Theorem 3.2.5 (Canonical Neighbourhoods [CHHW22, Corollary 1.18])**.** *Assume $X \in \mathfrak{S}$ is a neck singularity of the flow. Then for every $\delta > 0$ there exists a $R(X, \delta) > 0$ with the following significance. For any regular point $X' \in P(X, R)$ the flow $\mathcal{M}' = \mathcal{D}_\lambda(\mathcal{M} - X')$, obtained by parabolically rescaling the original flow around X' by $\lambda = |\mathbf{H}(X')|$, is δ-close in $C^{\lfloor 1/\delta \rfloor}$ in $B_{1/\delta}(0) \times (-1/\delta^2, 0]$ to a round shrinking sphere, round shrinking cylinder, a translating bowl soliton or ancient oval.*

Motivated by this theorem, we define the following open neighbourhood of the singular set of the flow $\mathcal{M}$.

Definition 3.2.6 ((α, β)-neighbourhood)**.** We fix

(i) $\alpha > 0, \text{with } \alpha < \min\{\alpha_{\text{sphere}}, \alpha_{\text{cylinder}}, \alpha_{\text{bowl}}, \alpha_{\text{oval}}\}$,

(ii) $\beta > 0, \text{with } 0 < \beta < \min\{\beta_{\text{sphere}}, \beta_{\text{cylinder}}, \beta_{\text{bowl}}, \beta_{\text{oval}}\}$,

(iii) $\gamma > 0$.

Here $\alpha_{\text{sphere}}, \alpha_{\text{cylinder}}, \alpha_{\text{bowl}}, \alpha_{\text{oval}}$ and $\beta_{\text{sphere}}, \beta_{\text{cylinder}}, \beta_{\text{bowl}}, \beta_{\text{oval}}$ are the respective optimal $\alpha > 0$ and $\beta > 0$ for the shrinking sphere, cylinder, translating bowl and ancient oval.

Let $\mathbb{\alpha} = (\alpha, \beta, \gamma)$. Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface with $|A| < \gamma$ and suppose $\mathcal{M}$ is a unit-regular, cyclic (mod 2) integral Brakke flow starting from M then encounters only (multiplicity-one) spherical and neck-pinch singularities. We fix an additional constant $H_{\text{bdd}} = H_{\text{bdd}}(\mathbb{\alpha})$. An (α, β)-neighbourhood, $\Omega_{(\alpha,\beta)}$, is an open space-time neighbourhood of the singular set $\mathfrak{S}$, composed of finitely many connected components, with the following properties.

(i) For every regular point $X \in \mathcal{M} \cap \Omega_{(\alpha,\beta)}$, $|H(X)| > H_{\text{bdd}}$.

(ii) If $X \in \mathcal{M} \cap \partial\Omega_i$, where Ω_i is a connected component of $\Omega_{(\alpha,\beta)}$, we require $|H(X)| = H_{\text{bdd}}$.

(iii) Furthermore, if $X \in \mathcal{M} \cap \partial\Omega_i$, then the flow is β-uniformly 2-convex in $P(X, 2\xi(|H(X)|)^{-1})$ and locally α-noncollapsed in $P(X, \xi(|H(X)|)^{-1})$.

(iv) $\mathcal{M}$ is locally α-noncollapsed in $\Omega_{(\alpha,\beta)}$ at regular points.

(v) $\mathcal{M}$ is β-uniformly 2-convex in $\Omega_{(\alpha,\beta)}$ at regular points.

The value of $\xi = \xi(\alpha, \Lambda)$ is that given by the curvature estimates of Haslhofer–Kleiner, and depends on some Λ, which will be derived later.

Remark 3.2.7. Observe, the mean curvature is uniform across the boundary.

Remark 3.2.8. The choice to have constant mean curvature along the boundary serves a practical purpose. Later, we will specify surgeries in a flow approximating $\mathcal{M}$ only occur as long as said flow is a small graph over $\mathcal{M}$ in some neighbourhood of the boundary. We will show knowledge of the boundary data of $\mathcal{M}$ in the above fashion guarantees in the flows with surgery, via the maximum principle, that the hypotheses of the curvature estimates (Theorem 3.1.8) are satisfied in the interior. To be explicit, at interior points X, the flow in $P(X, \xi(|H(X)|)^{-1})$ will be an (α, δ)-flow in the sense of [HK17b].

Lemma 3.2.9. *Let $\mathcal{M}$ be a Brakke flow with only spherical and neck-pinch singularities. For every $\boldsymbol{\alpha}$ as in Definition 3.2.6, there is a $H_0(\boldsymbol{\alpha}, \mathcal{M}) < \infty$ such that for all $H_{\mathrm{bdd}} > H_0$ an (α, β)-neighbourhood exists.*

Proof. Fix $\boldsymbol{\alpha}$ satisfying the assumptions of Definition 3.2.6, and take $\varepsilon < (2\xi)^{-1}$. Additionally, we take ε small enough that if a flow is ε-close an ancient, asymptotically cylindrical flow, then it is β-uniformly 2-convex.

By the canonical neighbourhood theorem, Theorem 3.2.5, and the compactness of the singular set, there is an $r > 0$ such that any regular point in the parabolic cylinder $P(Y, r)$, centred at $Y \in \mathfrak{S}$ is ε-close to one of the ancient, asymptotically cylindrical flows (at scale of the mean curvature).

This radius can be taken such that at any interior regular point the flow is locally α-noncollapsed.

The union of the above cylinders, $\cup_{Y \in \mathfrak{S}} P(Y, r)$, defines a cover of the singular set. Observe, in each connected component the mean curvature has a single sign (a local orientation). Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of regular points contained in a single connected component that accumulate in $\mathfrak{S}$. It is immediate from the canonical neighbourhood theorem that $H(X_i) \to \infty$.

Hence, we can fix a H_{bdd} sufficiently large that

$$\Omega := \{X \in \mathrm{reg}(\mathcal{M}) \text{ s.t. } |H(X)| > H_{\mathrm{bdd}}\} \Subset \cup_{Y \in \mathfrak{S}} P(Y, r).$$

Observe, $\mathrm{reg}(\mathcal{M})$ is relatively open in $\mathrm{supp}(\mathcal{M})$, so Ω is a relatively open set in $\mathrm{supp}(\mathcal{M})$. Moreover, the mean convex neighbourhood theorem shows that we can include singular points, provided they are spherical or neck-pinch singularities, i.e. $\Omega' = \{X \in \mathrm{reg}(\mathcal{M}) \mid |H(X)| > H_{\mathrm{bdd}}\} \cup \mathfrak{S}$ is open in $\mathrm{supp}(\mathcal{M})$. The topology of $\mathrm{supp}(\mathcal{M})$ is inherited from the standard parabolic topology of space-time, $\mathbb{R}^{n+1,1}$. Thus, there is an open set U in $\mathbb{R}^{n+1,1}$ such that $U \cap \mathrm{supp}(\mathcal{M}) = \Omega'$. $\Omega_{(\alpha,\beta)}$ can be taken as any collection of such open sets in space-time. Hence, $\Omega_{(\alpha,\beta)}$ is an open space-time neighbourhood of the singular set. We can assume this neighbourhood has finitely many connected components since the singular set is compact.

Finally, the β-uniform 2-convexity and α-noncollapsedness for $X \in \mathcal{M} \cap \partial\Omega_i$ is immediate from the choice of ε in the canonical neighbourhood theorem. QED

Definition 3.2.10 (Neighbourhood of the boundary)**.** For a connected component Ω_i, we define

$$N_i = \bigcup_{X \in \partial\Omega_i} P(X, 2\xi H_{\mathrm{bdd}}^{-1}).$$

Where $P(X, 2\xi H_{\mathrm{bdd}}^{-1})$ is the backwards parabolic cylinder centered at X. Observe, as specified in Definition 3.2.6, $\mathcal{M} \cap P(X, 2\xi H_{\mathrm{bdd}}^{-1})$ will be smooth and β-uniformly 2-convex.

We now define a flow similar to the mean convex (α, δ)-flows of [HK17b]. It is a unit-regular cyclic mod 2 Brakke flow with the replacement of (smooth) δ-necks by caps.

Definition 3.2.11 ((α, δ)-Brakke flow)**.** Compare definition 3.1.6.

Let $M^n \subset \mathbb{R}^{n+1}$ be a compact, smoothly embedded hypersurface. Let $\mathcal{M}$ a unit-regular, cyclic (mod 2) Brakke flow emerging from M that encounters only (multiplicity one) spherical and neck-pinch singularities.

An (α, δ)-Brakke flow is defined as the collection of unit-regular cyclic (mod 2) Brakke flows

$$\{\mathcal{M}^i\} = \{\mu_t^i\}_{t \in [t_{i-1}, t_i]}, (i = 1, \dots, k+1;\ 0 = t_0 < \cdots < t_k < t_{k+1} = t_{\mathrm{Ext}}),$$

with the following properties. We adopt the standard notation of 'calligraphic' $\mathcal{M}$ to denote flows, and 'roman' M_t the t-time slice of $\mathcal{M}$. Superscripts will remain consistent between flows and timeslices in flows with surgery.

(i) $\mathcal{M}^i$ is a smooth flow for $1 \leq i \leq k$. That is, surgery is only performed if the flow is smooth.

(ii) For each $i = 1, \dots k$, we identify in $M_{t_i}^i =: M_{t_i}^-$, the final time slice of the smooth mean curvature flow $\mathcal{M}^i$, a collection of disjoint strong δ-necks contained in $\Omega_{(\alpha,\beta)}$. Each neck is replaced, provided the next point is satisfied, by pairs of standard caps as in Definition 3.1.5, creating the possibly disconnected hypersurface $M_{t_i}^{\#}$.

(iii) Necks at time $t_i \in \{t_1, \dots, t_k\}$ contained in Ω_j, a connected component of $\Omega_{(\alpha,\beta)}$, are only replaced by caps if the flow $\mathcal{M}^i$ can be written as a δ-graph over $\mathcal{M}$ in the boundary neighbourhoods N_j at time t_i. This is to ensure that the curvature estimate of [HK17b] carries over to the surgery flow. See Remarks

3.2.13 and 3.2.14. If this condition fails, we treat the last time surgeries were successfully performed as t_k and we continue as in item (vi). Note, we allow the case where being a graph over the boundary at time t_i is 'vacuously true' i.e. $\mathcal{M}^i \cap \partial\Omega_j = \emptyset, \mathcal{M}^i \cap \Omega_j \neq \emptyset$. Indeed, if a component of the flow is contained entirely in $\Omega_{(\alpha,\beta)}$, then it satisfies the assumptions of α non-collapsedness and β-uniform 2-convexity by the maximum principle.

(iv) The initial timeslice of $\mathcal{M}^{i+1}$, $M^{i+1}_{t_i} := M^+_{t_i}$ is obtained from the post-surgery hypersurface $M^{\#}_{t_i}$ by dropping some connected components contained in $\Omega_{(\alpha,\beta)}$.

(v) There exists $s_\# > 0$ which depends only on the Brakke flow $\mathcal{M}$ such that all necks in item (i) have radius $s \in [\mu^{-1/2}s_\#, \mu^{1/2}s_\#]^*$.

(vi) We allow the flow $\mathcal{M}^{k+1}$ to develop as a unit-regular Brakke flow until its extinction at time $t_{k+1} = t_{\text{Ext}}$. Specifically, we choose the integral, unit-regular, cyclic (mod 2) Brakke flow whose support is the outer flow from the initial condition of $\mathcal{M}^k$. See Hershkovits–White, [HW20], where such a flow is constructed.

Remark 3.2.12. In item (i), we require that $M^i_{t_i}$ is a smooth hypersurface for neck replacement to occur. Thus, after neck replacement the flow can be continued as an integral, unit-regular, cyclic (mod 2) Brakke flow by elliptic regularisation. It should be possible to weaken this requirement to being an integral current, however, this is not needed for the purposes of the current work. The choice of outer flow is important later, for understanding barriers to flows with surgical modification.

Remark 3.2.13. Item (iii) requires the (α, δ)-Brakke flow can be written as a δ-graph over $\mathcal{M}$ in N_i. By this we mean, the surgery flow is δ-close to $\mathcal{M}$ in $C^{\lfloor \frac{1}{\delta} \rfloor}(N_i)$. Whilst imposing such a condition may seem unmotivated, it occurs naturally when considering sequences of smooth flows that converge to a smooth limit. We discuss how our flows with surgery converge in Section 3.3.

Remark 3.2.14. We use the δ-graphical condition to ensure that along the boundary of $\Omega_{(\alpha,\beta)}$, the surgery flow satisfies the β-uniformly 2-convex and α-noncollapsed conditions, provided $\delta > 0$ is taken sufficiently small. The size of the required δ will depend on H_{bdd} and, of course, our choice of α and β. We can then promote this to interior control by the maximum principle. Demanding control in a neighbourhood of the boundary (as opposed to just on the boundary) addresses two problems. Firstly, we need to use a two point maximum principle to show interior α-noncollapsedness as in[And12]. We discuss why this graphical condition in the boundary provides sufficient control of the geometry of the flows with surgery to apply a two point maximum principle in Remark 3.4.4. Secondly, by enforcing a boundary graphical

*$\mu \in [1, \infty)$ is a constant that quantifies the notation of surgeries at comparable scales. See [HK17b, Convention 1.2]

condition in the definition of the (α, δ)-Brakke flows, we ensure the hypotheses of the Haslhofer–Kleiner curvature estimate are satisfied at all interior points, before the final time of surgery. This follows essentially from the triangle inequality and the maximum principle. For details, see Theorem 3.A.1.

Remark 3.2.15. It is important to stress that the uniform backward control of 2-convexity and noncollapsedness along the boundary is fundamental in being able to apply the curvature estimate for our choice of Λ. Note, this control is not needed if the mean curvature tends to infinity, only when one expects the curvature to remain bounded. For example, this argument is not needed when applying the curvature estimates in the Canonical Neighbourhood Theorem of Haslhofer–Kleiner, but is needed for showing surgery accumulates in the singular set.

Remark 3.2.16. In the formalism of Haslhofer–Kleiner surgery, α and β are controlled by the initial condition. In this flow, these parameters are controlled locally from the values on the boundary by the maximum principle.

We now define the weak surgical flows. The key deviations are that (a) the flow can become singular, and (b) the requirement that surgery only takes place in a predetermined neighbourhood of the singular set of the flow $\mathcal{M}$. Whilst this initially may feel restrictive, it is entirely natural. See Section 3.3.

Definition 3.2.17 (Weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow)**.** Let $M^n \subset \mathbb{R}^{n+1}$ be a compact, smoothly embedded hypersurface be a γ-controlled initial condition. Let $\mathcal{M}$ be a unit-regular, cyclic (mod 2) Brakke flow emerging from M that encounters only (multiplicity one) spherical and neck-pinch singularities. For a fixed $\boldsymbol{\alpha}$ (as above), $\delta > 0$ and surgery parameters $\mathbb{H}$ we define $\mathcal{M}_{\mathbb{H}}$ as the *weak* $(\boldsymbol{\alpha}, \delta, \mathbb{H})$*-flow* or *weak surgery flow derived from* $\mathcal{M}$ as the (α, δ)-Brakke flow that satisfies the following conditions:

(i) All surgeries take place inside the (α, β)-neighbourhood of the singular set of $\mathcal{M}$, the region where the original flow is α-noncollapsed and β-uniformly 2-convex.

(ii) Surgeries and/or discarding takes place at times t when $|\mathbf{H}| = H_{\text{trig}}$ somewhere in $\Omega_{(\alpha,\beta)}$. Note, we actually allow $|\mathbf{H}|$ to exceed H_{trig} in the flow outside the region where we perform surgery.

(iii) The collection of necks is minimal, and the necks are of curvature $|H_{\text{neck}}|$. The necks separate the set $\{|\mathbf{H}| = H_{\text{trig}}\}$ from $\{|\mathbf{H}| \leq H_{\text{th}}\}$.

(iv) The smooth hypersurface M_t^+ is obtained from M_t^- by dropping some smooth components of mean curvature $|\mathbf{H}| > H_{\text{th}}$ contained in $\Omega_{(\alpha,\beta)}$. In particular, for each pair of facing surgery caps, precisely one is discarded.

(v) If a strong δ-neck is also a strong $\hat{\delta}$ neck for $\hat{\delta} < \delta$ then item (iv) of definition 3.2.11 holds with $\hat{\delta}$ instead of δ.

Remark 3.2.18. Item (v) is the stipulation that if a δ-neck sits inside a stronger $\hat{\delta}$-neck, then the surgery is performed in a 'better' way, that is closer to the ideal cylinder and cap. This is an essential component of self-improvement.

Remark 3.2.19. We allow the flow to continue as a unit-regular Brakke flow if a (possibly non-generic) singularity forms after the last surgery. Note that we cannot be certain such a continuation is unique. We gain control of the singular behaviour via the barriers constructed in Section 4, in particular showing that any singularities will be spherical or neck-pinch singularities (and thus the continuation is well defined). In Section 5, we will show that giving control back to H_{trig} gives a smooth surgery in the same sense as [HK17b].

Consider the following examples of weak surgery flows.

Example 3.2.20. The shrinking sphere is a weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-surgery flow for all values of $\mathbb{H}$, if one chooses not to drop components of high curvature.

Example 3.2.21. Fix $\mathbb{H}$. The shrinking sphere that vanishes once the mean curvature reaches H_{th} is a weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-surgery flow.

Example 3.2.22. Fix $\boldsymbol{\alpha}$ and $\delta > 0$. Let M be an $\boldsymbol{\alpha}$-controlled initial condition. Then, there is a $\mathbb{H}$ given by [HK17b] such that the $(\boldsymbol{\alpha}, \delta, \mathbb{H})$ mean curvature flow with surgery of [HK17b] exists. It is a weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-surgery flow.

3.3 Barriers and Stability

We now develop the tools for controlling the weak surgery flows. In the first half of this section, we show that the unit-regular Brakke flow from hypersurfaces equidistant to the initial hypersurface act as barriers to our weak surgery flows, provided the surgery scale is large enough. The existence of these barriers requires the recent technical result of [CCMS20], concerning the connectedness of the singular set for flows with singular set of small Hausdorff dimension. Indeed, such a result is critical as one needs a way to show higher multiplicities cannot develop. We then tackle the problem of stability of the surgery flows. The parabolic nature of mean curvature flow means that changing the flow in one location can affect other regions at infinite speed. Whilst this problem cannot be completely avoided, showing the surgery parameters can be chosen such that surgeries change the flow in a manner that is 'stable' with respect to the unmodified flow is sufficient. Recalling the definition of the (α, β)-neighbourhood, one can see that if we can show suitable control in N_i, a neighbourhood of the boundary of a connected component of the (α, β)-neighbourhood, then in the interior our flow with surgery will locally look like a $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow of Haslhofer—Kleiner. In Section 5, this is precisely how we will show that their theory can be applied directly to deduce existence of a smooth flow with surgery. Said boundary control is achieved by a local convergence result. In showing this, we additionally prove the stronger result that the weak flows with surgery converge to the unmodified flow as Brakke flows away from the singular set.

For the following, we will suppose that $M^n \subset \mathbb{R}^{n+1}$ is a closed, smoothly embedded hypersurface and that there is a unit-regular, cyclic (mod 2) Brakke flow $\mathcal{M}$ emerging from M that encounters only multiplicity one spherical and neck-pinch singularities. A priori, such a flow is not unique, however, combining recent results we get the following uniqueness result.

Theorem 3.3.1. *Let $M^n \subset \mathbb{R}^{n+1}$ be a closed, smoothly embedded hypersurface. If there is a unit-regular cyclic (mod 2) Brakke flow $\mathcal{M}$ emerging from M that encounters only multiplicity one spherical and neck-pinch singularities, then the level-set flow does not fatten. In particular, $\mathcal{M}$ is unique.*

Proof. Recall that the support of $\mathcal{M}$ defines a weak set flow, and thus is contained in the level-set flow of M. Let $\mathcal{N}$ be the unit-regular Brakke flow whose support is the outer flow $\{M_t\}$. The existence of such a flow is proven in [HW20]. The uniqueness of smooth mean curvature flow implies that $\mathcal{M}$ and $\mathcal{N}$ agree up to the first singular time. Thus, their supports agree at the first singular time. Since $\mathcal{M}$ has only spherical and neck-pinch singularities, the flow $\mathcal{N}$ cannot fatten at the first singular time, t_0, [HW20]. Moreover, stratification, [Whi97], yields that the singular set of $\mathcal{M}$ has parabolic Hausdorff dimension at most one. Hence, by Theorem 2.2.6, ([CCMS20, Theorem F.4]), the regular sets of $\mathcal{M}$ and $\mathcal{N}$ are connected, and thus we have unit density at smooth points. Thus, the flows agree as Brakke flows up to the first singular time. This argument can be iterated since the flow is compact. i.e. For the two flows to differ, the outer flow must encounter a non-spherical or non-neck-pinch singularity, which cannot happen as the flows agree back in time. Thus, $\mathcal{M} = \mathcal{N}$. In particular, the outer flow has only spherical and neck-pinch singularities and hence does not fatten, [CHHW22, Theorem 1.19].

Since the support of any Brakke flow defines a weak set flow, the non-fattening and connectedness of the regular set show that $\mathcal{M}$ is the unique unit-regular flow.

QED

Thus, it is sufficient to suppose $\mathcal{M}$ has only spherical and neck-pinch singularities.

We also pick a $\varepsilon_0 = \varepsilon_0(M) > 0$ sufficiently small, such that for $-\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$ the hypersurfaces $M_\varepsilon = \{\mathrm{dist}(\cdot, M) = \varepsilon\}$, where $\mathrm{dist}(\cdot, M)$ is the signed distance function to M, are smooth.

Lemma 3.3.2. *Let $\varepsilon < \varepsilon_0$, and let $\mathcal{M}_{\pm\varepsilon}$ be unit-regular cyclic (mod 2) Brakke flows emerging from the hypersurfaces $M_{\pm\varepsilon}$. Then,*

$$\lim_{\varepsilon \to 0} \mathcal{M}_{\pm\varepsilon} = \mathcal{M}$$

as Brakke flows.

Proof. We prove the statement for the $+\varepsilon$ flows, as the proof for the $-\varepsilon$ flows will be identical.

Smooth convergence of $\mathcal{M}_\varepsilon \to \mathcal{M}$ holds up to the first singular time of $\mathcal{M}$. For later times we consider the following.

Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a positive null sequence, and consider the flows $\mathcal{M}_{\varepsilon_i}$. By the convergence result of Ilmanen [Ilm94], there is a unit-regular flow $\tilde{\mathcal{M}}$ such that $\mathcal{M}_{\varepsilon_i} \rightharpoonup \tilde{\mathcal{M}}$. In particular, since the level-set flow from M does not fatten, we have $\mathrm{supp}(\tilde{\mathcal{M}}) \subseteq \mathrm{supp}(\mathcal{M})$.

We now proceed via the logic of Theorem 2.2.6 [CCMS20, Appendix F].

Since $\mathcal{M}$ has only spherical and neck-pinch singularities, stratification, [Whi97], yields that the singular set has parabolic Hausdorff dimension at most one, so by Theorem 2.2.6 $\mathcal{M}$ has connected regular set. Indeed, by considering paths that connect to the initial time avoiding the singular set and noting that $\tilde{\mathcal{M}}$ is unit regular, we see that the density of $\tilde{\mathcal{M}}$ is equal to that of $\mathcal{M}$ at all regular points. Since the singular set of $\mathcal{M}$ has small measure, we have $\tilde{\mathcal{M}} = \mathcal{M}$.

This is true for all null sequences $\{\varepsilon_i\}$, hence the above argument shows $\mathcal{M}_{+\varepsilon}$ converges to $\mathcal{M}$. QED

Remark 3.3.3. Note, for small $\varepsilon > 0$ the barrier flows have only spherical and neck–pinch singularities. This follows from the resolution of the mean convex neighbourhood conjecture, [CHH22, CHHW22] and the extension to near-by flows by Schulze–Sesum [SS20].

Lemma 3.3.4. *Let $M, M_{\pm\varepsilon}$ be as above. Then, for every t where both flows are defined, $|d(M_t, M_{\pm\varepsilon,t})| \geq \varepsilon$.*

Proof. Follows from the standard avoidance principle for Brakke flows, see [Ilm94]. QED

Definition 3.3.5. We will call the unit-regular Brakke flows $\mathcal{M}_{\pm\varepsilon}$ the *ε-barriers.*

We take the convention that $M_{+\varepsilon}$ is the hypersurface in the interior of M. $M_{-\varepsilon}$ is thus in the exterior.

Lemma 3.3.6. *($\mathcal{M}_{\pm\varepsilon}$ as Surgical Barriers) Let M be as above. Fix ε, with $0 < \varepsilon < \mu(M)$. Then, there exists a $H(\varepsilon) < \infty$ such that any weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$ surgical flow with $H_{\mathrm{th}} > H(\varepsilon)$ avoids $\mathcal{M}_{\pm\varepsilon}$. In particular, the distance between the barriers and surgery flow is non-decreasing.*

Proof. It is well known that the distance between two non-intersecting Brakke flows is non-decreasing, (avoidance principle [Ilm94]). Provided the distance is not decreased across surgery, the claim follows.

We hence check the behaviour at time of surgery. Without loss of generality, we consider only one of the barriers at inward and outward necks. The proof for the other barrier will follow identically.

Let $\mathcal{M}_{+\varepsilon}$ be the evolution of the hypersurface in the interior of M. We follow the argument as outlined in [HK17b].

Claim 3.3.1. Let t be a surgery time at an inward neck for the surgical flow $\mathcal{M}_{\mathbb{H}}$. For every $r > 0$, there is a $H_{\min}(r) < \infty$ such that if $H_{\text{th}} > H_{\min}$ and $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^-})$, then $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^+})$.

Proof. Fix $r > 0$. There are two regions one needs to check:

1. The collection of necks. For each neck we consider its interior K (See Definition 3.2.1). Following the argument of [HK17b, Theorem 1.25], for sufficiently large H_{th}, a ball of radius r cannot be contained in K, as it will be a long and thin neck.

2. The dropped components. If the ball were contained in the interior of a discarded component, then the discarded component would have a point with $|H| \leq nr^{-1}$. Discarded components have $|H| \geq H_{\text{th}}$, thus picking $H_{\text{th}} > nr^{-1}$ is sufficient to prove the claim.

QED

Claim 3.3.2. Let t be a surgery time at an outward neck for the surgical flow $\mathcal{M}_{\mathbb{H}}$. For every $r > 0$, there is a $H_{\min}(r) < \infty$ such that if $H_{\text{th}} > H_{\min}$ and $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^-})$, then $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^+})$.

Proof. Recall, at outward necks, the 'interior' of the neck is exterior to the flow. The caps are glued inside the cylinder. Thus, if $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^-})$, then we have $B(\mathbf{p}, r) \subset \text{int}(M_{\mathbb{H},t^+})$ for all values of H_{th}. QED

For the other barrier, we consider $B(\mathbf{p}, r) \subset \text{ext}(M_{\mathbb{H},t^-})$. The proofs are identical, but for the oppositely oriented necks.

To illustrate how the above claims prove the distance is non-decreasing, consider the following. Fix $\varepsilon > 0$ and choose the surgery parameter $\mathbb{H}$ such that $H_{\text{th}} > H_{\min}(\varepsilon)$. Let t be the first time of surgery. We now consider the balls $B(\mathbf{x}, d(\mathbf{x}, M_{\mathbb{H},t^-}))$, where $d(\cdot, M_{\mathbb{H},t^-})$ is the distance of a point to the hypersurface $M_{\mathbb{H},t^-}$, for each point $\mathbf{x}$ in the t timeslice of $\mathcal{M}_{\pm\varepsilon}$. Clearly any such ball will lie entirely on one side of $M_{\mathbb{H},t^-}$. Since flows with surgery are simply smooth flows up to time t, the avoidance principle shows that the radius, $r = r(\mathbf{x})$, of each ball must have $r \geq \varepsilon$. We deduce from the above claims that each of the discussed balls in the interior (resp. exterior) of $M_{\mathbb{H},t^-}$ will be interior (resp. exterior) to $M_{\mathbb{H},t^+}$ after surgery, as $H_{\text{th}} > H_{\min}$. Thus, the distance of $M_{\mathbb{H},t^+}$ to either barrier at time t cannot be less than that of $M_{\mathbb{H},t^-}$. Since a surgical flow is a Brakke flow between surgery times, the avoidance principle allows for the argument to be repeated at all later surgery times. We conclude the distance between the barriers and the surgical flow is non-decreasing along the entire flow.

QED

Remark 3.3.7. Interior and exterior are well defined because we are considering smooth hypersurfaces at times of surgery. Note, the property of 'separating' the inner and outer barriers is preserved through surgery, in the sense that at any time, any path connecting the inner and outer barriers must pass through the flow with surgery. In addition, such a separation property is valid for all times after the last surgery by our choice to continue the surgery flow as the unit-regular cyclic (mod 2) Brakke flow whose support is the outer flow.

Corollary 3.3.8 (Hausdorff Convergence)**.** *Taking the limit as $H_{\text{th}} \to \infty$, the weak flows with surgery from M converge to the level-set flow from M in the Hausdorff sense.*

Proof. Recall, we use the convention that $M_{+\varepsilon}$ is interior to M. Let U be the compact set bounded by M, and $U' = \overline{U^c}$. Similarly, denote $U_{\pm\varepsilon}$ as the compact sets with $\partial U_{\pm\varepsilon} = M_{\pm\varepsilon}$, and $U'_{\pm\varepsilon} = \overline{U^c_{\pm\varepsilon}}$. It is clear that for all $\varepsilon_1 > \varepsilon_2 > 0$ we have

$$
\begin{aligned}
&U_{-\varepsilon_1} \supset U_{-\varepsilon_2} \supset U \supset U_{+\varepsilon_2} \supset U_{+\varepsilon_1} \\
&U'_{+\varepsilon_1} \supset U'_{+\varepsilon_2} \supset U' \supset U'_{-\varepsilon_2} \supset U'_{-\varepsilon_2}
\end{aligned}
$$

Using the notation of [HW20], we denote the space-time track of the level-set flow from U, U' as $\mathcal{U}, \mathcal{U}'$. We have

$$
\begin{aligned}
&\mathcal{U}_{-\varepsilon_1} \supset \mathcal{U}_{-\varepsilon_2} \supset \mathcal{U} \supset \mathcal{U}_{+\varepsilon_2} \supset \mathcal{U}_{+\varepsilon_1} \\
&\mathcal{U}'_{+\varepsilon_1} \supset \mathcal{U}'_{+\varepsilon_2} \supset \mathcal{U}' \supset \mathcal{U}'_{-\varepsilon_2} \supset \mathcal{U}'_{-\varepsilon}
\end{aligned}
$$

By Lemma 3.3.2, we can take $\varepsilon > 0$ small enough such that $\mathcal{M}_{\pm\varepsilon}$ has only spherical and neck-pinch singularities. Thus, the level-set flow from $M_{\pm\varepsilon}$ does not fatten, and hence $\partial\mathcal{U}_{+\varepsilon} = \partial\mathcal{U}'_{+\varepsilon} = \operatorname{supp}(\mathcal{M}_{+\varepsilon})$.

We define the closed sets $\mathcal{K}_\varepsilon := \mathcal{U}'_{+\varepsilon} \cap \mathcal{U}_{-\varepsilon}$ and $K(t) := \{x \in \mathbb{R}^{n+1} \mid (x,t) \in \mathcal{K}\}$. Note, the space-time boundary of $\mathcal{K}_\varepsilon$ is $\partial\mathcal{K}_\varepsilon = \operatorname{supp}(\mathcal{M}_{+\varepsilon}) \sqcup \operatorname{supp}(\mathcal{M}_{-\varepsilon})$. Recall, these flows are disjoint by the avoidance principle.

By Lemma 3.3.6, for every $\varepsilon > 0$, we can find a $H(\varepsilon) < \infty$ such that any weak surgery flow $\mathcal{M}_{\mathbb{H}}$ with $H_{\text{th}} > H$ avoids $\mathcal{M}_{\pm\varepsilon}$. Indeed, we see that $\mathcal{M}_{\mathbb{H}} \subset \mathcal{K}_\varepsilon$ and at every time $t \geq 0$ where both $\mathcal{M}_{\pm\varepsilon}$ are non-empty, $\mathcal{M}_{\mathbb{H}}$ 'separates', in the sense that any (space-like) curve joining $M_{+\varepsilon}(t)$ to $M_{-\varepsilon}(t)$ must pass through $M_{\mathbb{H},t}$. The corollary will follow immediately from the following claim.

Claim 3.3.3. $\mathcal{K}_\varepsilon$ converges to $\operatorname{supp}(\mathcal{M}) = \{(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R} \text{ s.t. } x \in F_t(M)\}$ in the Hausdorff sense as $\varepsilon \to 0$.

Proof. By construction, $\operatorname{supp}(\mathcal{M}) \subset \mathcal{K}_\varepsilon$ for all $\varepsilon > 0$, i.e. for all $\xi > 0$, $\operatorname{supp}(\mathcal{M})$ is always in the ξ neighbourhood of $\mathcal{K}_\varepsilon$.

Observe, for $\varepsilon_1 > \varepsilon_2 > 0$, we have $\mathcal{K}_{\varepsilon_2} \subset \mathcal{K}_{\varepsilon_1}$. Thus, it is sufficient to show $\mathrm{supp}(\mathcal{M}) \supseteq \cap_{\varepsilon\to 0}\mathcal{K}_\varepsilon$. (Clearly the reverse inclusion is true). We do this by showing $\mathcal{K} := \cap_{\varepsilon\to 0}\mathcal{K}_\varepsilon$ defines a weak set flow from M.

Observe, at $t = 0$, we have $\cap_{\varepsilon\to 0}K_\varepsilon(0) = M$, as M is closed and

$$K_\varepsilon(0) = \{x \in \mathbb{R}^{n+1} \mid d(x, M) \le \varepsilon\}.$$

Given any smooth compact hypersurface N that is disjoint from M, we can find an $\varepsilon > 0$ such that $K_\varepsilon(0) \cap N = \emptyset$, simply by taking $\varepsilon \le d(M, N)$. It is immediate from the definition of $\mathcal{K}_\varepsilon$ that it will be disjoint from the space-time track of the mean curvature flow from N. Indeed, $\mathcal{K}$ must avoid every smooth mean curvature flow that is initially disjoint with M. Thus, $\mathcal{K}$ defines a weak set flow from M. Since $\mathrm{supp}(\mathcal{M})$ is the space-time track of the level-set flow, it must contain $\mathcal{K}$. This follows from the definition of the level-set flow as the maximal weak set flow, see [Ilm94].

QED

Indeed, we have shown that the 'gap' between $\mathcal{M}_{\pm\varepsilon}$, $\mathcal{K}_\varepsilon$, Hausdorff converges to $\mathrm{supp}(\mathcal{M})$ as $\varepsilon \to 0$. Since $\mathcal{M}_{\pm\varepsilon}$, the space-time boundary components of $\mathcal{K}_\varepsilon$, converge in the Brakke sense to $\mathcal{M}$, and any surgery flow with $H_{\mathrm{th}} > H(\varepsilon)$ will separate $\mathcal{M}_{\pm\varepsilon}$, we deduce $\lim_{H_{\mathrm{th}}\to\infty} \mathcal{M}_{\mathbb{H}} = \mathrm{supp}(\mathcal{M})$.

QED

Having shown Hausdorff convergence, our goal now is to establish graphical control of the weak surgery flows in the boundary neighbourhood of the (α, β) neighbourhood. This is achieved by establishing Brakke convergence in this region. We will actually show Brakke convergence on the full regular set. Consider for a moment a sequence of Brakke flows that converge in a Hausdorff sense to another Brakke flow. Improving the convergence to Brakke convergence is straight forward provided one can find a way to control multiplicity. See the proof of Proposition 3.3.17, claim 3.3.7 onwards. Recalling the definition of an (α, δ) Brakke-flow, Definition 3.2.11, inside any open space-time set that does not contain a surgery, an (α, δ)-Brakke flow is a unit-regular, cyclic (mod 2) Brakke flow. Thus, Brakke convergence will follow from understanding where, in a limiting sense, surgeries occur in our weak surgery flows. Indeed, we will show that the surgeries accumulate in the singular set of $\mathcal{M}$. Using what has been shown so far we can develop some intuition as to why this is expected behaviour.

Let $M^n \subset \mathbb{R}^{n+1}$ and $\mathcal{M}$ be as stated at the start of the current section. For the sake of simplicity, suppose further $\mathcal{M}$ encounters an isolated, non-degenerate neck-pinch singularity at the first singular time. Let $\mathcal{M}_{\mathbb{H}_i}$ be a sequence of weak flows with surgery starting from M, with $H^i_{\mathrm{th}} \to \infty$. At the first time of surgery in the flow $\mathcal{M}_{\mathbb{H}_i}$, we can identify a δ-neck with centre P_i and mean curvature $H_{\mathcal{M}_{\mathbb{H}_i}}(P_i) = H^i_{\mathrm{neck}}$

that is about to under-go surgery. The sequence $\{P_i\}_{i=1}^{\infty}$ can be treated as a sequence of points in $\mathcal{M}$ since, by definition, the weak flows with surgery must agree with $\mathcal{M}$ up to their respective first surgery time. Since $H^i_{\text{neck}} \to \infty$, it is clear that the points P_i must accumulate in the singular set at the first singular time. Whilst this argument works at the first time of surgery, it unfortunately cannot be applied at later surgery times, however, we can use the barriers begin to understand what is happening. In the following we develop a general intuition, though it may be informative for the reader to keep in mind the specific example of the classic 2-convex dumbbell as the initial condition M and $\mathcal{M}$ the outer flow from M. First, we note $\varepsilon > 0$ can be chosen small enough such that the barrier flows $\mathcal{M}_{\pm\varepsilon}$ also satisfy the canonical neighbourhood condition in our (α, β) neighbourhood. We may assume the barriers are moving monotonically towards their (global) interior inside $\Omega_{(\alpha,\beta)}$; in connected components of $\Omega_{(\alpha,\beta)}$ where flows are moving monotonically towards their exterior, simply exchange the roles of the inner and outer barriers. Secondly, we note that any weak flow with surgery (with sufficiently large H_{th}) separates $\mathcal{M}_{\pm\varepsilon}$. Indeed, we have the set of inclusions outlined in Corollary 3.3.8. Thus, by our avoidance principle, Lemma 3.3.6, surgeries can only occur in regions where the inner barrier is not present. Conversely, we see the outer barrier $\mathcal{M}_{-\varepsilon}$ can only pinch off into a cylindrical singularity or vanish in a spherical singularity in regions where the weak surgery flow is not present. From our canonical neighbourhood assumption, one expects the inner barrier to vacate the (ambient) interior of a δ-neck in the weak surgery flow by translating like a bowl or passing through a singularity. Similarly, we expect the weak surgery flow would vacate the interior of a neck-like region in the outer barrier developing into a singularity by surgery [†]. Indeed, this seems to indicate a correspondence of surgeries and singularities and thus one expects, along the sequence of weak surgery flows from M, for surgeries to accumulate in the singular set of $\mathcal{M}$.

Unfortunately, it is not clear that this picture is entirely correct. One possible issue is that there is no way to rule out a surgery neck developing in a weak surgery flow in such a way that is completely unrelated to the geometry of the barriers flows. This is possible as we have only shown the weak surgery flows (with large H_{th}) remain Hausdorff-close to the original weak flow after the first surgery time. For the above heuristic to have rigorous meaning we need to be able to relate the geometry of the weak surgery flows back to that of the original flow. Indeed, this would rule out 'gratuitous' surgery necks forming in regions where we would expect low curvature. One might hope to use pseudolocality to control the flow with surgery. Unfortunately, direct application of pseudolocality is obstructed by the surgeries, as the caps cannot be written as graphs over the necks they replace. We will show in Proposition 3.3.12 that the pseudolocality result as stated in [INS19] can be applied at a space-time point X_0 in a weak flow with surgery, with the caveat that surgeries

[†] that is, of course, presuming that surgery is permitted according to the Definition 3.2.11.

must be performed at a scale much larger than the curvature at the point X_0. In order to repeatedly apply pseudolocality one must introduce further ingredients (see Remark 3.3.16).

The purpose of the following lemma, Lemma 3.3.10, is to define a scaling factor $\lambda := \frac{|A|(\mathbf{x}_0)}{\tilde{C}_2}$, such that when the flow is dilated by λ, the hypotheses of the pseudolocality, Theorem 2.4.2, are satisfied, see Remark 3.3.11.

Remark 3.3.9. In Remark 3.2.14, we discussed how the canonical neighbourhoods had to be chosen careful such that we always satisfy the hypotheses of the Haslhofer–Kleiner curvature estimates, Theorem 3.1.8, in the interior for a particular choice of Λ. We now pause to start fixing the value of our constants so we can use them in the following arguments. In particular, we fix a value for the required Λ.

We fix $\eta > 0$ that satisfies the required gradient bound of the Ecker–Huisken graphical curvature estimate, Theorem 2.4.4. Taking this value of η into Pseudolocality, Theorem 2.4.2, fixes an initial Lipschitz bound $\varepsilon = \varepsilon(n, \eta) > 0$ and radius $\vartheta = \vartheta(n, \eta) > 0$. We hence take ϑ as the radius of the n-ball in the Ecker–Huisken estimate, Theorem 2.4.4, giving the constant $\tilde{C}_3 = \tilde{C}_3(n, \theta, \vartheta)$. We will only ever apply this graphical curvature bound to a point over the origin of the ball, so the value of θ does not matter, so for the sake of simplicity take $\theta = 1/2$. We can now fix $\Lambda = 10n \max\{\tilde{C}_3, 1\}$ for application of the Haslhofer–Kleiner curvature estimate. As was discussed in Remark 3.2.14, the value of Λ needs to be fixed so it is certain we can apply the estimate at interior points of $\Omega_{(\alpha,\beta)}$. The reasoning for this choice of value for Λ will become clear in the following theorems. Of course, fixing the value of Λ fixes the value of $\tilde{C}_0 = \tilde{C}_0(\alpha, \Lambda) < \infty$, the constant from the Haslhofer–Kleiner curvature estimate. Finally, taking ε given to us from pseudolocality and this value of $\tilde{C}_0$, we fix the value of $\tilde{C}_2 = \varepsilon/\tilde{C}_0$, as per Lemma 3.3.10.

In the following, constants will be denoted $\tilde{C}_k$ for some integer k and cylinders[‡] will be denoted $C(\mathbf{x}, r)$ for some point $\mathbf{x} \in \mathbb{R}^{n+1}$ radius $r > 0$. Note also, balls in the $(n+1)$-dimensional ambient space are denoted B, whilst balls of dimension n in an affine subspace (i.e. a tangent space) will be denoted B^n.

Lemma 3.3.10. *Let $\mathcal{M}_{\mathbb{H}}$ be a weak flow with surgery and suppose $X_0 = (\mathbf{x}_0, t_0) \in \mathcal{M}_{\mathbb{H}} \cap \overline{\Omega}_{(\alpha,\beta)}$. Suppose further $t_0 \leq t_F$, where t_F is the last surgery time.*

For every $\varepsilon > 0$, let $\tilde{C}_2(\alpha, \Lambda, \varepsilon) = \frac{\varepsilon}{\tilde{C}_0(\alpha,\Lambda)}$, where $\tilde{C}_0$ is the constant from the Haslhofer–Kleiner curvature estimate. Then the hypersurface $\lambda(M_{t_0} - \mathbf{x}_0)$, with $\lambda = \frac{|H|(\mathbf{x}_0)}{\tilde{C}_2}$, has

$$\sup_{\lambda M_{t_0} \cap B(\mathbf{0},1)} |A| \leq \varepsilon \tag{3.1}$$

$$\sup_{\lambda M_{t_0} \cap B(\mathbf{0},1)} \sqrt{1 + |Du|^2} < 1 + \varepsilon \tag{3.2}$$

[‡]The cylinder has been defined in Definition 2.4.1.

Where $u(\mathbf{x})$ is a function on the tangent space at 0 *such that* $\lambda(M_{t_0} - \mathbf{x}_0) \cap C(\mathbf{0}, 1) = \mathrm{graph}(u)$ *and* M_{t_0} *is the* $t = t_0$ *time-slice of* $\mathcal{M}_{\mathbb{H}}$*. In particular, we note that the above show that the Lipschitz constant of* u *is bounded by* ε*.*

Proof. Since $t_0 \leq t_F$, the surgery flow is certainly smooth, and thus we can apply the global curvature estimate, Theorem 3.1.8, with our choice of $\Lambda \geq 1$. The claim follows immediately. QED

Remark 3.3.11. The existence of such a $\tilde{C}_2$ is noteworthy, as it is uniform across any (α, δ)-flow that satisfies the assumptions of Theorem 3.1.8. Indeed, this shows that the $\vartheta > 0$ given to us in the following pseudolocality theorem (Theorem 3.3.12) is uniform, when working at the scale of mean curvature, across all weak surgery flows $\mathcal{M}_{\mathbb{H}}$ that satisfy the hypotheses of Theorem 3.3.12. This is required so limits may be taken.

As mentioned previously, the surgeries obstruct the use of pseudolocality as stated in [INS19]. Following their argument, the result is only valid until the next surgery is performed. In addition to their proof, we need to show that if any surgeries are performed in the forward time interval, then they are not performed in or near a large neighbourhood of the cylinder where we wish to apply pseudolocality. Indeed, this is true provided surgeries are done at a sufficiently large scale compared to the mean curvature of the point we wish to apply pseudolocality. The central idea is a combination of the Ecker–Huisken graphical curvature estimates and the Haslhofer—-Kleiner curvature estimate to bound the mean curvature in the cylinder below the surgery scale.

Proposition 3.3.12. *Let* $X_0 \in \mathcal{M}_{\mathbb{H}} \cap \Omega_{(\alpha,\beta)}$*,* $|A|(X_0) < \infty$*. Pseudolocality can be applied to the flow* $\mathcal{M}_{\mathbb{H}}$ *around* X_0*, provided the surgery is done with parameter* $H_{\mathrm{neck}} > \frac{\tilde{C}_0 \tilde{C}_3}{\tilde{C}_2} n^2 |H|(X_0)$*. That is,*

$$\mathcal{D}_\lambda(\mathcal{M}_{\mathbb{H}} - X_0) \cap C(\mathbf{0}, \vartheta), t \in [0, \vartheta^2) \cap [0, t_F] \tag{3.3}$$

is a smooth mean curvature flow, and can be written as a graph over ϑ *with Lipschitz constant less than* η *and height bounded by* $\eta\vartheta$*.* $\lambda = \lambda(\alpha, \Lambda, \varepsilon, X_0)$ *is as in the above claim.* t_F *denotes the final time of surgery in the dilated flow. Moreover, since* $\mathcal{M}_{\mathbb{H}}$ *is continued as a Brakke flow after the final time of surgery, we also deduce*

$$\mathcal{D}_\lambda(\mathcal{M}_{\mathbb{H}} - X_0) \cap C(\mathbf{0}, \vartheta), t \in [0, \vartheta^2) \cap [0, t_{\mathrm{Ext}}] \tag{3.4}$$

is a unit-regular, cyclic (mod 2), integral Brakke flow, and can be written as a graph over B^n_ϑ *with Lipschitz constant less than* η *and height bounded by* $\eta\vartheta$*.*

Remark 3.3.13. $\tilde{C}_0, \tilde{C}_3$ are expected to be large, $\tilde{C}_2$ is expected to be small. Thus, $\frac{\tilde{C}_0 \tilde{C}_3}{\tilde{C}_2}$ is very large. This may give the impression that the theorem is weak. Its strength will come once applied to points with bounded curvature in a sequence of flows with degenerating surgery parameters.

Proof. Suppose $X_0 \in \mathcal{M}_{\mathbb{H}} \cap \Omega_{(\alpha,\beta)}$, $|A|(X_0) < \infty$. We fix $\eta > 0$, and let $\vartheta(\eta), \varepsilon(\eta)$ be those given by the pseudolocality Theorem 2.4.2. Let λ be as in lemma 3.3.10 with $\varepsilon = \varepsilon(\eta)$.

If the surgery flow is a smooth mean curvature flow in the forward time interval given by Theorem 2.4.2, then there is nothing to check. Thus, let $\hat{\mathcal{M}}_{\mathbb{H}} = \mathcal{D}_\lambda(\mathcal{M}_{\mathbb{H}} - X_0)$, and suppose there are surgeries occurring in the time interval $[0, \vartheta^2)$. Note, there are only finitely many times to check in this interval, so we may enumerate them chronologically.

Let t_1 be the time of the first surgery in $\hat{\mathcal{M}}_{\mathbb{H}}$ after time $t = 0$. It is sufficient to show that all surgeries are performed far from the cylinder $C(\mathbf{0}, 1)$ at time t_1, as this demonstrates the flow is simply a smooth mean curvature flow in $C(\mathbf{0}, 1) \times [0, t_2)$ and thus the flow remains a graph in the cylinder $C(\mathbf{0}, \vartheta) \times [0, t_2)$, where t_2 is the next surgery time.

Remark 3.3.14. These times correspond to surgeries in the dilated flow, not the original time scale.

Since the flow is a mean curvature flow on $[0, t_1]$, we know from the classical pseudolocality result that $\hat{\mathcal{M}}_{\mathbb{H}} \cap C(\mathbf{0}, \vartheta)$ can be written as the graph of $u_t : B^n_\vartheta(\mathbf{0}) \to \mathbb{R}$, for $t \in [0, \vartheta^2) \cap [0, t_1]$.

Applying the Ecker–Huisken interior estimate for graphs, Theorem 2.4.4, to the function u_t we establish the following bounds on curvature

$$\sup_{B^n_{\theta\vartheta}(\mathbf{0}) \times [0,t_1]} |A| \leq \tilde{C}_3(n, \theta, \vartheta) \sup_{B^n_\vartheta(\mathbf{0}) \times \{0\}} |A| = \tilde{C}_3 \varepsilon \tag{3.5}$$

for some constant $\tilde{C}_3$ depending only on n, θ, ϑ.

Let $X = (\mathbf{0}, u_{t_1}(0), t_1) = (\mathbf{x}, t_1)$, i.e. the point in the flow above the origin at time t_1. Equation 3.5 shows $|A|(X) \leq \tilde{C}_3 \varepsilon$. Applying the Haslhofer–Kleiner curvature estimate, Theorem 3.1.8, at the point X, we deduce that in the backward parabolic cylinder $P(X, \Lambda r)$ the curvature is bounded by $\tilde{C}_0 r^{-1}$, where $r^{-1} = H(X) \leq \tilde{C}_3 \varepsilon n$ (and thus, $r \geq (\tilde{C}_3 \varepsilon n)^{-1}$). Note we have used the standard inequality $|H| \leq n|A|$.

As a simple consequence of the estimate in $P(X, \Lambda r)$, we have

$$\sup_{B_{\Lambda r}(\mathbf{x}) \cap \hat{M}_{t_1}} |A| \leq \tilde{C}_0 \tilde{C}_3 \varepsilon n,$$

where $\hat{M}_{t_1}$ denotes the $t = t_1$ time slice of $\hat{\mathcal{M}}_{\mathbb{H}}$. Moreover, using $|H| \leq n|A|$ once again, we see

$$\sup_{B_{\Lambda r}(\mathbf{x}) \cap \hat{M}_{t_1}} |H| \leq \tilde{C}_0 \tilde{C}_3 \varepsilon n^2.$$

We highlight that, since $\Lambda \geq 10n\tilde{C}_3$, the curvature bound holds in $B(\mathbf{x}, 10\varepsilon^{-1})$, moreover $B(\mathbf{x}, 10\varepsilon^{-1}) \supset C(\mathbf{0}, 1)$. That is to say, the curvature bound holds for the weak flow with surgery contained in the cylinder $C(\mathbf{0}, 1)$ at time t_1.

By definition, surgery in $\mathcal{M}_{\mathbb{H}}$ was done at scale H_{neck}. Scaling our parameters accordingly, we deduce surgery in $\hat{\mathcal{M}}_{\mathbb{H}}$ is done at scale $\hat{H}_{\text{neck}} = \lambda^{-1}H_{\text{neck}} = (\tilde{C}_2/|H|(\mathbf{x}_0))H_{\text{neck}} > \tilde{C}_0\tilde{C}_3n^2 > \tilde{C}_0\tilde{C}_3\varepsilon n^2$. Here, we have used our assumption that $H_{\text{neck}} > \frac{\tilde{C}_0\tilde{C}_3}{\tilde{C}_2}n^2|H|(X_0)$ and that $\varepsilon < 1$. Observe, from the bound on mean curvature in $B_{\Lambda r}(\mathbf{x})$, the mean curvature at every point $Y \in \hat{\mathcal{M}}_{\mathbb{H}} \cap (B_{10\varepsilon^{-1}}(x) \times \{t_1\})$ is below the threshold for surgery to be performed. In particular, any changes made at time t_1 do not affect the portion of the hypersurface $\hat{M}_{t_1}$ contained in $C(\mathbf{0}, 1)$. Hence, the flow $\hat{\mathcal{M}}_{\mathbb{H}} \cap (\overline{C}(\mathbf{0}, 1) \times [0, t_2])$ is a smooth mean curvature flow, and the flow is graphical over $B^n_{\vartheta}(\mathbf{0})$ in $C(\mathbf{0}, \vartheta) \times [0, t_2]$.

This argument is then repeated at all future surgery times in $[0, \vartheta^2) \cap [0, t_F]$. The second claim follows immediately from the Brakke form of Theorem 2.4.2, as $\mathcal{M}_{\mathbb{H}}$ is continued as a unit-regular integral Brakke flow after the final surgery time t_F. QED

We now have the tools necessary to show surgeries accumulate in the singular set.

Proposition 3.3.15. *Let $M^n \subset \mathbb{R}^{n+1}$ and $\mathcal{M}$ be as above. Then, for every open neighbourhood N of the singular set, there is a $H_{min}(N) < \infty$ such that if $\mathbb{H}$ has $H_{\text{th}} > H_{\text{min}}$, then all surgeries in $\mathcal{M}_{\mathbb{H}}$ occur inside this neighbourhood.*

Proof. The above statement is equivalent to the statement that, across a sequence of surgery flows with $H^i_{\text{th}} \to \infty$, any sequence of centres of surgery necks, $X_i \in \mathcal{M}_{\mathbb{H}_i}$, accumulates in the singular set $\mathfrak{S}$ of $\mathcal{M}$.

Suppose for contradiction that this is not the case. Let $\mathcal{M}_{\mathbb{H}_i}$ be a sequence of $(\boldsymbol{\alpha}, \delta, \mathbb{H}_i)$-flows evolving from M with $H^i_{th} \to \infty$. By the assumption we wish to contradict, we can find a sequence of points $X_i = (\mathbf{p}_i, t_i) \in \mathcal{M}_{\mathbb{H}_i}$ in δ-necks where surgery is performed, with $H(X_i) = H^i_{\text{neck}}$, that accumulate to some point $X_\infty = (\mathbf{x}_\infty, t_\infty) \in \mathfrak{S}^c$. It is clear that the sequence must accumulate to some point in $\text{supp}(\mathcal{M})$ from Hausdorff convergence. Note that $t_\infty \neq t_{\text{Ext}}$, as the regular set is empty at time of extinction.

Claim 3.3.4. $X_\infty \notin \partial\Omega_{(\alpha,\beta)}$

Proof. Suppose X_∞ were in the boundary of the chosen (α, β)-neighbourhood. Item (iii) of Definition 3.2.11 required a backward parabolic cylinder centred at each point in the boundary in which the surgery flow is a graph over the original flow. This immediately rules out surgeries being performed in this neighbourhood, and thus preventing accumulation forward in time (i.e. $t_i < t_\infty$, for infinitely many i) or 'spatially' ($t_i = t_\infty$, for infinitely many i) within a given time-slice to a point the boundary. Thus, it remains to check that surgeries cannot accumulate backward in time ($t_i > t_\infty$, for infinitely many i) to a point in the boundary.

We first prove a smooth convergence result. Again we recall Item (iii) of Definition 3.2.11. There is a backwards parabolic cylinder $P = P(X_\infty, 2\xi H_{\mathrm{bdd}})$ centred at X_∞ in which we can write $\mathcal{M}_{\mathbb{H}_i}$ as a graph over $\mathcal{M}$. This is true for all i. As mentioned above, being a small graph over the original flow rules out surgeries occurring in this parabolic cylinder. Clearly $\mathcal{M}_{\mathbb{H}_i} \cap P$ is a sequence of smooth unit-regular Brakke flows, and thus converge to some limiting Brakke flow $\mathcal{N}$ in P. Hausdorff convergence shows that the support of $\mathcal{N}$ is $\operatorname{supp}(\mathcal{M} \cap P)$. Finally, we note that being a small graph controls the multiplicity of the flows with surgery and thus the sequence converges locally smoothly in P to $\mathcal{M} \cap P$ by White regularity.

The smooth convergence is now used to show pseudolocality can be applied in such a way that is comparable across all the flows with surgery for sufficiently large i. Dilating by $\lambda = |H|(X_\infty)/\tilde{C}_2$ around the point X_∞, and following the proof of Lemma 3.3.10, we deduce $\tilde{M}_0$, the $t = 0$ time slice of the dilated flow $\tilde{\mathcal{M}} = \mathcal{D}_\lambda(\mathcal{M} - X_\infty)$, can be written as the graph of some smooth function u over $B = B_1^n(\mathbf{0})$, the ball of radius 1 in the tangent space at 0, with $|A| < \varepsilon$. Similarly, we set $\tilde{\mathcal{M}}_i = \mathcal{D}_{\lambda_i}(\mathcal{M}_{\mathbb{H}_i} - X_\infty)$, $\lambda_i = |H_i|(X_i)/\tilde{C}_2$. Since the (un-dilated) flows converged smoothly around X_∞, we deduce $\lambda_i \to \lambda$. Moreover, the dilated flows $\tilde{\mathcal{M}}_i$ converge smoothly to $\tilde{\mathcal{M}}$ in P, thus there is an $I < \infty$ such that for $i \geq I$, the time $t = 0$ time-slice, $\tilde{M}_{i,0}$, can be written as a graph of the function $u_i : B \to \mathbb{R}$, where B is the same ball in the tangent space to $\tilde{M}_0$ at 0, and $u_i \to u$ smoothly in B. Thus, by the Brakke form of the pseudolocality result for flows with surgery, Proposition 3.3.12, no surgeries of the flow $\mathcal{M}_{\mathbb{H}_i}$ occur in $\tilde{\mathcal{M}}_{\mathbb{H}_i} \cap C(\mathbf{0}, \vartheta_i) \times ([0, \vartheta_i^2) \cap [0, t_{\mathrm{Ext}}))$. Recall, ϑ_i essentially depended on the curvature at $u_i(0)$ and the dimension. Since the hypersurfaces at time $t = 0$ converge smoothly in some neighbourhood of the origin, there is a uniform $\vartheta > 0$ such that for every flow, $\tilde{\mathcal{M}}_i \cap C(\mathbf{0}, \vartheta) \times ([0, \vartheta^2) \cap [0, t_{\mathrm{Ext}}))$ is a unit-regular, cyclic (mod 2) Brakke flow. In particular, no surgeries occur. This contradicts our assumption that surgeries were accumulating from future times. QED

It remains to check regular points in the interior of $\Omega_{(\alpha,\beta)}$. In order to employ the above argument, we require knowledge that the weak surgery flows are graphical over $\mathcal{M}$ in some backwards parabolic cylinder. A priori, we have no control of the flow at points in the interior, other than information given by the maximum principle and Hausdorff convergence. To find such a neighbourhood, we will start at the boundary of $\Omega_{(\alpha,\beta)}$ and then repeatedly apply the pseudolocality theorem followed by the Haslhofer–Kleiner curvature estimate to work our way into the interior.

Claim 3.3.5. There is an open space-time neighbourhood of X_∞ such that the flows $\mathcal{M}_{\mathbb{H}_i}$ converge smoothly to $\mathcal{M}$.

Remark 3.3.16. If one were to just iterate pseudolocality, the forward time interval could shrink in a geometric progression. The essence of the argument presented below is, given a point of low curvature, we find our forward neighbourhood from pseudolocality. We deduce convergence of the sequence of surgery flows to $\mathcal{M}$ in this forward neighbourhood. Applying the Haslhofer–Kleiner curvature estimate we show, for large i, no surgeries will be performed in a larger backward neighbourhood (centred at some future time, compared to the point we applied pseudolocality), and we can deduce convergence on this larger set. One is then in a position to apply pseudolocality at the same scale.

Proof. Consider a path γ in $\mathrm{reg}(\mathcal{M}) \cap \Omega_{(\alpha,\beta)}$ connecting X_∞ to a point $X_0 \in \partial\Omega_{(\alpha,\beta)}$. Say $\gamma : [0, T] \to \mathrm{reg}(\mathcal{M})$, $\gamma(0) = X_0, \gamma(T) = X_\infty$. Since the flow is locally 2-convex, we can pick the point X_0 and translate in time such that $X_0 = (\mathbf{x}_0, 0)$, $\gamma(\tau) \in M_\tau$. We will write $\gamma(\tau) = (\mathbf{x}_\tau, \tau)$. The argument proceeds as follows:

- Since the path γ is compact, there exists some $\mathcal{A} < \infty$ such that

$$\max_{\tau\in[0,T]} |H_{\mathcal{M}}|(\gamma(\tau)) \leq \mathcal{A}.$$

- Fix a small constant $\zeta > 0$. Lemma 3.3.10 implies $\tilde{M}_\tau = \mathcal{D}_\lambda(M_\tau - \gamma(\tau))$ can be written in $C(\mathbf{0}, 1)$ as a graph over the ball $B_1^n(\mathbf{0})$ in the tangent space to $\tilde{M}_\tau$ at 0, where $\lambda = \frac{\mathcal{A}+\zeta}{\tilde{C}_2}$. In particular, the hypotheses of Theorem 2.4.2 are satisfied and hence we can apply the Brakke formulation of pseudolocality to $\tilde{M}_\tau$ at 0.

- We remark that the small constant $\zeta > 0$ is present so we can rescale each $\mathcal{M}_{\mathbb{H}_i}$ by the same factor. The plan is to use the same argument as in Claim 3.3.4, with the only complication coming from wanting to have the forward neighbourhood be comparable at every point along γ. Consider a sequence of points $Y_i \in \mathcal{M}_{\mathbb{H}_i}$ accumulating to $Y_\infty \in \gamma$, such that $|H_{\mathcal{M}_{\mathbb{H}_i}}(Y_i)| \to |H_{\mathcal{M}}(Y_\infty)|$. Then, there exists an $I = I(\zeta)$, such that $i \geq I$ implies $|H_{\mathcal{M}_{\mathbb{H}_i}}(Y_i)| < \mathcal{A} + \zeta$. The significance being one can choose a cylinder centred at Y_∞ in which the conclusion of pseudolocality (Theorem 2.4.2 and Proposition 3.3.12) is valid for $\mathcal{M}$ and all $\mathcal{M}_{\mathbb{H}_i}$ with $i \geq I(\zeta)$ after dilating by the common constant $\lambda = \frac{\mathcal{A}+\zeta}{\tilde{C}_2}$.

- Returning to our main argument, we transform back to the un-dilated flow and deduce there is a uniform ϑ such that at each point $\gamma(\tau) \in \mathrm{reg}(M)$, the flow $\mathcal{M} \cap \mathcal{C}(\tau)$ is graphical over the ball $B^n_{\lambda^{-1}\vartheta}(\mathbf{x}_\tau)$ in the tangent space to M_τ at $\gamma(\tau)$. Where $\mathcal{C}(\tau) = C(\mathbf{x}_\tau, \lambda^{-1}\vartheta) \times ([\tau, \tau + (\lambda^{-1}\vartheta)^2] \cap [0, T_{\mathrm{Ext}}))$.

- The path γ is continuous and compact. Hence, we can find finitely many times $0 = \tau_0 < \tau_1 < \cdots < \tau_N < T$ such that $\gamma([0, T]) \subset \cup_{j=0}^N \mathcal{C}(\tau_j)$. Note that $\tau_N < T$. This will be important for applying the curvature estimates to the flows with surgery $\mathcal{M}_{\mathbb{H}_i}$. Note further there must be 'overlap' of the cylinders, in the sense $\gamma(\tau_i) \in \mathcal{C}(\tau_{i-1}), i \geq 1$.

- By our assumption, $\gamma(0) \in \partial\Omega_{(\alpha,\beta)}$. Examining the proof of Claim 3.3.4, we can immediately deduce Brakke convergence of $\mathcal{M}_{\mathbb{H}_i} \to \mathcal{M}$ in $\mathcal{C}(0)$. Indeed, for sufficiently large i, $\mathcal{M}_{\mathbb{H}_i} \cap \mathcal{C}(0)$ is a Brakke flow (no surgeries occur in $\mathcal{C}(0)$). Multiplicity is controlled by our assumption the flows with surgery are graphical over the boundary.

- We can improve the regularity of the convergence. Recall, $\gamma(\tau_1) \in \mathcal{C}(0)$, thus $\mathcal{M}_{\mathbb{H}_i} \to \mathcal{M}$ in a Brakke sense in some small backwards parabolic cylinder P centred at $\gamma(\tau_1)$. We may suitably shrink P such that $P \cap \mathcal{M} \subset \mathrm{reg}(\mathcal{M})$. Since $\mathcal{M}$ is smooth in P, we deduce smooth convergence of $\mathcal{M}_{\mathbb{H}_i} \to \mathcal{M}$ in P by White regularity.

- We now prove an inductive step, allowing us to 'move along' the path γ. Smooth convergence in P centred at $\gamma(\tau_1)$ implies there is a sequence of points $Y_i = (\mathbf{y}_i, \tau_1) \in \mathcal{M}_{\mathbb{H}_i}, Y_i \to \gamma(\tau_1), H_{\mathcal{M}_{\mathbb{H}_i}}(Y_i) \to H_{\mathcal{M}}(\gamma(\tau_1))$. We can hence apply the Haslhofer–Kleiner curvature estimate to $\mathcal{M}_{\mathbb{H}_i}$ at Y_i as in Proposition 3.3.12 to deduce no surgeries occur in the backwards parabolic cylinder $P(Y_i, \Lambda(H_{\mathcal{M}_{\mathbb{H}_i}}(Y_i))^{-1})$. Applying the curvature estimate is permissible when i is taken sufficiently large: the surgery necks accumulate at some time T with $\tau_1 < T$, thus for large i we must have $\tau_1 < t_{F_i}$, where t_{F_i} is the final time of surgery in $\mathcal{M}_{\mathbb{H}_i}$.

- In particular, we deduce smooth convergence in $P(\gamma(\tau_1, (H_{\mathcal{M}}(\gamma(\tau_1)))^{-1})$ since $\Lambda > 1$. One is now in the position to apply the argument from Claim 3.3.4.

- This argument can be repeated at each τ_j, since $\tau_j \in \mathcal{C}(\tau_{j-1})$. In particular, we note that $\gamma(T) \in \mathcal{C}(\tau_N)$. Thus, again, taking i sufficiently large, we deduce no surgeries of the flow $\mathcal{M}_{\mathbb{H}_i}$ are performed near $\gamma(T)$, contradicting the claim that surgeries accumulated at $\gamma(T) = X_\infty$.

QED

This concludes the proof, as we have shown surgeries cannot accumulate to regular points.

QED

We now state and prove our crucial convergence result. Note, in the above proof we have already established convergence inside $\Omega_{(\alpha,\beta)}$.

Proposition 3.3.17 (Convergence away from singular set)**.** *Let $\mathcal{M}_{\mathbb{H}_i}$ be a sequence of $(\alpha, \delta, \mathbb{H}_i)$ surgical flows derived from M, and suppose $\mathbb{H}_i$ is a sequence of surgery parameters with $H^i_{th} \to \infty$. Then, $\mathcal{M}_{\mathbb{H}_i}$ converges to $\mathcal{M}$ as Brakke flows on the complement of the singular set of $\mathcal{M}$.*

Proof. Recall that the singular set $\mathfrak{S}$ is closed in space-time, thus its complement, $\mathfrak{S}^c$, is open. Recall further, the definition of convergence of Brakke flows [Bra78], [Ilm94], is with respect to compactly supported functions. If $f \in C^1_c(\mathfrak{S}^c)$, then by definition we have $\mathrm{supp}(f) \Subset \mathfrak{S}^c$. In particular, it is sufficient to verify the proposition on any connected open set $\Omega \Subset \mathfrak{S}^c$ that has non-trivial intersection with the initial timeslice. These properties are required to control the multiplicity of the Brakke flow as in Lemma 3.3.2.

Claim 3.3.6. For any open set $\Omega \Subset \mathfrak{S}^c$, there is an $I < \infty$ such that for $i > I$, no surgeries of the flow $\mathcal{M}_{\mathbb{H}_i}$ occur in Ω.

Proof. This follows from Proposition 3.3.15.

If $\Omega \cap \Omega_{(\alpha,\beta)} = \emptyset$, we immediately know surgeries are not present in a neighbourhood for all $i > 0$. It remains to check the case when $\Omega \cap \Omega_{(\alpha,\beta)} \neq \emptyset$. Without loss of generality, we consider $\Omega \subset \Omega_{(\alpha,\beta)}$. Since $\Omega \Subset \mathfrak{S}^c$, there is an open neighbourhood N of $\mathfrak{S}$, with $\Omega \cap N = \emptyset$.

Thus, by Proposition 3.3.15 we deduce all surgeries occur in N for sufficiently large i, and hence none occur in Ω. QED

Applying Ilmanen's compactness result for Brakke flows, [Ilm94], there is a limiting unit-regular Brakke flow $\mathcal{N}$ such that,

$$\lim_{i\to\infty} \mathcal{M}_{\mathbb{H}_i} \lfloor \Omega = \mathcal{N}.$$

Claim 3.3.7. $\mathrm{supp}(\mathcal{N}) = \mathrm{reg}(\mathcal{M}) \cap \Omega$

Proof. The claim follows immediately from Corollary 3.3.8. In particular, $\mathrm{supp}(\mathcal{N})$ is connected by the result of [CCMS20]. QED

Claim 3.3.8. $\mathcal{N} = \mathcal{M}\lfloor\Omega$ as unit-regular Brakke flows.

Proof. All that remains is to check $\mathcal{N}$ does not develop higher multiplicity. By the above, $\mathrm{supp}(\mathcal{N})$ is connected and has non-trivial intersection with the initial time-slice, thus $\mathcal{N}$ has unit density everywhere. QED

Thus, $\lim_{i\to\infty} \mathcal{M}_{\mathbb{H}_i} \lfloor \mathfrak{S}^c = \mathcal{M}$ as Brakke flows.

QED

As a corollary, one deduces the following results that control the behaviour of any potential singular points that form in weak surgery flows.

Corollary 3.3.18. *Let $\mathcal{M}_{\mathbb{H}_i}$ be a sequence of $(\alpha, \delta, \mathbb{H}_i)$ surgical flows derived from the flow $\mathcal{M}$, and suppose $\mathbb{H}_i$ is a sequence of surgery parameters with $H^i_{\mathrm{th}} \to \infty$. If $X_i \in \mathcal{M}_{\mathbb{H}_i}$ is a sequence of singular points (i.e. points with Gaussian density $\Theta_{\mathcal{M}_{\mathbb{H}_i}}(X_i) \geq 1 + \varepsilon_{White}$). Then X_i accumulate in $\mathfrak{S}$, the singular set of $\mathcal{M}$.*

Remark 3.3.19. Here $\varepsilon_{\text{White}}$ is the (dimension dependent) quantity of White regularity [Whi05].

Proof. Suppose for contradiction a sequence of points $\{X_i\}_i^\infty$, satisfying the above hypothesis, accumulates at $X_\infty \in \text{reg}(\mathcal{M})$. Then, by Proposition 3.3.17, the weak surgery flows converge to $\mathcal{M}$ in a neighbourhood of X_∞. In particular, $\Theta_{\mathcal{M}}(X_\infty) = 1$. This is in contradiction to the upper semi-continuity of the density; taking the limit of densities we should have $\Theta_{\mathcal{M}}(X_\infty) \geq 1 + \varepsilon_{\text{White}}$.

QED

Corollary 3.3.20. *The above corollary holds also for regular points $X_i \in \mathcal{M}_{\mathbb{H}_i}$ where*

$$\lim_{i\to\infty} |A(X_i)| = \infty$$

Proof. Following the above proof, we note that smooth convergence implies convergence of the second fundamental form. X_∞ is a smooth point, thus $|A| < \infty$, contradicting $\lim_{i\to\infty} |A(X_i)| \to \infty$. QED

3.4 Existence and Convergence of Smooth Flows with Surgery

Let $M^n \subset \mathbb{R}^{n+1}$ be a closed, smoothly embedded submanifold. Since M is compact and smooth, we can find a $\gamma > 0$ such that $|A| < \gamma$. We suppose there is a unique unit-regular Brakke flow $\mathcal{M}$ emerging from M that encounters only spherical and neck-pinch singularities. We fix

- $0 < \alpha < \min\{\alpha_{cyl}, \alpha_{sphere}, \alpha_{oval}, \alpha_{bowl}\}$.
- $0 < \beta < \min\{\beta_{\text{sphere}}, \beta_{\text{cylinder}}, \beta_{\text{bowl}}, \beta_{\text{oval}}\}$.

Let $\mathbb{\alpha} = (\alpha, \beta, \gamma)$. Additionally, we take $\delta > 0$ small enough that all the arguments of Haslhofer–Kleiner [HK17b] hold and to satisfy item (iii) of Definition 3.2.11 and Remark 3.2.14. For the sake of completeness, we also fix a suitable standard surgical cap, suitable cap separation parameter and the value of Λ as in Section 4.

Theorem 3.4.1 (Surgery at the first singular time)**.** *Let $\mathcal{M}$ be as above. Let Ω_1 be the union of the connected components of $\Omega_{(\alpha,\beta)}$ containing the first singular time. Let $T_1 > 0$ be the first singular time of the flow outside Ω_1. Then for every $\varepsilon > 0$, the parameters $H_{\min}(M) < \infty$ and $\Theta(M) < \infty$ can be chosen (depending only on the initial hypersurface) such that the $(\mathbb{\alpha}, \delta, \mathbb{H})$ weak surgery flow $\mathcal{M}_{\mathbb{H}}$ is a smooth mean curvature flow with surgery on $[0, T_1 - \varepsilon)$.*

Compare the result of Mramor, [Mra21], where similar ideas are discussed for surgery in mean convex ‘patches’ of non-compact flows.

Proof. Fix an $\varepsilon > 0$ and stipulate that surgeries may only be performed in Ω_1. By Corollary 3.3.20, we know the singularities of surgery flows converge to the singular set of $\mathcal{M}$ as $H_{\mathrm{th}} \to \infty$. Thus, we can choose $H_{\min} < \infty$ sufficiently large that all singularities of a weak surgery flow with $H_{\mathrm{th}} > H_{\min}$ occur within ε in time of the singularities of $\mathcal{M}$. Moreover, such singularities are contained in $\Omega_{(\alpha,\beta)}$ and are spherical or neck-pinch singularities.

We initially fix the surgery ratio $\Theta < \infty$, this will be changed in due course.

Claim 3.4.1. For sufficiently large $H_{\min}$, any $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow with $H_{\mathrm{th}} > H_{\min}$ is a δ-graph over $\mathcal{M}$ in N_1 along the boundary of Ω_1.

Proof. This is a consequence of Proposition 3.3.17 and its corollaries. Recall, N_1 is the open neighbourhood of the boundary of Ω_1 in which the flow $\mathcal{M}$ is smooth, locally α-noncollapsed and β-uniformly 2-convex, as defined in Definition 3.2.10. Since the boundary of N_1 is bounded away from the singular set, it is immediate from Proposition 3.3.17 and White regularity that, for sufficiently large H_{th}, the claim holds. QED

Remark 3.4.2. It is important to compare this claim with the definition of surgery. We only permit the surgery procedure to be applied when the flow is graphically over $\mathcal{M}$ along the boundary. Thus, we see the obstruction to the flow continuing as a smooth surgery flow is not from our definitions, but from a point with $H(X) = H_{\mathrm{neck}}$ that does not separate regions of curvature H_{th} and H_{trig} or is not a δ-neck. This is the same obstruction as is dealt with in the case for 2-convex flows in [HK17b].

Claim 3.4.2. Fix $H_{\min} < \infty$ to satisfy claim 5.1. Then if $H_{\mathrm{th}} > H_{\min}$, we can directly apply the arguments of Haslhofer–Kleiner [HK17b] to establish a $\Theta < \infty$ such that $\mathbb{H} > \Theta$ implies the weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$ surgery flow is a smooth mean curvature flow up to time $t = T_1 - \varepsilon$.

Proof. Recall, the definition of an (α, δ)-Brakke flow only allowed surgery as long as the flow was smooth. Thus, since the singularities of the surgical flows can occur within ε of any singular time, $T_1 - \varepsilon$ is the best one can do without more information on the singular set.

By the first claim, $\mathcal{M}_{\mathbb{H}} \cap \partial\Omega_1$ is 2-convex and α-noncollapsed for all $\mathbb{H}$ with $H_{\mathrm{th}} > H_{\min}$. After doing one surgical neck replacement, the maximum principle gives that the flow remains 2-convex and α-noncollapsed inside Ω_1. The same argument holds across any number of neck replacements, so every surgical flow with $H_{\mathrm{th}} > H_{\min}$ is 2-convex and α-noncollapsed inside Ω_1.

We now stipulate that the flow is stopped once $|H| = H_{\mathrm{trig}}$ is achieved inside Ω_1. [HK17b, Theorem 1.21] and [HK17b, Theorem 1.22] can now be applied directly find the desired $\Theta < \infty$ which establishes the existence of a weak flow with surgery that is smooth inside Ω_1 up to time $T_1 - \varepsilon$. We note that Corollary 3.3.20 prevents points of high curvature accumulating on the boundary of Ω_1 along sequences of surgical flows. This is important for the proof of [HK17b, Theorem 1.22]. QED

This completes the proof of the theorem. QED

Remark 3.4.3. We stop only if H_{trig} is achieved in Ω_1.

Remark 3.4.4. One should note that Andrews' maximum principle proof of α-noncollapsing for mean convex mean curvature flow, [And12], makes use of a 2-point maximum principle for a function $Z(x, y, t)$. The positivity of $Z(x, y, t)$ is equivalent to being α-noncollapsed. This argument can be suitably localised to the above situation by observing that along the boundary of Ω_1, the flows will be close to one of the canonical flows (sphere, cylinder, bowl, and oval). Indeed, we know for points in the boundary the 'touching points' of tangential spheres will be in our neighbourhood of the boundary, N_1. Since the interior mean curvature is larger than the boundary mean curvature, and surgery flows are Hausdorff close to the original flow, we see touching points of tangential spheres to interior points will be in $\Omega_1 \cup N_1$. That is, one only needs to consider the function $Z(x, y, t)$ for points $((\mathbf{x}, t), (\mathbf{y}, t)) \in \Omega_1 \times \{\Omega_1 \cup N_1\}$. This is similar to the argument presented in Theorem 3.A.1.

Theorem 3.4.5 (Existence of a smooth flow with surgery)**.** *Let $\mathcal{M}$ be as above. Then, the parameters $H_{\min}(M) < \infty$ and $\Theta(M) < \infty$ can be chosen (depending only on the initial hypersurface) such that every weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow, $\mathcal{M}_{\mathbb{H}}$, with $H_{\text{th}} > H_{\min}$, $\mathbb{H} > \Theta$ satisfies:*

- *$|H| \leq H_{\text{trig}} < \infty$ everywhere,*
- *$\mathcal{M}_{\mathbb{H}}$ vanishes in finite time.*

i.e. $\mathcal{M}_{\mathbb{H}}$ is a smooth mean curvature flow with surgery.

Remark 3.4.6. The weak surgery flows were unit-regular away from surgery, so sudden vanishing is not permitted. The second item is thus non-trivial.

Proof. $\Omega_{(\alpha,\beta)}$ has finitely many components, thus it is sufficient to argue inductively.

We show that given Theorem 3.4.1, we have the respective statement for Ω_2, the union of connected components of $\Omega_{(\alpha,\beta)}$ containing time T_1. Recall time T_1 was the first singular time that occurs outside Ω_1. We will establish that for every $\varepsilon > 0$ the parameters can be chosen such that there is a smooth flow with surgery up to time $T_2 - \varepsilon$. Here, T_2 the first singular time outside of $\Omega_1 \cup \Omega_2$

Remark 3.4.7. The time interval over which Ω_2 exists may overlap with that of Ω_1. Surgeries in Ω_2 can affect the surgeries that occur in Ω_1, since mean curvature flow is parabolic. This is not an issue as the convergence results still hold. We may require a larger $H_{\min}$ and/or Θ for the same conclusion to hold.

Pick $H_{\min}$, $\Theta < \infty$ such that the conclusion of Theorem 3.4.1 holds, and consider the boundary of Ω_2. Once again, the logic of Proposition 3.3.17 controls the behaviour in a neighbourhood of the parabolic boundary, N_2. We may take $H_{\min}$ large enough that the flow is β-uniformly 2-convex and α-noncollapsed in N_2. Proceeding exactly as in claim 3.4.2, we conclude the same result for $\Omega_1 \cup \Omega_2$.

This argument can be repeated for each connected component of $\Omega_{(\alpha,\beta)}$. Since there are only finitely many components, $H_{\min}$ and Θ stay bounded as they can only be changed a finite number of times.

Observe, the flow $\mathcal{M}$ will be entirely contained within the final connected component of $\Omega_{(\alpha,\beta)}$. Thus, there will be no singular times outside the final connected component, as there is no flow. The flow inside this final component will be a 2-convex surgery of [HK17b]. QED

We restate the canonical neighbourhood theorem of Haslhofer–Kleiner.

Theorem 3.4.8 (Canonical Neighbourhoods, Theorem 1.22 [HK17b])**.** *For all $\varepsilon > 0$, there exist $\overline{\delta} = \overline{\delta}(\boldsymbol{\alpha}) > 0$, $H_{\text{can}}(\varepsilon) = H_{\text{can}}(\boldsymbol{\alpha}, \varepsilon) < \infty$ and $\Theta_\varepsilon(\delta) = \Theta_\varepsilon(\boldsymbol{\alpha}, \delta) < \infty$ ($\delta \leq \overline{\delta}$) with the following significance. If $\delta \leq \overline{\delta}$ and $\mathcal{M}_{\mathbb{H}}$ is an $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow with $\mathbb{H} \geq \Theta_\varepsilon(\delta)$, then any $(\mathbf{p}, t) \in \mathcal{M}_{\mathbb{H}}$ with $|H(\mathbf{p}, t)| \geq H_{\text{can}}(\varepsilon)$ is ε-close to either (a) a β-uniformly 2-convex ancient α-noncollapsed flow, or (b) the evolution of a standard cap preceded by the evolution of a round cylinder.*

Proof. The proof is identical to that of Haslhofer–Kleiner [HK17b], for we only do surgery in 2-convex connected components. QED

The canonical neighbourhood theorem gives the following topological result concerning the dropped components.

Theorem 3.4.9 (Discarded components, [HK17b, Corollary 1.25])**.** *For all $\varepsilon > 0$ small enough, there are parameters $\Theta_\varepsilon(\delta) < \infty$, $H_{\text{can}}(\varepsilon)$ such that any weak $(\boldsymbol{\alpha}, \delta, \mathbb{H})$ surgical flow with $\mathbb{H} > \Theta_\varepsilon(\delta)$, and $H_{\text{th}} > H_{\text{can}}(\varepsilon)$, has all discarded components are diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Remark 3.4.10. The parameters are derived from the canonical neighbourhood theorem.

Proof. This follows from the canonical neighbourhood theorem [HK17b, Theorem 1.22]. The argument is identical to that in [HK17b], for components are only dropped if they are contained in $\Omega_{(\alpha,\beta)}$. QED

We conclude with a result similar to that of Lauer and Head, [Lau13, Hea13]. Note we also establish the stronger result that the convergence away from the singular set is smooth.

Theorem 3.4.11. *Taking the limit as $H_{\text{th}} \to \infty$, the weak $(\alpha, \delta, \mathbb{H})$ surgical flows converge in the Hausdorff sense to the level-set flow. Furthermore, away from the singular set of $\mathcal{M}$ the convergence is smooth.*

Proof. This is an immediate consequence of Proposition 3.3.17 and White regularity [Whi05]. QED

3.5 Applications of the Surgery

We now apply the above surgery formalism to prove a Schoenflies type theorem for hypersurfaces of entropy less than $\lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, without having to manually construct the isotopies. Such a proof was conjectured in [CCMS21, Conjecture 1.9]. The previous best bound on the entropy was $\lambda(\mathbb{S}^2 \times \mathbb{R}^1)$ and was achieved independently by Bernstein–Wang [BW22a] and Chodosh–Choi–Mantoulidis–Schulze [CCMS20].

Recall the definition of entropy for a hypersurface from [CM15].

Definition 3.5.1. The Entropy of a hypersurface Σ is

$$\lambda(\Sigma) = \sup_{\mathbf{x}_0, t_0} \left(\frac{1}{4\pi t_0}\right)^{\frac{n}{2}} \int_{\Sigma} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4t_0}\right) d\mu,$$

i.e. the supremum of the Gaussian densities over all scales and base-points. It can be considered a measure of the complexity of an embedding.

We first discuss the topological consequences of surgery. Recall, from Theorem 3.1.15 we know discarded components will be diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Moreover, we have the following

Lemma 3.5.2. *Let $\mathcal{M}_{\mathbb{H}}$ be a smooth mean curvature flow with surgery from the smooth initial condition M. Then,*

(i) The flow $\mathcal{M}_{\mathbb{H}}$ is a smooth isotopy between times of surgery.

(ii) Let $\tilde{M}$ be a connected component of the timeslice M_t, for any t, $0 < t < T_{\text{Ext}}$. The size of the fundamental group of $\tilde{M}$ satisfies $|\pi_1(\tilde{M})| \leq |\pi_1(M)|$.

Proof. (i) It is immediate from the definition that smooth mean curvature flow is an isotopy. The flow $\mathcal{M}_{\mathbb{H}}$ is a smooth flow with surgery, and thus a mean curvature flow between times of surgery. This proves the first statement.

(ii) From part (i), we know that any topological changes that occur must happen at surgeries. It is sufficient to show the claim at the first surgery time, as at future surgery times we can treat each connected component present before surgery as a separate flow.

Let t be the first time of surgery. We denote the pre-surgery hypersurface by M_t^- and post neck-replacement, but pre-component dropping, by $M_t^{\#}$. Note that it is possible for $M_t^{\#}$ to be disconnected. By item (i), we have $\pi_1(M_t^-) = \pi_1(M)$. We need only to consider connected components of $M_t^{\#}$ as clearly any component present at time between the first and second times of surgery must have evolved from some connected component of $M_t^{\#}$. Thus, it is sufficient to show $|\pi_1(M_t^-)| \geq |\pi_1(\tilde{M})|$, where $\tilde{M}$ is a connected component of $M_t^{\#}$. This follows immediately by [HS09, Proposition 3.23], which shows M_t^- is diffeomorphic to the connected sum (reversing the neck-replacement) of the connected components of $M_t^{\#}$. For completeness we prove our claim directly, by showing every non-trivial element of $\pi_1(\tilde{M})$ corresponds to a non-trivial element of $\pi_1(M_t^-)$.

Let $P_i = (p_i, t), i \in \{1, 2, \ldots, N\}, N < \infty$ be the centre of each δ-neck that is about to be replaced by caps at time t. We know all modifications are made in $B = \cup_i^N B(p_i, 5\Gamma H_{\text{neck}}^{-1})$ (see Definition 3.1.5 with $s = H_{\text{neck}}^{-1}$).

Let $\gamma \in \pi_1(\tilde{M})$ be a non-trivial element. We can take this element to be represented by a curve $\tilde{\gamma}$ lying entirely in $\tilde{M}\backslash\{\tilde{M} \cap B\}$. This follows as each connected component of $\tilde{M} \cap B$ is diffeomorphic to our standard cap. Since the cap is simply connected, any portion of the curve that enters a cap is homotopic to a curve on the boundary. Morally, we can consider this curve as detecting some topology unaffected by our surgery at time t.

Since $\tilde{\gamma} \cap B = \emptyset$, we can consider it as a curve in M_t^-, since $\tilde{M}\backslash\{\tilde{M} \cap B\} \subset M_t^-$. Clearly this curve cannot represent the trivial homotopy class as the connected sum operation cannot 'remove topology'. Consequently, $|\pi_1(\tilde{M})| \leq |\pi_1(M)|$).

QED

Remark 3.5.3. It is of note that the surgery procedure detailed above can break handles in two ways. This is best illustrated by the following examples.

1. Consider the 2-convex embedding of the torus known as the 'wedding band'. Deform it in a 2-convex manner such that one region is a much tighter neck than other regions. This flow will develop an inward neck pinch under mean curvature flow. If surgery is performed once, we are left with a 'sausage', smoothly isotopic to a sphere.

2. Consider a sphere with small holes drilled in around the poles, that has had the ends of a cylinder attached smoothly to each hole. This is a smooth embedding of the torus. This cylinder is a long thin neck which, heuristically, one expects would form an outward neck pinch under mean curvature flow. If one were to replace this neck by surgery, the resulting hypersurface is a sphere with the poles (smoothly) pushed in. This hypersurface is smoothly isotopic to a sphere.

Theorem 3.5.4 (Low-entropy Schoenflies for $\mathbb{R}^4$)**.** *Let $\Sigma^3 \subset \mathbb{R}^4$ be a hypersurface homeomorphic to $\mathbb{S}^3$ with entropy $\lambda(\Sigma) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. Then M is smoothly isotopic to the round $\mathbb{S}^3$.*

Proof. $\Sigma^3 \subset \mathbb{R}^4$ be a hypersurface homeomorphic to $\mathbb{S}^3$ with entropy $\lambda(\Sigma) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. By [CCMS21], there is a small (isotopic) perturbation of Σ, $\hat{\Sigma}$, such that the unit-regular Brakke flow, $\mathcal{M}$, emerging from $\hat{\Sigma}$ is unique and encounters only spherical and neck-pinch singularities. We find $\gamma > 0$ such that $\max_{\mathbf{x}\in\hat{\Sigma}}\{|\mathbf{A}(\mathbf{x})|\} < \gamma$ and fix α, β and $\delta > 0$ as discussed in section 5. By Theorem 3.4.5, the parameters H_{trig} and Θ can be chosen such that there is smooth $(\boldsymbol{\alpha}, \delta, \mathbb{H})$-flow with surgery $\mathcal{M}_{\mathbb{H}}$ that approximates the flow $\mathcal{M}$. In addition, we suppose H_{th} and Θ are large enough that the conclusion of Theorem 3.4.9 holds.

It remains to show that all the dropped components of $\mathcal{M}_{\mathbb{H}}$ are not tori and no handles are broken.

Claim 3.5.1. The topological constraint that Σ is homeomorphic to $\mathbb{S}^3$ rules out

(a) Dropped components being diffeomorphic to tori, $\mathbb{S}^2 \times \mathbb{S}^1$.

(b) The breaking of a handle during surgery.

Proof. We prove (a), (b) follows identically. Suppose for contradiction that there is at least one dropped component that is a torus. Let t be the first time a torus is dropped in surgery. It is clear that some component of the pre-surgery hypersurface M_{t^-} would have a non-trivial fundamental group (i.e. size greater than 1). By Lemma 3.5.2, the initial condition $\hat{\Sigma}$ must also have had non-trivial fundamental group. This is a contradiction to Σ being homeomorphic to $\mathbb{S}^3$. QED

Thus, all dropped components are isotopic to spheres and no handles are broken.

We now use backward induction to deduce $\hat{\Sigma}$ is smoothly isotopic to the round $\mathbb{S}^3$. There are finitely many surgeries, thus, there is a finite set of times $t_1 < \ldots < t_n$ when the flow is stopped.

Observe, at t_n^-, the final non-empty time slice of $\mathcal{M}_{\mathbb{H}}$, we have a collection of 2-convex components diffeomorphic to spheres. Each connected component is smoothly isotopic to a sphere. (Such an isotopy can be found in [BHH21].) Following the flow back to the $(n-1)^{\text{th}}$ surgery, item (i) of Lemma 3.5.2 shows each connected component of the t_{n-1}^+ time slice is smoothly isotopic to spheres. Reversing the surgery, the t_{n-1}^- time-slice is obtained by connecting the components present in $t_{n-1}^{\#}$) with smooth necks. Explicitly, we have the connected components present in t_{n-1}^+ and a collection of dropped components.

Claim 6.1 shows that these dropped components are diffeomorphic to spheres. No handles will be introduced when we reverse the surgery. Thus, the reversing of the surgery is a connected sum of spheres. In particular, t_{n-1}^- is smoothly isotopic to some sub-collection of the connected components, and thus isotopic to a collection of round $\mathbb{S}^3$.

By reverse induction, this is true for the initial time-slice. Since there is only one connected component, the hypersurface $\hat{\Sigma}$ is smoothly isotopic to the round $\mathbb{S}^3$. QED

Appendix

3.A Boundary Technicalities

Theorem 3.A.1. *Let $\mathcal{M}'$ be a (α, δ)-Brakke flow. Suppose $X = (\mathbf{x}, t_{\mathbf{x}}) \in \mathcal{M}' \cap \Omega_{(\alpha,\delta)}$ with $t_x \leq t_F$, where t_F is the final time surgeries are performed. Then, $\mathcal{M}' \cap P(X, \xi|H(X)|^{-1})$ is a smooth (α, δ)-flow in the sense of Haslhofer–Kleiner.*

Proof. Note, we do not need to check $Y \in P(X, \xi|H(X)|^{-1}) \cap \Omega_{(\alpha,\beta)}$, by our strict definitions of how and when surgery is performed. Since no surgeries occur outside of $\Omega_{(\alpha,\delta)}$ it is sufficient to check the flow is β-uniformly 2-convex and α-noncollapsed.

Suppose $X \in \mathcal{M}' \cap \Omega_{(\alpha,\beta)}$ and $Y = (\mathbf{y}, t_y) \in P(X, \xi|H(X)|^{-1}) \cap \Omega^c_{(\alpha,\beta)} \neq \emptyset$. From the definition of a backwards parabolic cylinder, we have that $\mathbf{y} \in B(\mathbf{x}, \xi|H(X)|^{-1})$. We may presume $(\mathbf{x}, t_{\mathbf{y}}) \in \Omega_{(\alpha,\beta)}$, as if we ever have $(\mathbf{x}, t) \in \partial\Omega_{\alpha,\beta}$ for some $t \in (t_{\mathbf{x}}, t_{\mathbf{x}} - (\xi|H(X)|^{-1})^2)$, we can use $Z = (\mathbf{x}, t)$ in the following argument.

Let L be the line segment joining $\mathbf{x}$ to $\mathbf{y}$ in the time-slice $\mathbb{R}^{n+1} \times \{t_y\}$. This line segment must pass through $\partial\Omega_{(\alpha,\beta)}$. Let $Z = (\mathbf{z}, t_y)$ denote the point on L intersecting $\partial\Omega_{(\alpha,\beta)}$. Clearly we have $|\mathbf{z} - \mathbf{y}| < |\mathbf{x} - \mathbf{y}| \leq \xi|H(X)|^{-1}$. By the maximum principle, we have $|H_{\text{bdd}}| \leq |H(X)|$, and so $Y \in P(Z, \xi|H_{\text{bdd}}|^{-1})$. By the assumption $t_x \leq t_f$, we know that at $t = t_y$, the flow $\mathcal{M}'$ remains δ-graphical over $\mathcal{M}$ in the neighbourhood of the boundary N. By the definition of N, Definition 3.2.10, we have $P(Z, \xi|H_{\text{bdd}}|^{-1}) \subset N$. In particular, by our choice of δ, at the point $Y \in \mathcal{M}'$, the flow is β-uniformly 2-convex and α-noncollapsed. QED

Chapter 4

Mean Curvature Flow through Conical Singularities

4.1 Preliminaries

Definition 4.1.1. We say a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is C^k asymptotically conical if there exists a C^k cone $\mathcal{C}$ such that

$$\lim_{\rho \to 0^+} \rho\Sigma = \mathcal{C}$$

in $C^k_{loc}(\mathbb{R}^{n+1}\backslash\{0\})$. We write $\mathcal{C}(\Sigma) = \mathcal{C}$.

Definition 4.1.2. Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone with fattening level-set flow. Suppose the outermost flows from $\mathcal{C}$ are smooth expanders. Given an orientation on $\mathcal{C}$, we label these expanders Σ^- and Σ^+ such that.

$$\begin{aligned} \Sigma^+ &\subset \mathrm{Ext}(\Sigma^-) \\ \Sigma^- &\subset \mathrm{Int}(\Sigma^+). \end{aligned}$$

We refer to Σ^-, Σ^+ as the inner and outer expanders respectively.

Definition 4.1.3. Let Σ be an asymptotically conical expander. We split Σ into two regions:

$$\begin{aligned} \Sigma_R &:= \Sigma \cap B(0,R) \\ E_R &:= \Sigma\backslash\overline{B}(0,R). \end{aligned}$$

Clearly, $\Sigma = \overline{\Sigma_R} \cup E_R$.

As we deal only with smooth expanders, we may define a simplified trace at infinity of Berstein–Wang, [BW21].

Definition 4.1.4 (d-trace at infinity)**.** Let Σ be a smooth expander, asymptotic to the cone $\mathcal{C}$. Suppose E_r can be parameterised as a graph over the cone $\mathcal{C}\backslash\overline{B}(0,R)$. Let $f \in C^\infty_{\text{loc}}(\Sigma)$ and let $g := f\circ(\pi_{\mathcal{C}})^{-1} \in C^\infty_{\text{loc}}(\mathcal{C}\backslash\overline{B}(0,R))$. We say f is asymptotically homogeneous of degree d if

$$h(p) := \lim_{\rho\to 0} \rho^d g(\rho^{-1}p) \in C^\infty_{\text{loc}}(\mathcal{C})$$

is homogeneous of degree d.

We define the d-trace at infinity of f to be

$$\text{tr}^d_\infty[f] := h|_{\mathcal{L}} \in C^\infty(\mathcal{L}(\Sigma)),$$

where $\mathcal{L}$ denotes the link of the cone $\mathcal{C}$.

Expanders have (forward) rescaled mean curvature equal to 0, and are thus fixed points of the forward rescaled flow; equivalently, they are minimal surfaces in $(\mathbb{R}^{n+1}, g_{\text{Gauss}})$. Thus, we may discuss the stability of an expander.

Definition 4.1.5 (Second Variation and Stability Operator)**.** Let $\Sigma^n \subset \mathbb{R}^{n+1}$, and let ϕ be a compactly supported normal variation, then

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \text{Vol}_{\text{Gauss}}(\phi(\Sigma, s)) = -\left\langle \phi', L_\Sigma \phi' \right\rangle,$$

where $\text{Vol}_{\text{Gauss}}$ is the Gaussian volume of Σ, $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on Σ with Gaussian weight, ϕ' is the velocity of the variation at time s and L_Σ is the operator

$$L_\Sigma = \Delta_\Sigma + \frac{\mathbf{x}}{2}\cdot\nabla_\Sigma - \frac{1}{2} + |A_\Sigma|^2.$$

L_Σ is called the *Stability* or *Jacobi* operator for Σ.

Definition 4.1.6. We say that Σ is stable if, for all compactly supported variations ϕ, we have

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \text{Vol}_{\text{Gauss}}(\phi(\Sigma, s)) = -\left\langle \phi', L_\Sigma \phi' \right\rangle \geq 0.$$

4.2 Motivation and Setup

We state our core assumptions for demonstrating uniqueness. Before doing so, we provide motivation from the existing literature as to why these are natural assumptions to make.

In [CCMS20], Chodosh–Choi–Mantoulidis–Schulze showed the outermost flows from a cone (in low dimensions) are modelled on smooth, outwardly minimising expanders. We will refer to these as the *outermost expanders*.

Theorem 4.2.1 ([CCMS20]). *For $2 \leq n \leq 6$, suppose $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a smooth cone. Then, there exists smooth expanders $\Sigma^{\pm}$, smoothly asymptotic to $\mathcal{C}$, such that the outermost flows of $\mathcal{C}$ are given by $\{\sqrt{t}\Sigma^{\pm}\}_{t\in(0,\infty)}$. Moreover, the expanders $\Sigma^{\pm}$ minimise the expander energy from the outside relative to compact perturbations.*

Remark 4.2.2. In dimensions $n \geq 7$ smoothness is not immediate, as the singular set can have co-dimension 7. Co-dimension is taken relative to the dimension of the hypersurface. The proof of Theorem 4.2.1 in [CCMS20] demonstrates the above result holds in dimensions $n \geq 7$, if the expanders happen to be smooth.

4.2.1 Jacobi Fields

In [DS20], Deruelle–Schulze show the existence of a positive Jacobi field over an expander Σ with linear growth at infinity, provided Σ can be embedded in a 1-parameter family of expanders with varying asymptotic cone. We recall their construction:

"Let $\mathcal{L}_0^{n-1} \subset \mathbb{S}^n$ be a smooth hypersurface in $\mathbb{S}^n$ and suppose Σ_0 is an expander asymptotic to the cone $\mathcal{C}(\mathcal{L}_0)$. Let $(\mathcal{L}_s)_{-\epsilon<s<\epsilon}$ be a continuously differentiable family of C^5 hypersurfaces of $\mathbb{S}^n$ and assume $(\Sigma_s)_{-\epsilon<s<\epsilon}$ is a continuously differentiable family of expanders such that $\mathcal{C}(\Sigma_s) = \mathcal{C}(\mathcal{L}_s)$. Let $\psi : \mathcal{L}_0 \to \mathbb{R}$ denote the normal variation speed at $s = 0$ of the family $\mathcal{L}_s$. Further, let $\pi_{\mathcal{L}_0} : \mathbb{R}^{n+1} \to \mathcal{L}_0$ denote the composition of the closest point projection $\pi_{C(\mathcal{L}_0)} : \mathbb{R}^{n+1} \to C(\mathcal{L}_0)$ composed with the projection $\pi : C(\mathcal{L}_0) \to \mathcal{L}_0$ on to the link."

Lemma 4.2.3 ([DS20]). *Let v be the Jacobi field induced on Σ_0 by the above variation. Then,*

$$v = r \cdot \psi \circ \pi_{\mathcal{L}_0} + w$$

where r is the ambient radius function and w satisfies

$$|\nabla^l w| \leq \frac{c}{r^{1+l}} \text{ for } l = 0, 1.$$

Remark 4.2.4. Suppose the outermost expanders $\Sigma^{\pm}$ are smooth. By varying the link of the cone to one side, we can construct 1-parameter families

$$(\Sigma_s^+)_{0\leq s<\varepsilon},\ (\Sigma_s^-)_{-\varepsilon<s\leq 0},$$

with $\Sigma_0^{\pm} = \Sigma^{\pm}$. Whilst it is not clear that either of these families are continuously differentiable in s, we can consider sequences $\{s_i\}, s_i \to 0, s_i > 0$. The outwardly minimising property of $\Sigma^{\pm}$ shows $\Sigma^{\pm}_{\pm s_i}$ converges to $\Sigma^{\pm}$ locally smoothly. This is sufficient to construct a signed Jacobi field $v^{\pm}$ on $\Sigma^{\pm}$ with the same asymptotics as Deruelle–Schulze.

4.2.2 Eigen-functions of the Linearised Operator

Suppose the expander Σ is outwardly minimising with respect to compact perturbations. This can be restated as a property of the spectrum of the stability operator.

Proposition 4.2.5 (Positivity of the first eigen-function and eigen-value)**.** *Let Σ be a smooth, asymptotically conical expander and suppose Σ is outwardly minimising with respect to compact perturbations. Then, for every $\mathfrak{R} \in (0, \infty)$, there exists a function $\phi_1^{\mathfrak{R}} \in C^\infty(\Sigma_{\mathfrak{R}})$, solving the Dirichlet eigen-value problem*

$$\begin{cases} L_\Sigma \phi_1^{\mathfrak{R}} = -\mu_1^{\mathfrak{R}} \phi_1^{\mathfrak{R}} & \text{in } \Sigma_{\mathfrak{R}} \\ \phi_1^{\mathfrak{R}} = 0 & \text{on } \Sigma \cap \partial B(0, \mathfrak{R}), \end{cases}$$

where $\mu_1^{\mathfrak{R}} > 0$ is the first eigen-value for $\Sigma_{\mathfrak{R}}$ and $\left\|\phi_1^{\mathfrak{R}}\right\|_{W^{1,2}_{L_\Sigma}(\Sigma_{\mathfrak{R}})} = 1$.

4.2.3 Assumptions

We now distil the above discussion regarding the properties of the outermost expanders into our assumptions. The properties of the expander are used to construct the barriers in Section 4.4.

Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone. We recall the definition of a smooth hypersurface with a conical singularity.

Definition 4.2.6. We say a closed set $M \subset \mathbb{R}^{n+1}$ is a smooth hypersurface with a conical singularity at 0 modelled on the cone $\mathcal{C}$ if:

1. $M \backslash \{0\}$ is a smooth hypersurface,
2. $\lim_{\rho \to \infty} \rho M = \mathcal{C}$,

where the convergence is taken in $C^\infty_{loc}(\mathbb{R}^{n+1} \backslash \{0\})$.

The work of Chodosh–Schulze, [CS21], demonstrates such hypersurfaces can occur at the singular time of a mean curvature flow with conical singularities. These hypersurfaces will be our main object of study. We state,

Assumption 4.2.1. $M_0 \subset \mathbb{R}^{n+1}$ is compact hypersurface with an isolated conical singularity at 0 modelled on $\mathcal{C}$ in the sense of Definition 4.2.6.

We next make an assumption on the Type-I behaviour of the flow from a compact initial condition, M_0, satisfying Assumption 4.2.1. Note, we do not assume that every flow from M_0 satisfies 4.2.2, however, we will only consider flows that do.

Assumption 4.2.2. Fix a smooth expander, Σ, asymptotic to $\mathcal{C}$. Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0. We suppose the tangent flow at $(\mathbf{0}, 0)$

of $\mathcal{M}$ is modelled on Σ (with multiplicity 1). That is, if $\{\lambda_i\}_{i\in\mathbb{N}}$ is a sequence with $\lambda_i \to \infty$, then for every subsequence $\{i_k\}$

$$\lim_{k\to\infty} \mathcal{D}_{\lambda_{i_k}}(\mathcal{M}) = \mathcal{M}_\Sigma.$$

This is a natural assumption to make if the only possible flow from the cone is generated by a smooth expander. For fattening cones, we demonstrate this property is satisfied by the outermost flows from M_0. In Section 4.3.2, we show flows satisfying Assumption 4.2.2 are in fact smooth and satisfy a Type-I curvature bound, Lemma 4.3.13. Readers interested only in the smooth case are invited to instead use the following assumption.

Assumption 4.2.2 B. Fix a smooth expander, Σ, asymptotic to $\mathcal{C}$. Let $\mathcal{M}$ be a mean curvature flow from M_0, smooth on $(0, \hat{T})$, $\hat{T} > 0$. We suppose $\mathcal{M}$ satisfies a Type-I curvature bound on $(0, \hat{T})$, and assume every tangent flow at $(\mathbf{0}, 0)$ of $\mathcal{M}$ is modelled on Σ.

Remark 4.2.7. A tangent flow is a subsequential limit in the sense of Brakke. It is clear that if every subsequence converges to the same limit, then the whole sequence converges and we can talk about *the* tangent flow. When the limiting flow is known to be smooth, as in our case, the regularity theorems of Brakke and White, [Bra78, Whi05], show that the convergence can be considered locally smooth away from $t = 0$.

Remark 4.2.8. The assumption of multiplicity one is vacuous. Blow-up sequences centred at the space-time origin will 'see' the initial condition. As an immediate consequence of Assumption 4.2.1, the initial time-slice of the limit will be the cone $\mathcal{C}$ with multiplicity one. The monotonicity formula yields that the subsequent flow must be multiplicity one.

Our final two assumptions are on the linearised dynamics of the expander Σ.

Assumption 4.2.3. The expander Σ is minimising with respect to compact perturbations to one side.

Assumption 4.2.4. There exists a positive, smooth function $\phi_0 : \Sigma \to \mathbb{R}$ such that $\phi_0 \nu_\Sigma$ defines a Jacobi field over Σ and ϕ_0 has linear growth at infinity.

Remark 4.2.9. Let $\psi := \mathrm{tr}^1_\infty(\phi_0)$. We take the convention $\min_{\mathbf{x}\in\mathcal{L}} \psi(\mathbf{x}) = 1$. We see there exists a function $w : \Sigma \to \mathbb{R}$ such that

$$\phi_0 = r \cdot \psi \circ \pi_{\mathcal{L}} + w$$

where r is the ambient radius function and w satisfies

$$|\nabla^l w| \leq \frac{C}{r^{1+l}} \text{ for } l \in \mathbb{N}.$$

When the Jacobi field is generated by the variational construction of Deruelle–Schulze, the 1-trace at infinity is equal to the speed of the variation of the link at $s = 0$.

The discussion at the start of the current section shows the outermost flows from $\mathcal{C}$ satisfy Assumptions 4.2.3 and 4.2.4, provided they are smooth.

4.3 Regularity and Graphicality

The reader interested in smooth solutions from a compact hypersurface with conical singularity satisfying Assumption 4.2.2 B is invited to skip to Subsection 4.3.3. Any theorems, proofs, and definitions required from earlier subsections are trivially adapted to smooth flows. Subsections 4.3.1, 4.3.2 deal with the case of unit-regular, cyclic mod 2 Brakke flows from our initial condition. Working only with smooth flows is permissible, as we demonstrate unit-regular, cyclic mod 2 Brakke flows satisfying Assumption 4.2.2 are in fact smooth on some short time interval, and satisfy a Type-I curvature bound.

In this section, we fix a smooth cone $\mathcal{C}^n \subset \mathbb{R}^{n+1}$. We assume $M_0 \subset \mathbb{R}^{n+1}, n \geq 2$ is a compact hypersurface with conical singularity modelled on $\mathcal{C}$ and $\mathcal{M}$ is a unit-regular, cyclic mod 2 Brakke flow from M_0. This is the only assumption required in Section 4.3.1. In Section 4.3.2, we make the additional assumption that every tangent flow at $(\mathbf{0}, 0)$ of $\mathcal{M}$ is equal to $\mathcal{M}_\Sigma$, where Σ is only assumed to be a smooth expander asymptotic to $\mathcal{C}$.

4.3.1 Behaviour away from the singular point

Since we make no assumptions on the behaviour of the flow near the singularity, the results in this subsection hold for any hypersurface with a conical singularity.

We recall the following standard estimate for the curvature on a hypersurface with conical singularity.

Lemma 4.3.1. *For every $\varepsilon \in (0, \frac{1}{2})$, there exists an $R_0 = R_0(M_0, \varepsilon)$ such that for $R < R_0$, the hypersurface $\mathcal{D}_{R^{-1}}(M_0)$ is ε-close in $C^{\lfloor \frac{1}{\varepsilon} \rfloor}$ to the cone $\mathcal{C}$ in the annular region $\{\overline{B}(0, 2) \backslash B(0, \frac{1}{2})\}$.*

Moreover, we can presume R_0 has been chosen sufficiently small that the second fundamental form of M_0 satisfies

$$\sup_{\mathbf{x} \in M \backslash \overline{B}(0,R)} |A|(\mathbf{x}) \leq \frac{\mathcal{A}_\mathcal{L}(1+\varepsilon)}{R},$$

Where $\mathcal{A}_\mathcal{L} = \max_{\mathbf{x} \in \mathcal{L}} |A|(\mathbf{x})$ is the maximum of the curvature of $\mathcal{L}$, the link of the cone $\mathcal{C}$.

Proof. Follows immediately from the smooth convergence of dilations of M to the cone $\mathcal{C}$ specified Definition 4.2.6. QED

The initial condition M_0 is smooth away from the conical singularity, and thus we can apply pseudolocality at every point $\mathbf{x} \in M_0 \backslash \{0\}$. As a consequence of the nearly-conical geometry, the (parabolic) scale at which pseudolocality can be applied will be proportional to $|\mathbf{x}|$, the distance from the origin. We demonstrate that these 'neighbourhoods', which we will call *pseudolocal cylinders*, can be 'patched together' into a region with moving boundary. We recall the definitions of an *n-ball* and an *n-cylinder*.

Definition 4.3.2. Let $\mathbf{x} \in \mathbb{R}^{n+1}$ and let $\Pi_{\mathrm{x}} \subset \mathbb{R}^{n+1}$ be an n-plane passing through $\mathbf{x}$. Denote by ν the normal to $\Pi_{\mathbf{x}}$. We define the n-ball

$$B^n(\mathbf{x}, r) := B(\mathbf{x}, r) \cap \Pi_{\mathbf{x}}$$

and n-cylinder

$$C(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^{n+1} \text{ s.t. } \mathbf{y} = \mathbf{z} + \alpha\nu,\ \mathbf{z} \in B^n(\mathbf{x}, r),\ |\alpha| < r\}.$$

Definition 4.3.3. To avoid confusion, we distinguish between the function $w_{\mathbf{x}_0}$, and the parabolically scaled version, $\tilde{w}_{\mathbf{x}_0}$, as follows.

$$w_{\mathbf{x}_0} : B^n(\mathbf{x}_0, \delta) \to \mathbb{R}$$

is the function constructed by applying pseudolocality at scale $C_{\mathrm{scale}}|\mathbf{x}_0|$. Undoing the dilation to return to the original scale yields the function

$$\hat{w}_{\mathbf{x}_0} : B^n(\mathbf{x}_0, C_{\mathrm{scale}}|\mathbf{x}_0|\delta) \to \mathbb{R}.$$

Proposition 4.3.4 (Existence of Pseudolocal regions)**.** *Let M_0 be a compact hypersurface with isolated conical singularity at 0, modelled on the smooth cone $\mathcal{C}$. Let $\mathcal{M}$ be any unit-regular, cyclic mod 2 Brakke flow from M_0 and let M_t denote a time-slice, for $t \in (0, T)$.*

There exists a $R_{\min} = R_{\min}(M_0) \in (0, \infty)$ such that, for each $R > 2R_{\min}$ and $\eta > 0$, one can find a time, $T = T(M_0, R, \eta) \in (0, \infty)$, for which the following holds.

For every $\mathbf{x} \in M_t \backslash B(0, R\sqrt{t})$, $t \in (0, T)$, there exists $\mathbf{x}_0 \in M_0 \backslash B(0, \frac{R}{2}\sqrt{t})$, a radius $r = r(\mathbf{x}_0, \eta)$ and a Lipschitz function

$$\hat{w}_{\mathbf{x}_0}(\cdot, t) : B^n(\mathbf{x}_0, r(\mathbf{x}_0)) \subset T_{\mathbf{x}_0}M_0 \to \mathbb{R}$$

with Lipschitz constant bounded by η, such that

$$\begin{aligned} M_t \cap C(\mathbf{x}_0, &r(\mathbf{x}_0)) = \operatorname{graph}_{B^n(\mathbf{x}_0, r)} \hat{w}_{\mathbf{x}_0}(\cdot, t), \\ \mathbf{x} =& \mathbf{x}_0 + \hat{w}_{\mathbf{x}_0}(\mathbf{x}_0, t)\nu_{M_0}(\mathbf{x}_0). \end{aligned}$$

That is to say,

$$M_t \backslash \overline{B}(0, R\sqrt{t}) \subset \bigcup_{\mathbf{x}_0 \in M_0 \backslash \overline{B}(0, \frac{R}{2}\sqrt{t})} M_t \cap C(\mathbf{x}_0, r(\mathbf{x}_0)).$$

Definition 4.3.5 (Pseudolocal Region)**.** For $R, T > 0$, we define the subset of space-time

$$\mathcal{G}_t(R) := \mathbb{R}^{n+1} \backslash \overline{B}(0, R\sqrt{t}),$$
$$\mathcal{G}(R, T) := \cup_{t \in (0,T)} \mathcal{G}_t \times \{t\}.$$

When we select $R \geq 2R_{\min}$ and $T(R)$ as in Lemma 4.3.4, we call $\mathcal{G}(R, T)$ a *Pseudolocal region.*

Proof of Lemma 4.3.4. This is a natural consequence of pseudolocality applied to smooth cones, with minor modifications for compact hypersurfaces with isolated conical singularities.

Let M_0 be a compact hypersurface with conical singularity at 0 modelled on $\mathcal{C}$ and let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0. We fix $\varepsilon \in (0, \frac{1}{5})$ to control the closeness to the cone to at least second order. Let $R_0 = R_0(\varepsilon)$ be the constant from Lemma 4.3.1, that is, R_0 is such that $\mathcal{D}_{R_1} M_0$ is ε-close to the cone in $\overline{B}(0, 2) \backslash B(0, \frac{1}{2})$ for scales $R_1 \leq R_0$.

Claim 4.3.1. Nearest point projection to $M_0 \backslash \{0\}$ is well defined and smooth in a 'tapered tubular neighbourhood', $\mathcal{T}'$, defined below.

Proof. The cone $\mathcal{C}$ is smooth, and so satisfies an interior ball condition at $\mathbf{y}_0 \in \mathcal{C}$, with $|\mathbf{y}_0| = 1$, for some radius $\mu_{\mathcal{C}}$. We can suppose $\varepsilon > 0$ has been chosen such that, for $\mathbf{x}_0 \in M_0 \cap \overline{B}(0, R_0)$, M_0 satisfies an interior ball condition at $\mathbf{x}_0$ with radius $\frac{\mu_{\mathcal{C}}|\mathbf{x}_0|}{2}$. This follows from scaling and closeness to $\mathcal{C}$ at scales $R_1 < R_0$.

Outside $B(0, R_0)$, we note $M_0 \backslash B(0, R_0)$ is smooth, and compact (with boundary). It is clear that there exists some $\mu_{R_0} > 0$ such that the interior ball condition is satisfied at $\mathbf{x}_0 \in M_0 \backslash B(0, R_0)$ with radius μ_{R_0}. We write $\mu_{R_0} = \tilde{\mu} R_0$ for some constant $\tilde{\mu} > 0$. Let

$$\mu := \min\{\frac{\mu_{\mathcal{C}}}{2}, \tilde{\mu}, \frac{1}{2}\}.$$

Nearest point projection to M_0 is hence well defined in the 'tapered tubular neighbourhood'

$$\mathcal{T}'(M_0) := \left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x} = \mathbf{x}_0 + \alpha\mu R_0 \nu_{M_0}(\mathbf{x}_0), \mathbf{x}_0 \in M_0 \backslash B(0, R_0), |\alpha| < 1\right\}$$
$$\cup \left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x} = \mathbf{x}_0 + \alpha\mu|\mathbf{x}_0|\nu_{M_0}(\mathbf{x}_0), \mathbf{x}_0 \in M_0 \cap B(0, R_0) \backslash \{0\}, |\alpha| < 1\right\}.$$

Regularity follows from the regularity of the hypersurface, M_0. QED

Claim 4.3.2. Suppose $R_1 \leq R_0$ and $\mathbf{x} \in \mathcal{T}'(M_0)\backslash B(0, 2R_1)$, then $\pi_{M_0}(\mathbf{x}) \in M_0\backslash B(0, R_1)$.

Proof. This follows from an elementary contradiction argument. Suppose one had a point $\mathbf{x} \in \mathcal{T}'(M_0)\backslash B(0, 2R_1)$ with $|\pi_{M_0}(\mathbf{x})| < R_1$. Since $\mu \leq \frac{1}{2}$, we immediately deduce a contradiction. QED

We now prove the proposition. To find the function $w_{\mathbf{x}_0}(\cdot, t)$, we proceed by a pseudolocality argument. Fix $R_1 \leq R_0$ and take $\eta \in (0, \eta_1)$, where η_1 is as in Proposition 4.B.2. Let $\epsilon, \delta > 0$ be, respectively, the initial Lipschitz constant and radial parameters of pseudolocality from [INS19] for our choice of η. We note, by Items ii), iii) of Remark 1.6 in [INS19], if the flow $\mathcal{M}$ is presumed to be unit-regular, ϵ, δ will only depend only on n, η and we do not need to consider a bound on the area ratios.

We now find cylinders on which to apply pseudolocality. Since the cone $\mathcal{C}$ is smooth, we deduce there exists some, presumably small, constant $C_{\text{scale}} = C_{\text{scale}}(\mathcal{L}, M_0, \varepsilon, \epsilon, \eta, n)$ such that for each $\mathbf{x}_0 \in M_0\backslash \overline{B}(0, R_1)$,

$$\mathcal{D}_{(C_{\text{scale}}|\mathbf{x}_0|)^{-1}}(M_0 - \mathbf{x}_0) \cap C(0, 1)$$

can be parameterised as a graph of the function $w_{0,\mathbf{x}_0} : B^n(\mathbf{x}_0, 1) \to \mathbb{R}$, over $B^n(\mathbf{x}_0, 1)$, the n-ball of radius 1 in the tangent plane $T_{\mathbf{x}_0}M_0$. We may assume C_{scale} has been chosen such that

$$C(\mathbf{x}_0, C_{\text{scale}}|\mathbf{x}_0|) \subset \mathcal{T}'(M_0). \tag{4.1}$$

We additionally presume C_{scale} has been chose small enough that the graph function $w_{0,\mathbf{x}_0}$ satisfies $\|w_{0,\mathbf{x}_0}\|_{C^2} < \epsilon$. This guarantees the bound on the Lipschitz constant of $w_{0,\mathbf{x}_0}$ is bounded by ϵ, and hence we may apply pseudolocality.

We conclude,

$$\mathcal{D}_{(C_{\text{scale}}|\mathbf{x}_0|)^{-1}}(\mathcal{M} - \mathbf{x}_0) \cap C(\mathbf{x}_0, \delta) \times [0, \delta^2), \tag{4.2}$$

can be parameterised as the graph of the function

$$w_{\mathbf{x}_0} : B^n(\mathbf{x}_0, \delta) \times (0, \delta^2) \to \mathbb{R},$$

with

$$\begin{gathered} w_{\mathbf{x}_0}(\cdot, 0) = w_{0,\mathbf{x}_0}(\cdot), \\ \mathrm{Lip}(w_{\mathbf{x}_0}) \leq \eta. \end{gathered}$$

Applying the parabolic dilation $\mathcal{D}_{C_{\text{scale}}|\mathbf{x}_0|}$ to each 'pseudolocal cylinder', we find our function $\hat{w}_{\mathbf{x}_0}$ and deduce, for $t \in (0, (C_{\text{scale}}|\mathbf{x}_0|\delta)^2)$

$$M_t \backslash \overline{B}(0, 2R_1) \subset \bigcup_{\mathbf{x}_0 \in M_0 \backslash \overline{B}(0, R_1)} M_t \cap C(\mathbf{x}_0, C_{\text{scale}}|\mathbf{x}_0|\delta). \tag{4.3}$$

This follows from equation 4.1 and claim 4.3.2.

To complete the proof, we demonstrate how to switch to the moving boundary perspective. We choose $R_{\min} = (C_{\text{scale}}\delta)^{-1}$. Fix $R \geq 2R_{\min}$. To see R satisfies the claim in the Proposition, let t_1 be such that $R\sqrt{t_1} = 2R_1$ and observe

$$t_1 = \left(\frac{2R_1}{R}\right)^2 \leq (C_{\text{scale}}R_1\delta)^2 < (C_{\text{scale}}|\mathbf{x}_0|\delta)^2.$$

Thus, for each $t \in (0, t_1]$, we may rewrite equation 4.3 as

$$M_t \backslash \overline{B}(0, R\sqrt{t_1}) \subset \bigcup_{\mathbf{x}_0 \in M_0 \backslash \overline{B}(0, \frac{R}{2}\sqrt{t_1})} M_t \cap C(\mathbf{x}_0, C_{\text{scale}}|\mathbf{x}_0|\delta).$$

Recall, $R_1 \leq R_0$ was chosen arbitrarily and $R_{\min}$ was independent of R_1. We conclude,

$$M_t \backslash \overline{B}(0, R\sqrt{t}) \subset \bigcup_{\mathbf{x}_0 \in M_0 \backslash \overline{B}(0, \frac{R}{2}\sqrt{t})} M_t \cap C(\mathbf{x}_0, C_{\text{scale}}|\mathbf{x}_0|\delta),$$

for $t \in (0, T]$, where $T = \left(\frac{2R_0}{R}\right)^2$.

QED

Following Item i) of [INS19, Remark 1.6], a curvature bound on the initial condition may be propagated forward in time by applying the Ecker–Huisken interior estimates for graphs. In the language of Proposition 4.3.4:

Corollary 4.3.6. *Let $\mathcal{M}$ satisfy the assumptions of Proposition 4.3.4. For every $\varepsilon > 0$, there exists $R_{\text{AC}} \in (0, \infty)$, a constant $C_{\text{AC}} < \infty$ and a time $T > 0$ such that, for $R > R_{\text{AC}}$, $t \in (0, T]$, we have*

$$\max_{\mathbf{x} \in M_t \backslash \overline{B}(0, R\sqrt{t})} |A|(\mathbf{x}) \leq \frac{C_{\text{AC}}}{R\sqrt{t}}.$$

Explicitly, $C_{\text{AC}} = 2C_{\text{EH}}A_{\mathcal{L}}(1 + \varepsilon)$, where C_{EH} is the constant from Ecker–Huisken [EH91, Theorem 3.1] with $\theta = \frac{1}{2}$.

Proof. The curvature bound in each pseudolocal neighbourhood follows from the interior estimates of Ecker–Huisken [EH91] by showing we have a curvature bound on the initial condition that scales accordingly. Fix $\varepsilon > 0$ and let $R_0 = R_0(\varepsilon)$ be the radius from Lemma 4.3.1. As in the proof of Proposition 4.3.4, consider $R_1 \leq R_0$.

Using Lemma 4.3.1, we observe, for $\mathbf{x} \in M_0 \backslash \overline{B}(0, R)$,

$$\max_{\mathbf{x}' \in B(\mathbf{x}, \frac{R_1}{2}) \cap M_0} |A|(\mathbf{x}') \leq \frac{2\mathcal{A}_{\mathcal{L}}(1+\varepsilon)}{R_1}.$$

The statement of the corollary follows by the discussion at the end of the proof of Proposition 4.3.4 with $R_{\mathrm{AC}} = 2R_{\min}(\varepsilon)$. QED

Remark 4.3.7. By standard parabolic methods, and the above estimate, one can presume the domain of each of the functions $w_{\mathbf{x}_0}$ has been chosen such that the functions are smooth.

4.3.2 Results for flows satisfying the tangent flow assumption

We fix a smooth expander Σ, asymptotic to $\mathcal{C}$, and now assume that the flow $\mathcal{M}$ satisfies Assumption 4.2.2 with respect to Σ. We begin by showing $\mathcal{M}$ is smooth on some time interval.

Proposition 4.3.8. *Let Σ be a smooth expander with $\mathcal{C}(\Sigma) = \mathcal{C}$. Suppose $\mathcal{M}$ is a unit-regular, cyclic mod 2 Brakke flow from M_0 satisfying Assumption 4.2.2. Then, there exists some time interval $(0, T)$, $T = T(\mathcal{M})$, on which $\mathcal{M}$ is smooth.*

Proof. Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0 satisfying Assumption 4.2.2 and suppose for contradiction that the flow is not smooth on $(0, T)$ for any $T > 0$. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence monotonically converging to 0. By our assumption, for each T_i there is a non-smooth point $X_i = (\mathbf{x}_i, t_i) \in \mathcal{M}$, $0 < t_i \leq T_i$.

As a consequence of Proposition 4.3.6 (for some choice of $\varepsilon > 0$), each $\mathbf{x}_i$ must be contained in the ball $\overline{B}(0, R_{\mathrm{AC}}\sqrt{t_i})$, as at time t_i the flow is smooth outside this ball. We hence consider the sequence of Brakke flows $\mathcal{M}^i = \mathcal{D}_{\lambda_i}(\mathcal{M})$, defined by dilating $\mathcal{M}$ around $(\mathbf{0}, 0)$ by $\lambda_i := t_i^{-\frac{1}{2}}$. Under dilation, $X_i \mapsto \hat{X}_i := (\hat{\mathbf{x}}_i, 1)$, $|\hat{\mathbf{x}}_i| \leq R_{\mathrm{AC}}$, and as a consequence there must be an accumulation point $\hat{X} \in B(0, R_{\mathrm{AC}}) \times \{1\}$.

$\mathcal{M}^i$ is, up to a bounded space-time translation, a tangent flow sequence. Thus, $\mathcal{M}^i \to \mathcal{M}_{\Sigma}$ as Brakke flows, by Assumption 4.2.2. Moreover, the limiting flow, $\mathcal{M}_{\Sigma}$, is smooth away from $(\mathbf{0}, 0)$ and hence, by the regularity theorems of Brakke [Bra78] and White [Whi05], the convergence is locally smooth away from the space-time origin. In particular, there exists an $I < \infty$ such that for $i \geq I$, $\mathcal{M}^i$ is smooth in $B(0, 2R_{\mathrm{AC}}) \times [\frac{1}{2}, \frac{3}{2}]$, a contradiction. We conclude no such sequence of non-smooth points X_i exists. QED

Definition 4.3.9 (Expander Region)**.** Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0. For $R \in (0, \infty), T < \infty$, we define the subsets of space-time

$$\Omega(R, T) := \cup_{t \in (0,T)} B(0, \sqrt{t}R) \times \{t\},$$
$$\tilde{\Omega}(R, T) := \mathcal{R}(\Omega) = B(0, R) \times (-\infty, \log(T)).$$

Note, $\tilde{\Omega}$ is the image of Ω under the transformation $\mathcal{R} : \mathbb{R}^{n+1} \times [0,T) \to \mathbb{R}^{n+1} \times (-\infty, \log(T))$, transforming mean curvature flow to rescaled mean curvature flow.

For $\varepsilon > 0$, we say $\Omega(R,T)$ is a (Σ, ε, R)-*Expander Region for* $\mathcal{M}$ if there exists a function $u \in C^{\lfloor \frac{1}{\varepsilon} \rfloor}(\Sigma_{2R} \times (-\infty, \log(T))$ such that, for $\tau \in (-\infty, \log(T))$, we have

$$RM_\tau \cap B(0,R) = \mathrm{graph}_{\Sigma_{2R}} u(\cdot, \tau) \cap B(0,R),$$

with $\|u\|_{C^{\lfloor \frac{1}{\varepsilon} \rfloor}} < \varepsilon$. i.e. each time-slice of the rescaled flow $\mathcal{RM}\lfloor\tilde{\Omega}$ can be parameterised as an ε-graph over some portion Σ.

Remark 4.3.10. We may quantify 'portion of Σ' as follows. Let $\varepsilon \in (0,1)$ and assume $\Omega(R,T)$ is a (Σ, ε, R)-expander region. For $\tau \in (-\infty, \log(T))$, we have

$$\Sigma_{R-\varepsilon} \subset \pi_\Sigma(RM_\tau \cap B(0,R)) \subset \Sigma_{R+\varepsilon},$$

where π_Σ is the nearest point projection to Σ. This is an immediate consequence of $\|u\|_{C^0} < \varepsilon$. Our graph function u is defined on Σ_{2R}, meaning the radius R is far from the boundary, thus avoiding the discussion of any unnecessary technical difficulties.

Lemma 4.3.11 (Existence of Expander Regions)**.** *Suppose $\mathcal{M}$ is a unit-regular, cyclic mod 2 Brakke flow from M_0 satisfying Assumption 4.2.2. Then, for every $\varepsilon > 0$ and $R \in (0, \infty)$, there exists a $T = T(\varepsilon, R, M_0) < \infty$ such that $\Omega(R,T)$ is a (Σ, ε, R)-expander region for $\mathcal{M}$.*

Proof. Proposition 4.3.8 demonstrates the rescaled flow $\mathcal{RM}$ is well defined and smooth on $(-\infty, \log(T))$, where T is the time from Proposition 4.3.8. The statement of the lemma is equivalent to the local smooth convergence of the rescaled flow $\mathcal{RM}$ to the expander. This is an immediate consequence of Assumption 4.2.2, as the blow-up limit is assumed to be $\mathcal{M}_\Sigma$, irrespective of the sequence chosen. QED

Definition 4.3.12 (Collar Region)**.** For $0 < r < R$, $T > 0$, we define the '*Collar Region*' as the following subset of space-time

$$\mathrm{Col}(r,R,T) := \cup_{t \in (0,T)} \left(B(0, R\sqrt{t}) \backslash \overline{B}(0, r\sqrt{t}) \times \{t\} \right).$$

By choosing the radii appropriately, and T sufficiently small, the pseudolocal region can be made to overlap with an expander region on the time interval $(0,T)$. This region of overlap is a *Collar*. When a collar region arises in this fashion, $\mathcal{M}\lfloor\mathrm{Col}(r,R,T)$ can be considered either as pseudolocal graphs over M_0 or as a graph over the expander. We will perform the welding process in a collar region of a carefully chosen length.

As an immediate corollary of Lemmas 4.3.6, 4.3.8 and 4.3.11, we get the following Type-I curvature estimate for $\mathcal{M}$ on some finite time interval.

Lemma 4.3.13. *Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0 satisfying Assumption 4.2.2. Suppose $C_\Sigma < \infty$ satisifies* $\max_{\mathbf{x}\in\Sigma} |A|(\mathbf{x}) \leq C_\Sigma$*. Then, for every $\varepsilon > 0$ there exists a $T = T(\varepsilon, M_0) > 0$ such that, for $t \in (0, T)$,*

$$\max_{\mathbf{x}\in M_t} |A|(\mathbf{x}) \leq \frac{C_\Sigma(1+\varepsilon)}{\sqrt{t}}.$$

Proof. Fix $\varepsilon > 0$ and take

$$R \geq (C_\Sigma(1+\varepsilon))^{-1} \max\{R_{\mathrm{AC}}(\varepsilon), C_{\mathrm{AC}}(\varepsilon)\},$$

where $R_{\mathrm{AC}}, C_{\mathrm{AC}}$ are as in Corollary 4.3.6. Applying said result yields $T_1 > 0$ such that the claimed curvature bound holds in $\mathcal{G}(R, T_1)$.

Applying Lemma 4.3.11, we find $T_2 > 0$ such that $\Omega(4R, T_2)$ is a $(\Sigma, \varepsilon, 4R)$-expander region. By taking $T_2 > 0$ sufficiently small, the claimed curvature bound can be assumed to hold in this region, as RM_τ converges locally smoothly to Σ as $\tau \to -\infty$. Set $T_0 = \min\{T_1, T_2\}$, and observe the discussed regions overlap in the collar region $\mathrm{Col}(R, 4R, T_0)$. We conclude the claimed curvature bound holds on $(0, T_0)$.

QED

Remark 4.3.14. Again, by standard parabolic methods, we also have bounds on the derivatives $|\nabla^k A|$.

4.3.3 Separation and graphicality of smooth flows

Having demonstrated Assumption 4.2.2 implies Assumption 4.2.2 B, we consider smooth flows from our initial hypersurface with an isolated conical singularity, M_0, with tangent flows modelled on the flow $\mathcal{M}_\Sigma$. We demonstrate two such flows separate at rate $o(\sqrt{t})$, and that we may write these flows as a graph over each other for some short time. In the following, $\mathcal{M}$ will denote the smooth space-track of our flow, defined on some time interval $(0, \hat{T})$, i.e.

$$\mathcal{M} := \{M_t\}_{t\in(0,\hat{T})}$$
$$\lim_{t\to 0} M_t = M_0.$$

Demonstrating the $o(\sqrt{t})$ rate is critical to our main argument. We will construct super/sub-solutions from M_0 relative to some flow, $\mathcal{M}^1$, satisfying Assumption 4.2.2 B. By definition, these flows will separate from $\mathcal{M}^1$ at rate $O(\sqrt{t})$. Given another flow, $\mathcal{M}^2$, from M_0 that also satisfies Assumption 4.2.2 B with respect to the same expander, the $o(\sqrt{t})$ separation yields a short time for which these super/sub-solution flows must be disjoint from from $\mathcal{M}^2$ and are thus barriers on their (longer) time interval of definition.

If $\mathcal{M}^2$ did not satisfy Assumption 4.2.2 B with respect to Σ, then we expect these flows to separate at a rate $O(\sqrt{t})$. It is helpful to consider the case when the flows have tangent flows modelled on different expanders.

Proposition 4.3.15 (Separation at rate $o(\sqrt{t})$)**.** *Let $\mathcal{M}^j, j \in \{1,2\}$, be two mean curvature flows from M_0, smooth on $(0,\hat{T})$, $\hat{T} > 0$, satisfying Assumption 4.2.2 B with respect to Σ. For every $h > 0$, there exists a $T \in (0,\hat{T}]$ such that for $t \in (0,T)$*

$$d_H(M_t^1, M_t^2) \leq h\sqrt{t}.$$

Equivalently, as $t \to 0$, the Hausdorff distance between M_t^1 and M_t^2 decays at rate $o(\sqrt{t})$.

Proof. Fix $h > 0$ and set $\varepsilon < \frac{\min\{1,h\}}{2}$. Set $R \geq \max\{R_{\mathrm{AC}}, \frac{4n(1+\eta_1)C_{\mathrm{AC}}}{h}\}$, where η_1 is the constant from Lemma 4.B.2.

We find $T > 0$ such that $\Omega(4R, T)$ is a $(\Sigma, \varepsilon, 4R)$-expander region for both $\mathcal{M}^j$ and $\mathcal{G}(R,T)$ is a pseudolocal region on which the curvature estimate Corollary 4.3.6 holds for $\mathcal{M}^j \lfloor \mathcal{G}(R,T)$.

Fix $t_0 \in (0,T)$ and consider the portion of the flow contained inside the expander region. Our choice of expander region yields C^0 estimates for the rescaled flows as graphs over the expander Σ. For $\mathbf{x} \in N_i(t_0) := RM^i_{\log(t_0)} \cap B(0,4R)$ let $\mathbf{y} \in \Sigma$ be such that $\pi_\Sigma(\mathbf{x}) = \mathbf{y}$. Then, the (unsigned) distance satisfies

$$d(\mathbf{x}, \mathbf{y} + \nu_\Sigma(\mathbf{y})u_j(\mathbf{y}, t_0)) \leq h.$$

That is, we have explicitly found a point in $RM^j_{\log(t_0)}, j \neq i$ at most distance h away. Translating this back to a statement about the non-rescaled flows, we have, for $\mathbf{x} \in M^i_{t_0} \cap B(0, 4R\sqrt{t_0})$ and $j \neq i$

$$d(\mathbf{x}, M^j_{t_0}) \leq h\sqrt{t_0}.$$

We now turn our attention to the pseudolocal region $\mathcal{G}(R,T)$. Let $X = (\mathbf{x}, t_0) \in \mathcal{M}^i \lfloor \mathcal{G}(R,T)$. Denote by $\mathbf{x}_0$ the point $\pi_{M_0}(\mathbf{x})$. Recall the function $w_{\mathbf{x_0}} : B^n(\mathbf{x}_0, \delta) \times (0, \delta^2) \to \mathbb{R}$ given by pseudolocality in the proof of Proposition 4.3.4, parameterising the flow as a graph in the pseudolocal cylinder centred at $\mathbf{x}_0 \in M_0$. We have,

$$|w_{\mathbf{x}_0}(\mathbf{x}_0, (C_{\mathrm{scale}}|\mathbf{x}_0|)^{-2}t_0)| = (C_{\mathrm{scale}}|\mathbf{x}_0|)^{-1} d(\mathbf{x}, \mathbf{x}_0)$$

where C_{scale} is the constant from the proof of Lemma 4.3.4 used for rescaling.

Observe, $w_{\mathbf{x}_0}$ evolves under the evolution equation for graphs moving by mean curvature flow, derived by Ecker–Huisken in [EH91]. Our choice of R yields the curvature bound $|A| \leq \frac{h}{4n(1+\eta)\sqrt{t}}$ in $\mathcal{G}$. We note such a curvature bound is invariant under parabolic dilation.

We deduce

$$\left|\frac{dw_{\mathbf{x}_0}}{ds}\right|(\mathbf{x}_0,t) \le \sqrt{1+|Dw_{\mathbf{x}_0}|^2(x_0,s)}|H|(\mathbf{x}_0,s) \le \frac{h}{4\sqrt{s}},$$

where, we have used that the Lipschitz constant for $w_{\mathbf{x}_0}$ is bounded by $\eta < \eta_1$, and hence $\sqrt{1+|Dw_{\mathbf{x}_0}|^2(x_0,s)} < 1+\eta$. Integrating in the time parameter s from 0 to $(C_{\text{scale}}|\mathbf{x}_0|)^{-2}t_0$ and applying the fundamental theorem of calculus, we deduce, for $\mathbf{x} \in M^i_{t_0}\backslash\overline{B}(0,R\sqrt{t_0})$

$$d(\mathbf{x},\pi_{M_0}(\mathbf{x})) \le \frac{h\sqrt{t_0}}{2}.$$

The choice $\mathbf{x}$ was arbitrary, hence this holds for all $\mathbf{x} \in M^i_{t_0}\backslash B(0,R\sqrt{t_0})$.

Thus, for $\mathbf{x} \in M^i_{t_0}\backslash\overline{B}(0,R\sqrt{t_0})$ and $j \neq i$

$$d(\mathbf{x},M^j_{t_0}) \le h\sqrt{t_0}.$$

Since $t_0 \in (0,T)$ was chosen arbitrarily, and the regions $\Omega(4R,T)$, $\mathcal{G}(R,T)$ overlap in the collar region $C(R,4R,T)$ we conclude

$$d_H(M^1_t,M^2_t) \le h\sqrt{t},\ t \in (0,T).$$

QED

Proposition 4.3.16. *Let $\mathcal{M}^j, j \in \{1,2\}$ be mean curvature flows from M_0 smooth on $(0,\hat{T})$, $\hat{T} > 0$, satisfying Assumption 4.2.2 B with respect to Σ. There exists a time $T \in (0,\hat{T}]$ and a smooth function*

$$u : \mathcal{M}^1\lfloor(0,T) \to \mathbb{R}$$

such that, for $t \in (0,T)$, M^2_t can be parameterised as a normal graph of the function $u(\cdot,t)$ over M^1_t.

Proof. Using the results from Section 4.B.1 (to which the reader is directed for the definition of common graphical atlas), we will find a time $T \in (0,\hat{T}]$ such that, for each $t \in (0,T)$, M^2_t can be parameterised as the graph of the function $u_t(\cdot) : M^1_t \to \mathbb{R}$, given explicitly by the distance function* in the direction of the normal field of M^1_t. I.e. for $\mathbf{x} \in M^1_t$ we have

$$u_t(\mathbf{x}) = d(\mathbf{x},\nu_{M^1_t}(\mathbf{x}),M^2_t).$$

The function u defined on the space-time track of the flow $\mathcal{M}^1$ up to time T,

*the notion of distance in a given direction used here is detailed in Section 4.B.1

specified in the statement of the theorem, is hence given by

$$u: \ \mathcal{M}^1\lfloor(0,T) \to \mathbb{R}$$
$$u(\mathbf{x},t) := u_t(\mathbf{x}).$$

Using the definition of the function on each time-slice, we deduce

$$u((\mathbf{x},t)) = u_t(\mathbf{x}) = d(\mathbf{x}, \nu_{M_t^1}(\mathbf{x}), M_t^2) = d((\mathbf{x},t), \mathfrak{v}, \mathcal{M}^2)$$

where $\mathfrak{v}$ is the space-time vector-field on $\mathcal{M}^1\lfloor(0,T)$ given by

$$\mathfrak{v} : \mathcal{M}^1\lfloor(0,T) \to \mathbb{R}^{n+1,1},$$
$$\mathfrak{v}((\mathbf{x},t)) := (\nu_{M_t^1}(\mathbf{x}), 0).$$

It is trivial to verify $\mathfrak{v}$ is smooth on the space-time track of $\mathcal{M}^1$ up to time T, from which we conclude u is a smooth function in $\mathbf{x}$ and t on $\mathcal{M}^1$ on the time interval $(0,T)$.

We now detail the construction of the functions u_t. This is achieved by showing there exists a time $T > 0$ such that at each time $t \in (0,T)$, we can apply Proposition 4.B.2 in each of the charts of some common graphical atlas for M_t^1 and M_t^2.

One must find a T such that the following hold.

1. There exists a $\delta > 0$ such that, for $t \in (0,T)$, $\pi_{M_t^1}$ is well defined and $d(M_t^1, \cdot)$ is smooth in the tubular neighbourhood $\mathcal{T}_{\delta\sqrt{t}}(M_t^1)$.
2. For $t \in (0,T)$, $M_t^2 \subset \mathcal{T}_{\delta\sqrt{t}}(M_t^1)$.
3. For $t \in (0,T)$, there exists a 'Common Graphical Atlas' for M_t^1, M_t^2.
4. The height and gradient bounds required in Proposition 4.B.2 hold for each function in our common graphical atlas.

Let T_0 be as in Lemma 4.B.4, T_1 as in Lemma 4.B.6 and T_δ such that Lemma 4.3.15 holds with $h = \delta$. Set $T = \min\{T_0, T_1, T_\delta\}$.

Item (1) follows immediately from Lemma 4.B.4 with $\delta = \mu$ (see [GT83]).

Item (2) follows from Lemma 4.3.15 with our choice of $h = \delta$.

Item (3) is proven in Lemma 4.B.6.

Item (4) follows from Remark 4.B.7 and pseudolocality as follows. In charts constructed by pseudolocality, we have height and gradient bounds for $w^i_{\mathbf{x}_0}$. For any choice of $\eta > 0$, we have, by scaling,

$$\left\|\hat{w}^i_{\mathbf{x}_0}(\cdot,t)\right\|_{C^0(B^n(\mathbf{x}_0,\delta C_{\mathrm{scale}}|\mathbf{x}_0|))} < \eta\delta C_{\mathrm{scale}}|\mathbf{x}_0|,$$
$$\left\|\nabla\hat{w}^i_{\mathbf{x}_0}(\cdot,t)\right\|_{C^0(B^n(\mathbf{x}_0,\delta C|\mathbf{x}_0|))} < \eta.$$

In charts derived from local smooth convergence to the expanding flow $\mathcal{M}_\Sigma$, we use Remark 4.B.7 to deduce

$$\begin{aligned}\left\|\hat{w}^i_{\mathbf{x}_0}(\cdot,t)\right\|_{C^0(B^n(\mathbf{x}_0,r\sqrt{t}))} &< \eta r\sqrt{t},\\ \left\|\nabla\hat{w}^i_{\mathbf{x}_0}(\cdot,t)\right\|_{C^0(B^n(\mathbf{x}_0,r\sqrt{t}))} &< \eta.\end{aligned}$$

Thus, choosing $\eta < \eta_1$, Proposition 4.B.2 can be applied. We deduce, for every $t \in (0,T)$ and $\mathbf{y} \in M^2_t$, there exists an open set $V_{\mathbf{y},t}$, a point $\mathbf{x} \in M^1_t$ and open set $V'_{\mathbf{y},t}$ such that

$$M^2_t \cap V_{\mathbf{y},t} = \mathrm{graph}_{V'_{\mathbf{x},t}} u$$

where $u(\mathbf{x},t) = d(\mathbf{x}, M^2_t, \nu_{M^1_t}(\mathbf{x}))$. In each neighbourhood, the function is derived from only extrinsic data and does not depend on the open sets. We may apply a standard patching argument to deduce

$$M^2_t = \mathrm{graph}_{M^1_t} u(\cdot,t).$$

QED

4.4 Barriers from Linearised Dynamics

Our goal is to show two mean curvature flows from M_0 satisfying Assumption 4.2.2 are equal if Σ satisfies Assumptions 4.2.3 and 4.2.4. In this section, we will construct the barriers over the expander region using our assumptions on the linearised dynamics.

Recall, the stability of the expander Σ ensures that, on any compact domain, the first Dirichlet eigen-value of the stability operator L_Σ is positive, moreover, the associated eigen-function, $\phi^{\mathfrak{R}}_1$, is positive on said compact domain. We hence define, for a radius $\mathfrak{R} \in (0,\infty)$ and 'growth correction parameter' $\alpha \in (0,\infty)$, the functions

$$\begin{aligned}f^{\mathfrak{R}}_\alpha &: \Sigma_{\mathfrak{R}} \to \mathbb{R}\\ f^{\mathfrak{R}}_\alpha &:= \phi_0 + \alpha\phi^{\mathfrak{R}}_1,\end{aligned}$$

where ϕ_0 was the function defining a Jacobi field on Σ with linear growth. Clearly, such a function satisfies

$$L_\Sigma f^{\mathfrak{R}}_\alpha = -\alpha\mu^{\mathfrak{R}}_1\phi^{\mathfrak{R}}_1,$$

and by standard linearised methods, we see the hypersurfaces defined by the graphs of $sf^{\mathfrak{R}}_\alpha$ over Σ_R (for $R < \mathfrak{R}$) will have positive rescaled mean curvature for sufficiently small $s > 0$.

The reader familiar with such barrier constructions will have noted that stability alone is sufficient to construct barriers over a compact region, and hence may wonder why we add a Jacobi field. As briefly mentioned in the introduction, the linear growth afforded by the Jacobi field is required in order to employ a barrier welding argument. In the second part of this section, we demonstrate that the radii R, $\mathfrak{R}$, and parameter α may be selected to ensure the functions $f_\alpha^{\mathfrak{R}}$ grow sufficiently between R and $C_{\text{len}}R$. The details of the welding are contained in the uniqueness argument in Section 4.6 and the construction of entire barriers over Σ in Section 4.7.

Finally, we note the functions described above yield barriers relative to the expander Σ, a property we make use of in Section 4.7. This is, however, not sufficient for the proof of uniqueness, for which we require barriers relative to some smooth flow, $\mathcal{M}^1$, from M_0 satisfying our tangent flow assumption. To construct barriers that achieve this, at least in the expander region, we add the function u_1 to $f_\alpha^{\mathfrak{R}}$, where u_1 is the function parameterising the flow $\mathcal{RM}^1$ as a graph over Σ. We control the error introduced through this by noting the linearisation error splits 'binomially'.

4.4.1 Existence of Barriers

As commented above, we use the function $f_\alpha^{\mathfrak{R}}$ and the local graphical representation of some flow $\mathcal{RM}^1$ over the expander Σ to construct barriers. We first specify the region where the barriers will be defined.

Definition 4.4.1 (Length of the Collar Region)**.** We set

$$C_{\text{len}} := \frac{21 \max_{\mathbf{x}\in\mathcal{L}} \psi(\mathbf{x})}{4 \min_{\mathbf{x}\in\mathcal{L}} \psi(\mathbf{x})} = \frac{21 \max_{\mathbf{x}\in\mathcal{L}} \psi(\mathbf{x})}{4}$$

where ψ is 1-trace at infinity of ϕ_0. Recall, we take the normalisation convention that $\min \psi = 1$.

This choice of collar length is used in Lemma 4.6.3 to show the welding process 'hides' the boundaries of the barriers.

Definition 4.4.2 (Barrier functions)**.** Let $\mathcal{M}^1$ be a smooth flow from M_0 and suppose $\mathcal{M}^1$ satisfies Assumption 4.2.2. Fix $R \in (1,\infty)$, $\mathfrak{R} \in (2C_{\text{len}}R, \infty)$ and $\varepsilon_0 \in (0, \frac{1}{5})$. By Lemma 4.3.11, we find a $T_0 > 0$ such that $\Omega(C_{\text{len}}R, T_0)$ is a $(\Sigma, \varepsilon_0, C_{\text{len}}R)$-expander region for $\mathcal{M}$. Let $u_1 \in C^{\lfloor \frac{1}{\varepsilon} \rfloor}(\Sigma_{2C_{\text{len}}R} \times (-\infty, \log(T_0))$ be the function parameterising $\mathcal{RM}^1 \lfloor \tilde{\Omega}$ as a graph. For each $s \in (0,1)$ and $\alpha \in (0,\infty)$ we define the functions

$$\begin{aligned} u^{\pm}_{s,\alpha,R,\mathfrak{R}} &: \Sigma_{C_{\text{len}}R} \times (-\infty, \log(T_0)) \to \mathbb{R} \\ u^{\pm}_{s,\alpha,R,\mathfrak{R}}(\mathbf{x},\tau) &:= u_1(\mathbf{x},\tau) \pm s f_\alpha^{\mathfrak{R}}(\mathbf{x}). \end{aligned}$$

Note, it is not immediate that $u^{\pm}_{s,\alpha,R,\mathfrak{R}}$ defines a smooth hypersurface.

Remark 4.4.3. We suppress $\alpha, R, \mathfrak{R}$ in the notation, i.e. we write $u_s^{\pm}, \Gamma_s^{\pm}$, as these parameters will remain fixed once chosen.

Proposition 4.4.4. *Let $\mathcal{M}, \varepsilon_0, R, \mathfrak{R}$ be as in Definition 4.4.2. For every choice of $\alpha \in (0, \infty)$, there exists $s_0 = s_0(\alpha, \xi, R, \mathfrak{R}) \in (0,1)$, and $T_1 \in (0, T_0]$ such that for $s \in [0, s_0], \tau \in (-\infty, \log(T_1))$ the following holds.*

1. *The normal graph of $u^{\pm}_{s,\alpha,R,\mathfrak{R}}$ over $\Sigma_{C_{\text{len}}R}$,*

$$\Gamma_s^{\pm}(\tau) := \text{graph}_{\Sigma_{C_{\text{len}}R}} u^{\pm}_{s,\alpha,R,\mathfrak{R}}(\cdot, \tau),$$

   defines a smooth hypersurface.

2. *The hypersurfaces $\Gamma_s^{\pm}(\tau)$ satisfy*

$$v_{\Gamma_s^+}(\mathbf{x}, \tau)\left(\partial_\tau \mathbf{x}_{\Gamma_s^+} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{\Gamma_s^+}\right) \cdot \nu_{\Gamma_s^+} \geq 0 \tag{4.4}$$

$$v_{\Gamma_s^-}(\mathbf{x}, \tau)\left(\partial_\tau \mathbf{x}_{\Gamma_s^-} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{\Gamma_s^-}\right) \cdot \nu_{\Gamma_s^-} \leq 0. \tag{4.5}$$

Proof. Fix a choice of $\alpha \in (0, \infty)$. We prove the claims for u_s^+, Γ_s^+. We suppress the '+' in the proof to reduce notational complexity. The proof for u_s^-, Γ_s^- is similar.

Recall, from Appendix 4.A, in order for the graph of function $g : M \to \mathbb{R}$ to define a hypersurface and to calculate geometric quantities in terms of g, it is required Assumption 4.A.1 is satisfied for a chosen value of $\xi < 1$ †. That is, we require

$$\max\{|A||u_1|, |\nabla u_1|^2\} < \frac{\xi}{4} \tag{4.6}$$

$$\max\{|A||sf_\alpha^{\mathfrak{R}}|, |\nabla sf_\alpha^{\mathfrak{R}}|^2\} < \frac{\xi}{4} \tag{4.7}$$

$$\max\{|A||u_s|, |\nabla u_s|^2\} < \frac{\xi}{4} \tag{4.8}$$

where the maximum is taken over $\Sigma_{C_{\text{len}}R} \times (-\infty, \log(T_1)]$ for some $T_1 > 0$ and $\xi \in (0,1)$. We may presume 4.6 holds on $(-\infty, \log(T_0)]$ by the choice of ε_0. Since $f_\alpha^{\mathfrak{R}}$ is a smooth function, on the bounded set $\Sigma_{\mathfrak{R}}$, we can pick s_0 such that 4.7 holds in $\Sigma_{C_{\text{len}}R}$. Finally, having fixed s_0, 4.8 holds by taking T_1 sufficiently small since $u_1 \to 0$ locally smoothly as $\tau \to 0$.

It remains to show the inequality 4.4 holds if T_1, s_0 are taken sufficiently small. This is achieved by using the linearised rescaled mean curvature flow equation. Using Corollary 4.A.5, we calculate,

† The value of ξ will be fixed in Section 4.6.

$$
\begin{aligned}
v_{\Gamma_s}(\mathbf{x},\tau)\left(\partial_\tau \mathbf{x}_{\Gamma_s} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{\Gamma_s}\right)\cdot \nu_{\Gamma_s} &= \partial_\tau u(\mathbf{x},\tau) - L_\Sigma(u + s f_\alpha^{\mathfrak{R}}) \\
&\quad - E(u + s f_\alpha^{\mathfrak{R}}) \\
&= s\alpha\mu_1^{\mathfrak{R}}\phi_1^{\mathfrak{R}} - E(s f_\alpha^{\mathfrak{R}}) - Q_{\mathrm{mixed}}(u, s f_\alpha^{\mathfrak{R}}).
\end{aligned}
$$

Note, we have used $\partial_\tau u_1 = L_\Sigma(u) + E(u)$. In addition, Corollary 4.A.5 and Theorem 4.A.10 show

$$
\begin{aligned}
&|E(s f_\alpha^{\mathfrak{R}})| \le C_E C_{f_\alpha^{\mathfrak{R}}}^2 s^2 \\
&|Q_{\mathrm{mixed}}(u, s f_\alpha^{\mathfrak{R}})| \le C_Q C_{f_\alpha^{\mathfrak{R}}} \|u_1(\cdot,\tau)\|_{C^2(\Sigma_{C_{\mathrm{len}}R})} s,
\end{aligned}
$$

where $C_{f_\alpha^{\mathfrak{R}}} := \|f_\alpha^{\mathfrak{R}}\|_{C^2(\Sigma_{C_{\mathrm{len}}R})}$. Taking s_0 satisfying

$$
s_0 < \frac{\alpha\mu_1^{\mathfrak{R}}}{2C_E C_{f_\alpha^{\mathfrak{R}}}^2} \min_{\mathbf{x}\in\Sigma_{C_{\mathrm{len}}R}} \phi_1^{\mathfrak{R}}
$$

and T_1 sufficiently small that for $\tau \in (-\infty, \log(T_1))$

$$
\|u_1(\cdot,\tau)\|_{C^2(\Sigma_{C_{\mathrm{len}}R})} < \frac{\alpha\mu_1^{\mathfrak{R}}}{2C_Q C_{f_\alpha^{\mathfrak{R}}}} \min_{\mathbf{x}\in\Sigma_{C_{\mathrm{len}}R}} \phi_1^{\mathfrak{R}},
$$

we see $s\alpha\mu_1^{\mathfrak{R}}\phi_1^{\mathfrak{R}} > 0$ dominates and the claimed inequality holds.

QED

4.4.2 Asymptotic properties of the barriers

We record an elementary result demonstrating we may pick $\alpha \in (0,\infty)$ such that the function $f_\alpha^{\mathfrak{R}}$ has 'almost linear' behaviour in our chosen collar. This is important for the welding of barriers.

Definition 4.4.5. For each $\delta > 0$ we define $R_{\mathrm{decay}} = R_{\mathrm{decay}}(\delta) < \infty$ to be the radius such that for $\mathbf{x} \in E_{R_{\mathrm{decay}}}$,

$$
|\phi_0 - r\cdot\psi\circ\pi_{\mathcal{L}_0}|(\mathbf{x}) < \delta.
$$

The existence of such a radius is a consequence of the asymptotics of ϕ_0.

Lemma 4.4.6 (Linear growth in Collar region for $f_\alpha^{\mathfrak{R}}$)**.** *For every $\delta \in (0,\frac{1}{4})$, $R > R_{\mathrm{decay}}(\frac{\delta}{2})$ and $\mathfrak{R} \in (2C_{\mathrm{len}}R,\infty)$, there exists $\alpha_0 = \alpha_0(\delta, R, \mathfrak{R}) \in (0,\infty)$ such that for $\alpha \in (0,\alpha_0)$, we have*

$$
|f_\alpha^{\mathfrak{R}} - r\cdot\psi\circ\pi_{\mathcal{L}_0}|(\mathbf{x}) < \delta \text{ for } \mathbf{x}\in\Sigma_{C_{\mathrm{len}}R}\cap E_R.
$$

Remark 4.4.7. This lemma can be interpreted as a statement regarding the C^0 closeness of $\mathrm{graph}_\Sigma(\pm s f^{\mathfrak{R}}_\alpha)$ to the cone $\mathcal{C}^\pm(s)$ in the collar region.

Proof. Fix $\delta \in (0, \frac{1}{4})$ and $R > R_{\mathrm{decay}}(\frac{\delta}{2})$. By Definition 4.4.5, we see

$$|\phi_0 - r \cdot \psi \circ \pi_{\mathcal{L}_0}|(\mathbf{x}) < \frac{\delta}{2} \text{ for } \mathbf{x} \in E_R.$$

Let $C_{\phi_1^{\mathfrak{R}}} := \max_{\mathbf{x} \in \Sigma_{\mathfrak{R}}} \phi_1^{\mathfrak{R}}$ and pick $\alpha_0 = \frac{\delta}{2C_{\phi_1^{\mathfrak{R}}}}$. Thus, for $\alpha \in (0, \alpha_0)$ we have

$$|\alpha \phi_1^{\mathfrak{R}}|(\mathbf{x}) < \frac{\delta}{2} \text{ for } \mathbf{x} \in \Sigma_{C_{\mathrm{len}}R} \cap E_R.$$

The claim follows from the triangle inequality. QED

4.5 Separation Estimate

In Section 4.4, we constructed a graph over the expander Σ that acts as a barrier in the expander region relative to the flow $\mathcal{M}^1$. Since this definition explicitly depends on functions defined on a portion of Σ, the barrier is defined only locally, and it is not clear that there is a natural way to extend this to a barrier over the entire flow.

Instead, we construct a barrier in the pseudolocal region that can be welded to the other barrier in the collar region. Our barrier takes the form of separation estimate. We show that a bound on the height of the form $|u| \le h\sqrt{t}$ along the boundary of the ball $B(0, R\sqrt{t})$ on the time interval $(0, T)$ can be propagated to all of $\mathcal{M}^1 \lfloor \mathcal{G}(R, T)$. Here, u denotes the function parameterising M_t^2 as a graph over M_t^1.

This is motivated by the observation in Proposition 4.B.10, where it is shown the barriers from Section 4.4 provide a bound for function $\tilde{u}$ (parameterising $\mathcal{RM}^2$ as a graph over $\mathcal{RM}^1$) at radius R that is independent of rescaled time. Transforming back to the non-rescaled flow, this corresponds to a bound of the form $|u| \le h\sqrt{t}$ on the boundary of the ball $B(0, R\sqrt{t})$.

Theorem 4.5.1. *Suppose $M_0 \subset \mathbb{R}^{n+1}$ satisfies Assumption 4.2.1. Let $\mathcal{M}^1$ and $\mathcal{M}^2$ be mean curvature flows from M_0, smooth on $(0,T)$ and satisfying Assumption 4.2.2 B with respect to the chosen expander Σ. Suppose there exists a radius $R > 0$ such that the curvature of $\mathcal{M}^1$ satisfies*

$$\max_{(\mathbf{x},t) \in \mathcal{M}^1 \text{ s.t } |\mathbf{x}| \ge R\sqrt{t}} |A|^2(\mathbf{x}, t) < \frac{1}{2(1+D)t}, \; t \in (0, T), \tag{4.9}$$

where $D = D^{\ddagger}(\xi, n) \in (0, \infty)$ is as in Proposition 4.A.9.

‡Choice of ξ dictated in Section 4.6.

Finally, suppose $u : \mathcal{M}^1 \to \mathbb{R}$ *is a smooth function such that*

$$M_t^2 = \mathrm{graph}_{M_t^1} u(\cdot, t),\ t \in (0, T),$$

with $\max_{\mathbf{x} \in M_t^1} |A||u|(\mathbf{x}, t) < \xi < 1$ *on* $(0, T)$.

If the height of u *satisfies*

$$\max_{\{(\mathbf{x},t) \in \mathcal{M}^1 \text{ s.t } |\mathbf{x}| = R\sqrt{t}\}} |u|(\mathbf{x}, t) < h\sqrt{t},\ t \in (0, T), \tag{4.10}$$

where $h > 0$, *then,*

$$\max_{\{\mathbf{x} \in M_t^1 \text{ s.t } |\mathbf{x}| \geq R\sqrt{t}\}} |u|(\mathbf{x}, t) < h\sqrt{t},\ t \in (0, T).$$

Proof. One can verify the graph of $\pm h\sqrt{t}$ over $M_t^1 \backslash B(0, R\sqrt{t})$ is a super/sub-solution to mean curvature flow starting from M_0 using the linearisation of the geometric quantities. The claim then follows from the avoidance principle with boundary data; a full proof is provided for completeness.

We proceed using a maximum principle argument. We seek to show contradiction if $|u| \geq h\sqrt{t}$ occurs at an interior point. To do this, one first needs to show that the inequality is not immediately violated. We recall, by Lemma 4.3.15, u decays like $o(\sqrt{t})$ as $t \to 0$. That is, for any sufficiently small $\vartheta > 0$ there is some time interval $(0, t_0)$, $t_0 = t_0(\vartheta)$, for which

$$|u|(\cdot, t) < (h - \vartheta)\sqrt{t}, t \in (0, t_0].$$

Suppose for contradiction that the claimed bound does not hold for all $t \in (0, T)$. By the above reasoning, there must be some first time, $t_1 \in (t_0, T)$ at which the inequality is violated in $M_{t_1}^1 \backslash \overline{B}(0, R\sqrt{t_1})$. We call such a time the 'first touching time'. By Assumption 4.10, this must occur at an interior point of $M_{t_1}^1 \backslash \overline{B}(0, R\sqrt{t_1})$. This point is hence a local, spatial, critical point of u. We suppose this is a positive local maximum of u. Denote this point $(\mathbf{x}_{\max}(t_0), t_0)$. The proof for a negative minimum follows similarly.

Since $|A||u| < \xi$, Proposition 4.A.9 shows that at a positive local maximum of u, the bound on the error term takes a particularly useful form. At a local, positive, maximum of u we have

$$\frac{\partial u}{\partial t}(x_{\max}(t_1), t_1) \leq (1 + D)(|A|^2 u)(\mathbf{x}_{\max}(t_1), t_1).$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}(u - h\sqrt{t})(\mathbf{x}_{\max}(t_1), t_1) \leq &(1+D)|A|^2 u(\mathbf{x}_{\max}(t_1), t_1) - \frac{h}{2\sqrt{t_1}} \\ < &\frac{1}{2t_1} u(\mathbf{x}_{\max}(t_1), t_1) - \frac{h}{2\sqrt{t_1}}.\end{aligned}$$

Where we have used the curvature bound, Assumption 4.9. Factoring the above, we deduce

$$\frac{\partial}{\partial t}(u - h\sqrt{t})(\mathbf{x}_{\max}(t_1), t_1) < \frac{1}{2t_1}(u(\mathbf{x}_{\max}(t_1), t_1) - h\sqrt{t_1}) = 0,$$

with the final equality coming from $u(\mathbf{x}_{\max}(t_1), t_1) - h\sqrt{t_1} = 0$.

Recall, we supposed $t_1 > 0$ was the first touching time. Thus, we must have

$$\frac{\partial}{\partial t}(u - h\sqrt{t})(\mathbf{x}_{\max}(t_1), t_1) \geq 0,$$

a contradiction.

The theorem follows by noting the same argument applied to $h\sqrt{t} - u$ establishes a lower bound at a negative, local minima of u. QED

4.6 Uniqueness

We now prove our uniqueness theorem. In the following, we suppose $\mathcal{C}$ is a smooth cone and M_0 satisfies Assumption 4.2.1. That is, M_0 is a compact hypersurface with isolated conical singularity modelled on $\mathcal{C}$. We fix Σ, a smooth expander with $\mathcal{C}(\Sigma) = \mathcal{C}$, satisfying Assumptions 4.2.3 and 4.2.4. Finally, we assume $\mathcal{M}^1$ is a mean curvature flow from M_0, smooth on some time interval $(0, \hat{T})$ satisfying Assumption 4.2.2 B with respect to Σ, i.e. the tangent flow of $\mathcal{M}$ at $(\mathbf{0}, 0)$ is $\mathcal{M}_\Sigma$, and satisfies a Type-I curvature estimate. Recall, in Lemma 4.3.13, we demonstrated this includes any unit-regular, cyclic mod 2 Brakke flow from M_0, satisfying Assumption 4.2.2.

In order to apply the tools developed in the previous sections, we need to fix our free parameters. For the results contained in the appendices, we fix the graph height parameter, $\vartheta \in (1, 2)$ and 'low-energy' parameter $\xi \in (0, 1)$. We take $\delta \in (0, \frac{1}{4})$, which in turn fixes the radius $R_{\text{decay}}(\delta/2)$ in Definition 4.4.5. Fix $\varepsilon_0 \in (0, \frac{1}{5})$ and let $R_{\text{AC}}(\varepsilon_0)$ be as in Corollary 4.3.6.

We now take

$$R > \max\left\{1, R_{\text{decay}}, R_{\text{AC}}, \frac{\sqrt{2(1+D)}}{C_{\text{AC}}}\right\} \tag{4.11}$$

$$\mathfrak{R} > 2C_{\text{len}} R \tag{4.12}$$

where D is as in Proposition 4.A.9. We additionally assume $\varepsilon \in (0, \varepsilon_0)$ is taken small enough so we may apply the results of Section 4.B.2. Having fixed $\delta, R, \mathfrak{R}$, we find $\alpha_0 \in (0,\infty)$ satisfying the conclusions of Lemma 4.4.6, allowing us to fix $\alpha \in (0,\alpha_0)$. We take $T \in (0,\hat{T})$ such that Corollary 4.3.6 holds on $(0,T)$ and $\Omega(C_{\text{len}}R, T)$ is a $(\Sigma, \varepsilon, C_{\text{len}}R)$-expander region for $\mathcal{M}^1$. The above choice of R and T ensures Corollary 4.3.6 holds with at least the curvature bound required in Theorem 4.5.1 and the function $f_\alpha^{\mathfrak{R}}$ satisfies Lemma 4.4.6 for $\alpha \in (0,\alpha_0)$.

Finally, with the above choices of $\varepsilon, R, \mathfrak{R}$ and α, we use Definition 4.4.2 to define the barrier functions $u^{\pm}_{s,\alpha,R,\mathfrak{R}}$ and use Proposition 4.4.4 to pick $T_1 > 0$ and $s_0 > 0$ such that $\{\Gamma^{\pm}_{s,R,\mathfrak{R}}(\tau)\}_{\tau\in(-\infty,\log(T_1))}$ satisfies the conclusion of Proposition 4.4.4 for $s \in (0, s_0]$. For simplicity, we may assume T has been taken sufficiently small $T = T_1$.

Definition 4.6.1 (Barriers for Rescaled Flows)**.** Let $\mathcal{M}'$ be another mean curvature flow from M_0, smooth on $(0,T)$, satisfying Assumption 4.2.2 B with respect to Σ. Suppose that $\Omega(C_{\text{len}}R, T)$ is a $(\Sigma, \varepsilon, C_{\text{len}}R)$-expander region for $\mathcal{M}'$. Let

$$u' : \Sigma_{C_{\text{len}}R} \times (-\infty, \log(T)) \to \mathbb{R}$$

be the function parameterising $\mathcal{RM}'$ as a graph over the expander in the chosen expander region.

We say the family $\{\Gamma^{\pm}_s(\tau)\}_{\tau\in(-\infty,\tau_0)}$ is a *barrier for $\mathcal{RM}'$ on the time interval* $(-\infty, \tau_0)$, $\tau_0 \leq \log(T)$ if

$$u^-_s(\mathbf{x}, \tau) < u'(\mathbf{x}, \tau) < u^+_s(\mathbf{x}, \tau)$$

holds for each $(\mathbf{x}, \tau) \in \overline{\Sigma_{C_{\text{len}}R}} \times (-\infty, \tau_0)$.

We now show that for any other flow, $\mathcal{M}'$, from M_0 satisfying Assumption 4.2.2 B with respect to Σ, there exists a time interval $I = (-\infty, \tau_{s_0}(\mathcal{M}'))$ such that $\mathcal{M}'$ remains between the barriers $\{\Gamma^{\pm}_{s_0}(\tau\}_{\tau\in I}$.

Lemma 4.6.2. *Suppose $\mathcal{M}^2$ is another mean curvature flow from M_0, smooth on $(0,T)$, satisfying Assumption 4.2.2 B with respect to Σ. Then, for each $s \in (0, s_0]$ there exists a rescaled time $\tau_s \in (-\infty, \log(T)]$ such that $\{\Gamma^{\pm}_s(\tau)\}_{\tau\in I}$ is a barrier for $\mathcal{RM}^2$ on $(-\infty, \tau_s)$.*

Proof. This follows from the avoidance principle for ancient rescaled flows, provided we can find some time τ_s, before which $\mathcal{RM}^2$ and $\{\Gamma^{\pm}_s(\tau)\}_{\tau\in(-\infty,\tau_s)}$ are disjoint along the boundary $\partial\Sigma_{C_{\text{len}}R} \times (-\infty, \tau_s)$. Clearly such a τ_s exists: the rescaled flow $\mathcal{RM}^2$ converges locally smoothly to Σ as $\tau \to -\infty$, and $u^{+(-)}_s \to +(-)sf^{\mathfrak{R}}_\alpha > 0$ (resp. < 0) as $\tau \to -\infty$, on its domain of definition. QED

Using a welding argument, we can improve this time interval to the entire interval on which $\Gamma^{\pm}_s(\tau)$ satisfies Proposition 4.4.4.

Lemma 4.6.3. *Suppose $\mathcal{M}^2$ is another mean curvature flow from M_0, smooth on $(0,T)$ satisfying Assumption 4.2.2 B with respect to Σ. Suppose T is such that $\Omega(C_{\mathrm{len}}R, T)$ is a $(\Sigma, \varepsilon, C_{\mathrm{len}}R)$-expander region for $\mathcal{M}^2$. Further, we suppose M_t^2 can be parameterised as a normal graph over M_t^1, with $|u||A_{M_t^1}| < \xi$, for $t \in (0,T)$. Then, for every $s \in (0, s_0]$, $\{\Gamma_s^{\pm}(\tau)\}_{\tau \in (-\infty, \log(T))}$ is barrier for $\mathcal{RM}^2$ on $(-\infty, \log(T))$.*

Remark 4.6.4. Of course, by Lemma 4.3.13, Proposition 4.3.15 and Proposition 4.3.16, we can always find a $T > 0$ such that M_t^2 can be parameterised as normal graph over M_t^1 with $|u||A| < \xi$ for $t \in (0,T)$.

Proof. Fix $s \in (0, s_0]$ and write $\log(T) = \tau_0$. We suppose for contradiction that $\{\Gamma_s^{\pm}(\tau)\}_{\tau \in (-\infty, \tau_0)}$ is not a barrier for $\mathcal{RM}^2$ on $(-\infty, \tau_0)$. Appealing to Lemma 4.6.2, there exists a time τ_s on which $\{\Gamma_s^{\pm}(\tau)\}_{\tau \in (-\infty, \tau_s)}$ is a barrier to $\mathcal{RM}^2$ on $(-\infty, \tau_s)$. We may presume $\tau_s < \tau_0$, else there is an immediate contradiction. We deduce there must be a first time, $\tau_1 \in (\tau_s, \tau_0)$, such that either

$$u_2(\mathbf{x}_1, \tau_1) = u_s^-(\mathbf{x}_1, \tau_1)$$

or

$$u_2(\mathbf{x}_1, \tau_1) = u_s^+(\mathbf{x}_1, \tau_1)$$

for some $\mathbf{x}_1 \in \Sigma$, $|\mathbf{x}_1| = C_{\mathrm{len}}R$.

Equivalently, there exists a point $\mathbf{x}_1 \in \Sigma$, $|\mathbf{x}_1| = C_{\mathrm{len}}R$ with

$$|u_2 - u_1|(\mathbf{x}_1, \tau_1) = s f_\alpha^{\mathfrak{R}}(\mathbf{x}_1),$$

from which we deduce

$$\begin{aligned} |u_2 - u_1|(\mathbf{x}_1, \tau_1) &= s f_\alpha^{\mathfrak{R}}(\mathbf{x}_1) \\ &\geq s \min_{|\mathbf{y}| = C_{\mathrm{len}}R} f_\alpha^{\mathfrak{R}}(\mathbf{y}) \\ &> s(C_{\mathrm{len}}R \min_{\mathbf{x} \in \mathcal{L}} \psi(\mathbf{x}) - \frac{1}{4}), \end{aligned}$$

where the last inequality comes from Lemma 4.4.6. Observe, using Definition 4.4.1 and the normalisation of ψ, we have

$$C_{\mathrm{len}}R \min_{\mathbf{x} \in \mathcal{L}} \psi(\mathbf{x}) - \frac{1}{4} = \frac{1}{4}(21R \max_{\mathbf{x} \in \mathcal{L}} \psi(\mathbf{x}) - 1) \geq 5R \max_{\mathbf{x} \in \mathcal{L}} \psi(\mathbf{x}).$$

We will deduce a contradiction to the last inequality using our separation estimate. By the above reasoning, $\{\Gamma_s^{\pm}(\tau)\}_{\tau \in (-\infty, \tau_1)}$ is a barrier to $\mathcal{RM}^2$ on $(-\infty, \tau_1)$.

For $\tau \in (-\infty, \tau_1]$, we have

$$\max_{|\mathbf{x}|=R} |u_2 - u_1|(\mathbf{x}, \tau) \leq s \max_{|\mathbf{x}|=R} f_\alpha^{\mathfrak{R}}(\mathbf{x}).$$

Using Theorem 4.B.10, we deduce, for $t \in (0, \exp(\tau_1)]$,

$$\max_{\mathbf{x} \in M_t^1, |\mathbf{x}|=R\sqrt{t}} |u|(\mathbf{x}, t) \leq \vartheta s \max_{|\mathbf{y}|=R} f_\alpha^{\mathfrak{R}}(\mathbf{y})\sqrt{t},$$

where $u : \mathcal{M}^1 \to \mathbb{R}$ is the function parameterising $\mathcal{M}^2$ as a normal graph.

Recall, R and T were chosen at the start of this section such that we may apply Theorem 4.5.1. Recalling our choice of $\vartheta \in (1, 2)$, we set $h = 2s \max_{|\mathbf{y}|=R} f_\alpha^{\mathfrak{R}}(\mathbf{y})$ and deduce, for $t \in (0, \exp(\tau_1)]$,

$$\max_{\mathbf{x} \in M_t^1, |\mathbf{x}|\geq R\sqrt{t}} |u|(\mathbf{x}, t) < h\sqrt{t}.$$

Appealing to Theorem 4.B.11, we deduce, for $\mathbf{x} \in \Sigma, |\mathbf{x}| = C_{\text{len}}R, \tau \in (-\infty, \tau_1]$,

$$\begin{aligned} |u_2 - u_1|(\mathbf{x}, \tau) &< \vartheta h \\ &= 2\vartheta s \max_{|\mathbf{y}|=R} f_\alpha^{\mathfrak{R}}(\mathbf{y}) \\ &< s(4R \max_{\mathbf{z} \in \mathcal{L}} \psi(\mathbf{z}) + 1) \\ &< 5sR \max_{\mathbf{z} \in \mathcal{L}} \psi(\mathbf{z}). \end{aligned}$$

A contradiction for $\tau = \tau_1$. QED

Remark 4.6.5. The above proof demonstrates that the welding of the barriers, $\Gamma_s^+(\tau)$ with $h\sqrt{t}$, and $\Gamma_s^-(\tau)$ with $-h\sqrt{t}$, yields barriers, in the standard sense, relative to $\mathcal{M}^1$. Here, the welding process occurs in the collar $\mathrm{Col}(R, C_{\text{len}}R, T)$. We use the reparameterisation results from Section 4.B.2 demonstrate the boundaries of each of the barriers components are hidden above the other barrier, and thus the welding is well defined. See Figure 1.3.2.

We now state our uniqueness theorem.

Theorem 4.6.6. *Suppose $\mathcal{M}^1$, $\mathcal{M}^2$ are two smooth (or unit-regular, cyclic mod 2) flows from M_0 satisfying Assumption 4.2.2 B (resp. Assumption 4.2.2) with respect to Σ. Then, for as long as the flows are smooth,*

$$\mathcal{M}^1 \equiv \mathcal{M}^2.$$

Proof. This follows immediately from Lemma 4.6.3 and Remark 4.6.4. Indeed, using the assumptions from the start of the section, we deduce $\{\Gamma_s^\pm(\tau)\}_{\tau \in (-\infty, \tau_0)}$ is a barrier to $\mathcal{RM}^2$ on $(-\infty, \tau_0)$ for every $s \in (0, s_0]$, with $\tau_0 = \log(T)$. Moreover, this implies we can apply the separation estimate, Theorem 4.5.1, with $h > 0$ arbitrarily small. We deduce the flows $\mathcal{M}^1$ and $\mathcal{M}^2$ must agree on $(0, T)$. By standard theory for smooth mean curvature flow, we deduce equality up to the first singular time, $\hat{T} > T$. QED

Theorem 4.6.7. *Let $\mathcal{C}$ be a smooth cone, and suppose Σ is a smooth expander, with $\mathcal{C}(\Sigma) = \mathcal{C}$. Suppose the level-set flow from $\mathcal{C}$ does not fatten and the only mean curvature flow from the cone $\mathcal{C}$ is $\mathcal{M}_\Sigma$. If M_0 is a compact hypersurface with conical singularity modelled on $\mathcal{C}$, the level-set flow from M_0 does not fatten instantaneously. Moreover, there is a unique, unit-regular, cyclic mod 2 Brakke flow from M_0, smooth until the next singular time.*

Proof. Following the motivating discussion at the start of Section 4.2, we note that the uniqueness of Σ implies Σ satisfies Assumptions 4.2.3 and 4.2.4. Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 flow from M_0. Any Type-1 blow-up of $\mathcal{M}$ around $(\mathbf{0}, 0) \in \mathbb{R}^{n+1} \times [0, \infty)$ sub-converges to a unit-regular, cyclic mod 2 Brakke flow on $(0, \infty)$. This limit can be extended to include the cone $\mathcal{C}$ at time 0, as dilations of the initial condition converge locally smoothly to the cone away from 0. $\mathcal{M}_\Sigma$ is assumed to be the only flow from $\mathcal{C}$, and thus any Type-I blow-up around the space-time origin is modelled on Σ. $\mathcal{M}$ hence satisfies Assumption 4.2.2. Since $\mathcal{M}$ was chosen arbitrarily, every unit-regular, cyclic mod 2 flow from M_0 satisfies Assumption 4.2.2. We apply Theorem 4.6.6 to deduce $\mathcal{M}$ is the unique unit-regular, cyclic mod 2 flow from M_0 until the first singular time $\tilde{T} > 0$. Recall, the inner and outer Brakke flows of Hershkovits–White are unit-regular and cyclic mod 2, and so the inner and outer flows from M_0 are equal until $t = \tilde{T}$. We conclude the level-set flow from M_0 does not fatten instantaneously. QED

In ambient dimensions $(n + 1) \in [3, 7]$, all expanders are automatically smooth, by classical results for minimal surfaces. We deduce,

Corollary 4.6.8. *For $n \in [2, 6]$, let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone. Suppose $M_0^n \subset \mathbb{R}^{n+1}$ satisfies Assumption 4.2.1. If the level-set flow from $\mathcal{C}$ does not fatten, then the level-set flow from M_0 does not fatten instantaneously. Moreover, there is a unique, unit-regular, cyclic mod 2 Brakke flow from M_0, smooth until the next singular time.*

4.7 Fattening

We now discuss how the functions $f_\alpha^{\mathfrak{R}}$ introduced in Section 4.4 can be welded to a constant to yield global barriers over the expander. We then use these barriers to demonstrate that if the mean curvature flow from the asymptotic cone fattens, then so does the flow from a compact hypersurface with conical singularity modelled on said cone. We work with the inner flow oriented with the outward pointing normal.

The same results hold for the outer flow, though one must change the relevant signs to get barriers on the correct side.

Lemma 4.7.1. *Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone, and suppose the outermost expanders are smooth. Let Σ be the inner expander asymptotic to $\mathcal{C}$. There exists a radius $R \in (0,\infty)$ such that for any $h \in (0,1)$, the hypersurface*

$$\Gamma_{R,h} := \mathrm{graph}_{E_R} h \tag{4.13}$$

defines a sub-solution to rescaled mean curvature flow.

Proof. The graph of $u = h \in (0,1)$ is well defined over the expander Σ in the region where $|A| \leq 1$. Since the asymptotic geometry is conical, there exists R_1 such that the graph over E_{R_1} of $h \in (0,1)$ defines a smooth hypersurface. We now show this hypersurface is a sub-solution. As in the proof of Theorem 4.4.4, we calculate

$$\begin{aligned} v_{\Gamma_{R,h}}(\mathbf{x},\tau)\left(\partial_\tau \mathbf{x}_{\Gamma_{R,h}} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{\Gamma_{R,h}}\right)\cdot \nu_{\Gamma_{R,h}} &= -L_\Sigma(h) + E(h) \\ &= \left(\frac{1}{2} - |A|^2\right) h + E(h). \end{aligned}$$

Since h is constant its derivatives vanish. We can apply the same reasoning as in the proof of Theorem 4.A.9 to bound the error in terms of h and $|A|^2$. Note, the contributions to the error from the drift term vanish, as they depend on $\nabla h = 0$. We have

$$v_{\Gamma_{R,h}}(\mathbf{x},\tau)\left(\partial_\tau \mathbf{x}_{\Gamma_{R,h}} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{\Gamma_{R,h}}\right)\cdot \nu_{\Gamma_{R,h}} \geq \left(\frac{1}{2} - (1+D)|A|^2\right) h.$$

Once again using the asymptotic geometry, we find R_2 such that

$$\max_{\mathbf{x}\in E_{R_2}} |A|^2(\mathbf{x}) < \frac{1}{2(1+D)}.$$

We conclude, for $R \geq R_2$

$$\Gamma_{R,h} := \mathrm{graph}_{E_R} h \tag{4.14}$$

is a sub-solution to the rescaled mean curvature flow. QED

In the following proposition, we use the choice of parameters, $R, \mathfrak{R}, \alpha$, discussed at the start of Section 4.6. In addition, we assume R has chosen large enough to satisfy Lemma 4.7.1.

Proposition 4.7.2. *Let $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ be a smooth cone, and suppose the outermost* *expanders are smooth. Let Σ be the inner expander. Fix*

$$h = \max_{\mathbf{x}\in\Sigma\cap\partial B(0,R)} f_\alpha^{\mathfrak{R}}(\mathbf{x}),$$

and set

$$u_s(\mathbf{x}) = \begin{cases} s f_\alpha^{\mathfrak{R}}(\mathbf{x}) & \mathbf{x}\in\overline{\Sigma_R} \\ s\min\{f_\alpha^{\mathfrak{R}}(\mathbf{x}), h\} & \mathbf{x}\in E_R\backslash\Sigma_{C_{\mathrm{len}}R} \\ sh & \mathbf{x}\in E_{C_{\mathrm{len}}R}. \end{cases}$$

Then, there exists $s_0\in(0,\frac{1}{h})$ such that for $s\in[0,s_0]$ the graph

$$\Gamma_s = \mathrm{graph}_\Sigma u_s$$

defines a sub-solution to rescaled mean curvature flow.

Remark 4.7.3. The above theorem shows $\{\sqrt{t}\Gamma_s\}_{t\in(0,\infty)}$ defines a sub-solution to mean curvature flow starting from the cone $\mathcal{C}(\Sigma)$.

Proof. Using the argument of Proposition 4.4.4, we find s_0 such that for $s\in[0,s_0]$, $\mathrm{graph}_{\Sigma_{C_{\mathrm{len}}R}} s f_\alpha^{\mathfrak{R}}$ is a smooth hypersurface and defines a sub-solution (over $\Sigma_{C_{\mathrm{len}}R}$). It follows from a standard argument that the minimum of two sub-solutions is itself a sub-solution, provided they intersect in the correct manner. This is guaranteed by our choice of collar region and parameter α. By definition we have $h\geq f_\alpha^{\mathfrak{R}}(\mathbf{x})$ for $\mathbf{x}\in\Sigma, |\mathbf{x}|=R$ and, by Lemma 4.4.6 and the choice of C_{len}, $h\leq f_\alpha^{\mathfrak{R}}(\mathbf{x})$ for $\mathbf{x}\in\Sigma, |\mathbf{x}|=C_{\mathrm{len}}R$. QED

We are now able to state and prove our key theorem for singularities modelled on cones with fattening level-set flow.

Theorem 4.7.4. *Let $\mathcal{C}^n\subset\mathbb{R}^{n+1}$ be a smooth cone, with fattening level-set flow. Suppose the outermost flows from $\mathcal{C}$ are modelled on smooth expanders. Let M_0 be a smooth hypersurface with a conical singularity modelled on $\mathcal{C}$. Then, every tangent flow at the origin of any unit-regular, cyclic mod 2 Brakke flow supported on the inner flow from M_0 is modelled on the inner expander of $\mathcal{C}$. The same relation holds for the outer flow from M_0 and the outer expander.*

Remark 4.7.5. We must choose the orientation of the cone $\mathcal{C}$ so that the notion of 'inside' agrees with the 'inside' of the hypersurface M_0. This is so that the inner flow from M_0 will have tangent flows modelled on the inner flow from $\mathcal{C}$.

Proof. Let $\mathcal{C}^n\subset\mathbb{R}^{n+1}$ be a smooth cone, and fix the appropriate orientation. Suppose the outermost flows from $\mathcal{C}$ are modelled on the smooth expanders Σ^+,Σ^-, the outer and inner expanders respectively. We define

$$\mathrm{Gap}(\mathcal{C}) := \mathrm{Int}(\Sigma^+)\cap\mathrm{Ext}(\Sigma^-).$$

Observe, $\{\sqrt{t}\mathrm{Gap}(\mathcal{C})\}_{t\in(0,\infty)}$ defines the level-set flow from $\mathcal{C}$, and thus every flow from $\mathcal{C}$ is contained in this region. We note, as a consequence of the work of Deruelle–Schulze, the inner and outer expanders of $\mathcal{C}$ decay towards each other at rate $O(r^{-n-1}\exp(-\frac{r^2}{4}))$.

Suppose $\mathcal{M}$ is any unit-regular, cyclic mod 2 Brakke flow starting from M_0. Using the same argument as in the proof of Theorem 4.6.7, we see any forward tangent flow of $\mathcal{M}$ must be some flow starting from the cone. Whilst the tangent flow may not be unique, every tangent flow will be contained in $\{\sqrt{t}\mathrm{Gap}(\mathcal{C})\}_{t\in(0,\infty)}$. We deduce the following claim:

Claim 4.7.1. Let $\mathcal{M}$ be a unit-regular, cyclic mod 2 Brakke flow from M_0. For every $\varepsilon > 0$, $\tilde{R} < \infty$, there exists a time $T = T(\varepsilon, \tilde{R}, \mathcal{M}) > 0$ such that the rescaled flow $\mathcal{R}\mathcal{M}\lfloor B(0,\tilde{R}) \times (-\infty, \log(T))$ is contained in the ε-thickening of $\mathrm{Gap}(\mathcal{C})$.

We now consider a unit-regular, cyclic mod 2 Brakke flow supported in the inner flow from M_0. Recalling the construction of Hershkovits–White, [HW20], we approximate M_0 by a sequence of smooth hypersurfaces, M_i, from the inside, with $\lim_{i\to\infty} M_i = M_0$ in the varifold sense. Additionally, we assume local smooth convergence away from the singular point. Moreover, if $\mathcal{M}^i$ are unit-regular, cyclic mod 2 Brakke flows starting from M_i, then there exists a subsequential limit $\mathcal{M}_{\text{inner}}$, supported on the inner flow from M_0.

We note at this point it is not clear $\mathcal{M}_{\text{inner}}$ is the only unit-regular, cyclic mod 2 Brakke flow supported on the inner flow. Once we show $\mathcal{M}_{\text{inner}}$ is modelled on the inner expander near the cone point, we can apply Proposition 4.3.8 to see the support is smooth fon some time interval. We can then appeal to [CCMS20, Corollary F.4] (Theorem 2.2.6) to conclude $\mathcal{M}_{\text{inner}}$ is the only unit-regular, cyclic mod 2 Brakke flow supported on the inner flow on this time interval.

Let Γ_s be the global sub-solutions constructed in Proposition 4.7.2 for each $s \in [0, s_0]$ relative to the inner expander Σ^-. Recall, they lie on the outside of Σ^-.

Claim 4.7.2. For each $s \in (0, s_0]$, there exists a radius R_s such that, for each $\tilde{R} \geq R_s$, one may find a time $T = T(s, \tilde{R}, \mathcal{M}_{\text{inner}}) > 0$ so that $\mathcal{M}_{\text{inner}}\lfloor\Omega(\tilde{R}, T)$ lies strictly on the inside of $\{\sqrt{t}\Gamma_s\}_{t\in(0,T)}$.

Proof. Fix $s \in (0, s_0]$. The flow $\{\sqrt{t}\Gamma_s\}_{t\in(0,\infty)}$ is a sub-solution to mean curvature flow starting from $\mathcal{C}$ by Proposition 4.7.2. Indeed, any Brakke flow starting from a compact, smooth hypersurface lying interior to the cone $\mathcal{C}$ will never touch $\{\sqrt{t}\Gamma_s\}_{t\in(0,\infty)}$ by the avoidance principle.

Fix $\varepsilon = \frac{s}{4}$ and let $R_s < \infty$ be the radius such that the distance between Γ_s and the ε-thickening of $\mathrm{Gap}(\mathcal{C})$ is at least ε outside $B(0, R_s)$. Such a radius exists by the decay proved by Deruelle–Schulze [DS20]. Fix $\tilde{R} \geq R_s$. Let $T = T(\varepsilon, \tilde{R}, \mathcal{M}_{\text{inner}}) > 0$ be the time from Claim 4.7.1 for $\mathcal{M}_{\text{inner}}$. By checking the boundary data, it is straightforward to adapt the above avoidance argument to see that the flows

$\mathcal{M}^i\lfloor\Omega(\tilde{R},T)$ avoid $\{\sqrt{t}\Gamma_s\}_{t\in(0,T)}$ in $B(0,\tilde{R}\sqrt{t})$. We conclude that $\mathcal{M}_{\text{inner}}\lfloor\Omega(\tilde{R},T)$ must also avoid the barrier by taking limits. QED

Using this observation, any tangent-flow of $\mathcal{M}_{\text{inner}}$ must lie on one-side of the barriers $\sqrt{t}\Gamma_s$ for every $s\in(0,s_0]$. The only flow from $\mathcal{C}$ for which this is true is $\{\sqrt{t}\Sigma^-\}_{t\in(0,\infty)}$. We conclude every tangent flow of $\mathcal{M}_{\text{inner}}$ is modelled on the inner expander.

QED

As an immediate corollary, we may apply Theorem 4.6.6 to deduce the following.

Corollary 4.7.6. *The unit-regular, cyclic mod 2 Brakke flow supported on the inner flow from M_0 is smooth until the next singular time and is the unique flow with tangent flows modelled on the inner expander. The analogous statement holds for the outer flow from M_0.*

Appendix

4.A Linearisation of Geometric Quantities

In the following, let $M_t^1, M_t^2 \subset \mathbb{R}^{n+1}$ be smooth, 1 parameter families of smooth hypersurfaces for $t \in I \subset \mathbb{R}$. Suppose there exist open sets $U_1, U_2 \subset \mathbb{R}^{n+1}$ such that $M_t^2 \cap U_2$ can be parameterised as a graph over $M_t^1 \cap U_1$. That is, for every $t \in I$, there is a smooth function $u(\cdot, t) : M_t^1 \cap U_1 \to \mathbb{R}$ such that

$$M_t^2 \cap U_2 = \mathrm{graph}_{M_t^1 \cap U_1} u(\cdot, t)$$

We restate the following theorems from the recent work of Chodosh–Choi–Schulze [CCS23]. Similar results have been shown throughout the literature. We compute standard geometric quantities on M_t^2 in terms of those on M_t^1 and the function u. We note the convention $\mathbf{H} = H\nu$ (which yields a scalar mean curvature with opposite sign than is typical). We begin by defining

$$v(\mathbf{x}, t) = (1 + |(\mathrm{Id} - uS_{M_t^1})^{-1}(\nabla_{M_t^1} u)|^2)^{\frac{1}{2}}.$$

Theorem 4.A.1. *The upwards pointing normal along M_t^2 can be written*

$$\nu_{M_t^2}(\mathbf{x} + u\nu_{M_t^1}) = v^{-1}(-(\mathrm{Id} - uS_{M_t^1})^{-1}\nabla_{M_t^1} u + \nu_{M_t^1}). \tag{4.15}$$

In particular

$$v = (\nu_{M_t^1} \cdot \nu_{M_t^2})^{-1}$$

Theorem 4.A.2. *For $\mathbf{x} \in M_t^1 \cap U_1$, the mean curvature of M_t^2 at $\mathbf{x} + u(\mathbf{x}, t)\nu_{M_t^1}(\mathbf{x})$ satisfies*

$$v(\mathbf{x}, t)H_{M_t^2}(\mathbf{x} + u(\mathbf{x}, t)\nu_{M_t^1}) = H_{M_t^1}(\mathbf{x}) + (\Delta_{M_t^1} u + |A_{M_t^1}|^2 u)(\mathbf{x}, t) + E^H$$

Furthermore, E^H can be decomposed into

$$E = uE_1^H + E_2^H(\nabla_{M_t^1} u, \nabla_{M_t^1} u)$$

where $E_1^H \in C^\infty(M_t^1)$ *and* $E_2^H \in C^\infty(M_t^1; T^*M_t^1 \otimes T^*M_t^1)$

Theorem 4.A.3. *The support function along* M_t^2 *satisfies*

$$v(\mathbf{x},t)(\mathbf{x}_{M_t^2} \cdot \nu_{M_t^2}) = \mathbf{x} \cdot \nu_{M_t^1} + u(\mathbf{x},t) - \mathbf{x}^T \cdot \nabla_{M_t^1} u + uE^{\mathbf{x}\cdot\nu}$$

Corollary 4.A.4. *If* M_t^1 *is a smooth mean curvature flow, then*

$$v(\mathbf{x},t)(\partial_t \mathbf{x}_{M_t^2} - \mathbf{H}) \cdot \nu_{M_t^2} = \partial_t u - (\Delta_{M_t^1} u + |A_{M_t^1}|^2 u)(\mathbf{x},t) + E.$$

Moreover, there exists $C_E = C_E(\xi, n) \in (0,\infty)$ *such that*

$$\|E(u)(\cdot,t)\|_{C^1(M_t^1)} \leq C_E \|u(\cdot,t)\|^2_{C^2(M_t^1)}.$$

Additionally, if M_t^2 *is a smooth mean curvature flow, then*

$$\partial_t u = (\Delta_{M_t^1} u + |A_{M_t^1}|^2 u)(\mathbf{x},t) + E$$

Corollary 4.A.5. *If* M_t^1 *is smooth rescaled mean curvature flow then*

$$v(\mathbf{x},t)(\partial_t \mathbf{x}_{M_t^2} - \mathbf{H} + \frac{1}{2}\mathbf{x}_{M_t^2}) \cdot \nu_{M_t^2} = \\ \partial_t u - \left(\Delta_{M_t^1} u + \frac{1}{2}\mathbf{x}^T \cdot \nabla_{M_t^1} u + \left(|A_{M_t^1}|^2 - \frac{1}{2}\right) u\right)(\mathbf{x},t) + E.$$

Moreover, there exists $C_E = C_E(\xi, n) \in (0,\infty)$ *such that*

$$\|E(u)(\cdot,t)\|_{C^1(M_t^1)} \leq C_E \|u(\cdot,t)\|^2_{C^2(M_t^1)}.$$

Additionally, if M_t^2 *is a smooth rescaled mean curvature flow, then*

$$\partial_t u = \left(\Delta_{M_t^1} u + \frac{1}{2}\mathbf{x}^T \cdot \nabla_{M_t^1} u + \left(|A_{M_t^1}|^2 - \frac{1}{2}\right) u\right)(\mathbf{x},t) + E.$$

We note, [CCS23] prove better estimates for the error term E than those stated here. In particular, they show control of higher spatial and time derivatives of $E(u)(\mathbf{x},t)$. In the remainder of this section, we expand on their exposition to discuss how the error terms depend on the function u and its derivatives.

We state the Sherman–Morrison formula .

Lemma 4.A.6 (Sherman–Morrison)**.** *Let* V *be a vector space and suppose* A *is an invertible linear map* $A : V \to V$. *Suppose* $u \in V$ *and* $v \in V^*$. *Then, the linear map*

$$A + u \otimes v : V \to V$$

is invertible if and only if $1 + vA^{-1}u \neq 0$. *Moreover, the inverse has the following explicit form.*

$$(A + u \otimes v)^{-1} = A^{-1} - \frac{A^{-1} \circ u \otimes v \circ A^{-1}}{1 + vA^{-1}u}.$$

Let S denote the shape operator of M_t^1. We recall the following linear map discussed in [CCS23].

$$\mathcal{G} : T_{\mathbf{x}}M \to T_{\mathbf{x}}M \tag{4.16}$$
$$\mathcal{G} := (I - uS)^2 + \nabla u \otimes \mathrm{d}u. \tag{4.17}$$

When $|u||A| < 1$, $(I - uS)$ is invertible, with an explicit formula via Taylor series. Provided the first derivative exists, we may apply the Sherman–Morrison formula to $\mathcal{G}$, as

$$du((I - uS)^{-1}\nabla u) > 0.$$

Remark 4.A.7. We note there is no requirement for ∇u to be bounded for the inverse to exist, only finite at each point. We will use this fact in the proof of Proposition 4.A.9, however, we will later impose bounds on the first and second derivative so that we may apply the Taylor expansion of $1/(1 + t)$ to understand how the error decomposes.

We assume $|u||A| < 1$. We then may apply the Taylor expansion of $1/(1 - t)$ to $(I - uS)^{-1}$ to deduce

$$(I - uS)^{-1} = \mathrm{Id} + \sum_{n=1}^{\infty}(uS)^n$$

and thus

$$\mathcal{G}^{-1} = \mathrm{Id} + \mathfrak{L} \tag{4.18}$$

where we have used the Shermann–Morrison formula to define

$$\mathfrak{L} := 2\sum_{n=1}^{\infty}(uS)^n + \left(\sum_{n=1}^{\infty}(uS)^n\right)^2 - \frac{(I - uS)^{-1} \circ \nabla u \otimes du \circ (I - uS)^{-1}}{1 + du((I - uS)^{-1}\nabla u)}. \tag{4.19}$$

4.A.1 Derivation of Linearisation

We repeat the derivation contained in [CCS23], without dropping 'quadratic' terms. In doing so, we show the explicit form of the error operator seen in the linearisation of the mean curvature.

By integrating the first variation formula by parts, Chodosh–Choi–Schulze demonstrate

$$vH_{M_t^2} = \frac{v^2}{\sqrt{\det \mathcal{G}}}\mathrm{div}(\sqrt{\mathcal{G}}\mathcal{G}^{-1}\nabla u) + v^2(\mathrm{tr}(\mathcal{G}^{-1}S) - \mathrm{tr}(\mathcal{G}^{-1}S^2)u).$$

We first note

$$\begin{aligned} v^2(\mathbf{x}, t) &= 1 + |(\mathrm{Id} - uS_{M_t^1})^{-1}(\nabla_{M_t^1} u)|^2 \\ &= 1 + |(\Sigma_{k=0}^{\infty} u^k S_{M_t^1}^k)(\nabla_{M_t^1} u)|^2. \end{aligned} \tag{4.20}$$

Now, analysing the divergence term in more detail, we have

$$\frac{1}{\sqrt{\det \mathcal{G}}}\mathrm{div}(\sqrt{\det \mathcal{G}}\mathcal{G}^{-1}\nabla u) = \mathrm{div}(\mathcal{G}^{-1}\nabla u) + \frac{1}{2}\nabla \log\det(\mathcal{G}) \cdot \mathcal{G}^{-1}\nabla u. \tag{4.21}$$

$$\mathrm{div}(\mathcal{G}^{-1}\nabla u) = \mathrm{div}((\mathrm{Id} + \mathfrak{L})\nabla u) \tag{4.22}$$

$$= \Delta u + \mathrm{div}(\mathfrak{L}\nabla u). \tag{4.23}$$

To analyse the $\nabla \log\det$ term, we recall the Jacobi Formula,

$$\frac{d}{dt}\det A(t) = \det A(t) \cdot \mathrm{tr}\left(A^{-1}(t) \cdot \frac{dA(t)}{dt}\right).$$

We calculate,

$$\begin{aligned} \nabla \log\det(\mathcal{G}) \cdot \mathcal{G}^{-1}\nabla u &= \frac{1}{\det(\mathcal{G})}\nabla_i \det(\mathcal{G})(\mathcal{G}^{-1}\nabla u)_i = \mathrm{tr}(\mathcal{G}^{-1}\nabla_i \mathfrak{L})(\mathcal{G}^{-1}\nabla u)_i \\ &= \mathrm{tr}(\nabla_i \mathfrak{L})((\mathrm{Id} + \mathfrak{L})\nabla u)_i + \mathrm{tr}(\mathfrak{L}\nabla_i \mathfrak{L})((\mathrm{Id} + \mathfrak{L})\nabla u)_i \end{aligned} \tag{4.24}$$

where $(\mathbf{v})_i$ denotes the i^{th} component of the vector $\mathbf{v}$.

Finally, we analyse the terms involving the trace. We see

$$\mathrm{tr}(\mathcal{G}^{-1}S) - \mathrm{tr}(\mathcal{G}^{-1}S^2 u) = H_{M_t^1} + |A_{M_t^1}|^2 u + \mathrm{tr}((\mathfrak{L} - 2uS)S - \mathfrak{L}S^2 u). \tag{4.25}$$

Definition 4.A.8. For future reference, we group the higher order terms from the divergence and trace terms as follows.

$$\begin{aligned} E_{\mathrm{div}}(u) &:= \mathrm{div}(\mathfrak{L}\nabla u) + \mathrm{tr}(\nabla_i \mathfrak{L})((\mathrm{Id} + \mathfrak{L})\nabla u)_i + \mathrm{tr}(\mathfrak{L}\nabla_i \mathfrak{L})((\mathrm{Id} + \mathfrak{L})\nabla u)_i \\ E_{\mathrm{tr}}(u) &:= \mathrm{tr}((\mathfrak{L} - 2uS)S - \mathfrak{L}S^2 u). \end{aligned}$$

4.A.2 Error at a critical point

We now consider the structure of the error at a critical point of u.

By definition, at a critical point of the function u, we have $\nabla u = 0$. Considering equations 4.19 and 4.20, we see the error terms, Definition 4.A.8, simplify. From this, we can deduce bounds on how fast the function u is increasing or decreasing.

This observation was noted for minimal surfaces in [CM11], and for mean curvature flow in [Her17]. We provide a proof below for completeness.

Proposition 4.A.9. *For $i = 1, 2$, Let $\{M_t^i\}_{t\in(0,T)}$ be a smooth mean curvature flow. Suppose there exists a smooth, 1-parameter family of smooth functions*

$$\left\{u(\cdot, t) : M_t^1 \to \mathbb{R}\right\}_{t\in(0,T)},$$

with $|u||A| < \xi < 1$ such that M_t^2 can be parameterised as the normal graph of $u(\cdot, t)$ over M_t^1.

Then, at a positive spatial maximum of $u(\cdot, t)$, we have

$$\frac{\partial u}{\partial t} \leq (1 + D)|A|^2 u,$$

and at a negative spatial minimum of $u(\cdot, t)$, we have

$$\frac{\partial u}{\partial t} \geq (1 + D)|A|^2 u,$$

where $D = D(\xi, n) \in (0, \infty)$.

Proof. Let X_0 be a spatial critical point of u, we have $\nabla u(X_0) = 0$, and hence equation 4.20 immediately shows $v^2 = 1$. We also note

$$\begin{aligned}
\mathfrak{L} &= 2\sum_{n=1}^{\infty}(uS)^n + \left(\sum_{n=1}^{\infty}(uS)^n\right)^2 \\
&= 2uS + (uS)^2\left(\sum_{n=0}^{\infty}(uS)^n + \left(\sum_{n=0}^{\infty}(uS)^n\right)^2\right).
\end{aligned}$$

Using equation 4.A.4 and the derivation in Section 4.A.1, we see u satisfies the following differential equation at X_0.

$$\frac{\partial u}{\partial t}(X_0) = \mathrm{div}_{M_t^1}((I + \mathfrak{L})\nabla u) + |A|^2 u + E_{\mathrm{tr}}(u).$$

Substituting the above formula for $\mathfrak{L}$ at a critical point into the equation for $E_{\mathrm{tr}}(u)$, we deduce

$$|E_{\mathrm{tr}}| \leq D|A|^3|u|^2,$$

for some constant D depending only on ξ and the dimension n. This follows from $|u||A| < \xi$ and the Cauchy-Schwartz inequality.

The linear map $\mathcal{G}$ is symmetric and positive definite, and hence $\mathcal{G}^{-1} = I + \mathfrak{L}$ is symmetric and positive definite. Thus, by a standard argument (see for example [Eva10, Section 6.4.1. Theorem 1]), $\mathrm{div}_{M_t^1}((I + \mathfrak{L})\nabla u)$ will have the same sign as $\Delta_{M_t^1} u$ at a critical point. Note, terms from $\mathrm{div}(\mathfrak{L}\nabla u)$ involving derivatives of $\mathfrak{L}$ vanish, as any such term will also involve ∇u, which is equal to 0.

If X_0 is a positive maximum, or negative minimum, we conclude

$$\left|\frac{\partial u}{\partial t}\right|(X_0) \leq |A|^2|u| + D|A|^3|u|^2.$$

The proposition follows by recalling $|A||u| < 1$. QED

4.A.3 Decomposition of the Error Operator

We wish to show the Error Operator decomposes binomially when applied to the sum of functions. These results can be interpreted as 'local continuity' of E as an 'operator' at u with respect to perturbations by functions with small C^2 norm.

Let S_{ij}, $1 \leq i, j \leq n$, denote the components of the shape operator of M_t^1. We recall the linear map $\mathcal{G} : T_{\mathbf{x}}M_t^1 \to T_{\mathbf{x}}M_t^1$. We write

$$\mathcal{G}_{ij} = \delta_{ij} - 2uS_{ij} + u^2 S_{ik}S_{jk} + u_i u_j.$$

We set $Q_{ij}(u) := u^2 S_{ik}S_{jk} + u_i u_j$, $B_{ij} := -2uS_{ij} + Q_{ij}$:

$$\begin{aligned}\mathcal{G}_{ij} =& \delta_{ij} - 2uS_{ij} + Q_{ij}(u)\\ =& \delta_{ij} + B_{ij}(u).\end{aligned}$$

In the following, we directly apply the Taylor expansion of $1/(1 + t)$ to $\mathcal{G} = I + B$, rather than applying the Sherman–Morrison formula. For the Taylor series to converge, we make the following assumption.

Assumption 4.A.1. Let $\xi \in (0, 1)$. To ensure all of the Taylor series converge, we make the assumption

$$\max\left\{|u||A|, |\nabla u|^2\right\} < \frac{\xi}{4}.$$

This ensures

$$\begin{aligned}|-2uA + u^2A^2 + \nabla u \otimes du| \leq& 2|u||A| + |u^2A^2| + |\nabla u|^2\\ <& \xi.\end{aligned}$$

By Taylor expansion, we deduce

$$\mathcal{G}^{-1} = \mathrm{Id} + \sum_{n=1}^{\infty} (-1)^n B^n \tag{4.26}$$

$$\mathfrak{L}(u) = \sum_{n=1}^{\infty} (-1)^n (B(u))^n. \tag{4.27}$$

Consider $g = u + f$ and assume u, f and g satisfy Assumption 4.A.1. We calculate

$$\begin{aligned} Q(g) &= g^2 S_{ik} S_{jk} + g_i g_j \\ &= Q(u) + Q(f) + 2(uf) S_{ij} S_{jk} + u_i f_j + u_j f_i \\ &:= Q(u) + Q(f) + M(u, f). \end{aligned}$$

Hence,

$$B(g) = B(u) + B(f) + M(u, f).$$

Having decomposed $B(g)$, we look to decompose $B(g)^n$. Applying the binomial theorem, we deduce

$$\begin{aligned} B(g)^n &= (B(u) + B(f) + M(u, f))^n \\ &= \sum_{p=0}^{n} \sum_{q=0}^{p} \binom{n}{p} \binom{p}{q} B(u)^{n-p} B(f)^{p-q} M(u, f)^q \\ &= B(u)^n + B(f)^n + \sum_{p=1}^{n} \sum_{q=1}^{p} \binom{n}{p} \binom{p}{q} B(u)^{n-p} B(f)^{p-q} M(u, f)^q. \end{aligned}$$

Substituting the above formula into the Taylor series for $\mathfrak{L}(g)$, equation 4.27, we calculate

$$\begin{aligned} \mathfrak{L}(g) &= \sum_{n=1}^{\infty} (-1)^n (B(u) + B(f) + M(u, f))^n \\ &= \sum_{n=1}^{\infty} (-1)^n B(u)^n + \sum_{n=1}^{\infty} (-1)^n B(f)^n \\ &+ \sum_{n=1}^{\infty} (-1)^n \sum_{p=1}^{n} \sum_{q=1}^{p} \binom{n}{p} \binom{p}{q} B(u)^{n-p} B(f)^{p-q} M(u, f)^q \\ &= \mathfrak{L}(u) + \mathfrak{L}(f) + E_{\mathrm{mixed}}(u, f). \end{aligned}$$

We note that the series $E_{\mathrm{mixed}}(u, f)$ converges, in at least a C^1 sense, as

$$E_{\mathrm{mixed}}(u, f) = \mathfrak{L}(g) - \mathfrak{L}(u) - \mathfrak{L}(f),$$

where the right hand side is well defined, since u, f, g satisfy Assumption 4.A.1.

Substituting this formula for $\mathfrak{L}(g)$ into equations 4.23, 4.24,4.25 and 4.20, we deduce the following theorem.

Theorem 4.A.10. *Let $M_i, i \in \{1,2,3\}$ be smooth hypersurfaces. Suppose there exist open sets $U_i \subset \mathbb{R}^{n+1}, i \in \{1,2,3\}$ and $C^2(M_1)$ functions*

$$\begin{aligned} u_2 &: M_1 \cap U_1 \to \mathbb{R} \\ u_3 &: M_1 \cap U_1 \to \mathbb{R}, \end{aligned}$$

such that

$$M_i \cap U_i = \operatorname{graph}_{M_1 \cap U_1} u_i,\ i \in \{2,3\}.$$

Suppose further

$$\max\{|u_i||A|, |\nabla u_i|^2\} < \frac{\xi}{4},\ i \in \{2,3\}.$$

Then, for $\mathbf{x} \in M_1 \cap U_1$, the scalar mean curvatures of M_2 and M_3 at the image of $\mathbf{x}$ under u_i satisfies

$$\begin{aligned} &v_{M_3} H_{M_3}(\mathbf{x} + u_3 \nu_{M_1}(\mathbf{x})) - v_{M_2} H_{M_2}(\mathbf{x} + u_2 \nu_{M_1}(\mathbf{x})) && (4.28) \\ &\quad = \Delta_{M_1} f + |A_{M_1}|^2 f + E(f) + Q_{\text{mixed}}(u_2, f), && (4.29) \end{aligned}$$

where $f := u_3 - u_2$ and Q_{mixed} is a degree 1 homogeneous polynomial with bounded coefficients in

$$\{u_2, \nabla_i u_2, \nabla^2_{i,j} u_2\} \times \{f, \nabla_i f, \nabla^2_{i,j} f\},$$

for $i, j \in \{1, \dots, n\}$. That is, a polynomial in the totally mixed terms of order 2. Moreover,

$$\|Q_{\text{mixed}}(u_2, f)\|_{C^1(M_1)} \leq C_Q \|u_2\|_{C^2(M_1)} \|f\|_{C^2(M_1)},$$

where $C_Q = C_Q(\xi, n) \in (0, \infty)$.

Remark 4.A.11. We note that the related decomposition of the error holds for the linearisation of the rescaled mean curvature. In [CCS23, Lemma A.3], it is shown

$$v(\mathbf{x}, \tau)(\mathbf{x}_{M^2_\tau} \cdot \nu_{M^2_\tau}) = \mathbf{x} \cdot \nu_{M^1_\tau} + u - \mathbf{x} \cdot \nabla u - u \sum_{k=1}^{\infty} u^{k-1} \mathbf{x} \cdot S^k(\nabla u).$$

For functions $g = u + f$, one decomposes the infinite series via the binomial theorem as above to get a mixed term of the same form as the mixed term in Theorem 4.A.10.

4.B Cartographic considerations

We prove a collection technical results for graphs over hypersurfaces.

4.B.1 New coordinates from Old

Definition 4.B.1 (Signed distance in a given direction)**.** Let v be a unit vector, $M \subset \mathbb{R}^{n+1}$ and $\mathbf{x} \in \mathbb{R}^{n+1}$. Let

$$B(\mathbf{x}, v, M) := \{\alpha \in \mathbb{R}, \mathbf{x} + \alpha v \in M\}.$$

Let $\alpha_0 \in B(\mathbf{x}, v, M)$ be the element such that

$$|\alpha_0| = \min_{B(\mathbf{x},v,M)} |\alpha|.$$

We define the signed distance from $\mathbf{x} \in \mathbb{R}^{n+1}$ to a subset $M \subset \mathbb{R}^{n+1}$ in the direction v as

$$d(\mathbf{x}, v, M) := \alpha_0.$$

That is, the signed length of the shortest line segment parallel to the direction v connecting $\mathbf{x}$ to a point in M. We take the convention $d(\mathbf{x}, v, M) = \infty$ if there is no intersection.

Proposition 4.B.2. *Let M_1, M_2 be smooth, closed hypersurfaces in $\mathbb{R}^{n+1}$. Let $\mathcal{T}_\delta$ denote a tubular neighbourhood of M_1 with radius $\delta > 0$ on which $\pi_{M_1} : \mathcal{T}_\delta(M_1) \to M_1$ is well defined and $d(\cdot, M_1) : \mathcal{T}_\delta(M_1) \to \mathbb{R}$ is a smooth function. Suppose further that there exist a point $\mathbf{x} \in \mathbb{R}^{n+1}$ and open sets of M_i, $U_{i,\mathbf{x}} := C(\mathbf{x}, r) \cap M_i$ such that*

$$U_{i,\mathbf{x}} = \mathrm{graph}_{B^n(\mathbf{x},r)} u_{i,\mathbf{x}},$$

where $B^n(\mathbf{x}, r)$ is the n-ball of radius r in some n-plane through $\mathbf{x}$ and $u_{i,\mathbf{x}} \in C^1(B^n(\mathbf{x}, r))$. Let $\mathcal{T}(M_1 \cap U_1)$ denote the subset of $\mathcal{T}_\delta(M_1)$ that projects to $M_1 \cap U_1$. Additionally, we define

$$V_2 := U_{2,\mathbf{x}} \cap \mathcal{T}(M_1 \cap U_{1,\mathbf{x}})$$
$$V_1 := \pi_{M_1}(V_2).$$

Given the above, there exists an $\eta_1 > 0$ such that, if

$$\|u_{i,\mathbf{x}}\|_{C^0(B^n(\mathbf{x},r))} < \eta_1 r.$$

$$\|\nabla u_{i,\mathbf{x}}\|_{C^0(B^n(\mathbf{x},r))} < \eta_1.$$

Then,

1. *V_2 is an open subset of M_2,*

2. *V_2 is non-empty and contains the image of* 0 *under $u_{2,\mathbf{x}}$,*

3. *V_2 can be parameterised as a normal graph over V_1 i.e.*

$$M_2 \cap V_2 = \mathrm{graph}_{V_1} u,$$

where the function $u : V_1 \to \mathbb{R}$ is given explicitly by the signed distance in ν_{M_1} to M_2.

Proof. Let $\mathbf{x} \in \mathbb{R}^{n+1}$ be as in the statement of the proposition and write $U_i = U_{i,\mathbf{x}}$. To simplify our discussion, we scale to take $r = 1$, translate $\mathbf{x}$ to the origin, and rotate such that the normal of the plane containing $B^n(\mathbf{x}, 1)$ is e_{n+1}.

Item (1) is trivial from the definition of open sets in the submanifold topology.

Item (2) follows from standard trigonometry. It is clear we may choose $\eta_1 > 0$ such that the image of the origin under u_2 is in V_2.

To prove Item (3), we show there exists a choice of $\eta_1 > 0$ such that π_{M_1} restricted to $M_2 \cap V_2$ is injective. The moral of the proof is that the gradient bound on u_2 is violated if two points project to the same point on M_1, provided the normal at each point of $M_1 \cap U_1$ is sufficiently close to vertical.

Using equation 4.15, it is straightforward to show that if $\|\nabla u_1\|_{C^0} < \eta_1$, there exists $C < \infty$ depending only on the dimension such that

$$\|\nu_{M_1}(\mathbf{x}) - e_{n+1}\| < C\eta_1. \tag{4.30}$$

We hence prove the following.

Claim 4.B.1. If $\eta_1 < (1 + C)^{-1}$, then π_{M_1} restricted to $M_2 \cap V_2$ is injective.

Suppose for contradiction this is false. Fix $\eta_1 < (1 + C)^{-1}$. By our assumption, there exists 2 non-equal points $\mathbf{y}_1, \mathbf{y}_2 \in M_2 \cap V_2$, and $\mathbf{x} \in M_1$ such that $\pi_{M_1}(\mathbf{y}_1) = \pi_{M_1}(\mathbf{y}_2) = \mathbf{x}$. Let $\mathbf{y}'_j = \pi_{B^n(0,1)}(\mathbf{y}_j)$.

Using the definition of π_{M_1} we deduce

$$\frac{\mathbf{y}_1 - \mathbf{y}_2}{|\mathbf{y}_1 - \mathbf{y}_2|} = \pm\nu_{M_1}(\mathbf{x}).$$

Thus, considering the e_{n+1} component of $\frac{\mathbf{y}_1 - \mathbf{y}_2}{|\mathbf{y}_1 - \mathbf{y}_2|}$ and using equation 4.30, we have

$$1 - C\eta_1 < \frac{|u_2(\mathbf{y}'_1) - u_2(\mathbf{y}'_2)|}{|\mathbf{y}_1 - \mathbf{y}_2|}. \tag{4.31}$$

We now show equation 4.31 is bounded above by η_1. Let $b := |\mathbf{y}_1' - \mathbf{y}_2'|$ and define the function

$$\begin{aligned} f :& [0,b] \to \mathbb{R} \\ f(z) :&= u_2\big(\frac{z}{b}\mathbf{y}_1' + \frac{b-z}{b}\mathbf{y}_2'\big). \end{aligned}$$

The regularity of u_2 implies f is at least once differentiable, hence we may apply the mean value theorem for functions of one variable. There exists $z' \in [0,b]$ such that

$$|\mathbf{y}_1' - \mathbf{y}_2'|\frac{df}{dz}(z') = f(b) - f(0) = u_2(\mathbf{y}_1') - u_2(\mathbf{y}_2').$$

Observe,

$$\frac{df}{dz}(z') = Du_2\left(\frac{z'}{b}\mathbf{y}_1' + \frac{b-z'}{b}\mathbf{y}_2'\right)\left(\frac{\mathbf{y}_1' - \mathbf{y}_2'}{|\mathbf{y}_1' - \mathbf{y}_2'|}\right).$$

Using $\|\nabla u_2\|_{C^0} < \eta_1$, we deduce

$$\left|\frac{df}{dz}(z')\right| < \eta_1.$$

With this bound, we calculate

$$\begin{aligned} \frac{|u_2(\mathbf{y}_1') - u_2(\mathbf{y}_2')|}{|\mathbf{y}_1 - \mathbf{y}_2|} \cdot \frac{|\mathbf{y}_1' - \mathbf{y}_2'|}{|\mathbf{y}_1' - \mathbf{y}_2'|} &= \left|\frac{df}{dz}(z')\right| \frac{|\mathbf{y}_1' - \mathbf{y}_2'|}{|\mathbf{y}_1 - \mathbf{y}_2|} && (4.32) \\ &= \left|\frac{df}{dz}(z')\right| \left(\sqrt{1 + \left(\frac{df}{dz}(z')\right)^2}\right)^{-1} < \eta_1, && (4.33) \end{aligned}$$

where we have used $|\mathbf{y}_1 - \mathbf{y}_2|^2 = |\mathbf{y}_1' - \mathbf{y}_2'|^2 + |u_2(\mathbf{y}_1') - u_2(\mathbf{y}_2')|^2$.

Combining 4.31 and 4.32,

$$1 < (1 + C)\eta_1.$$

This contradicts our assumption $\eta_1 < (1 + C)^{-1}$. QED

Definition 4.B.3 (Interior/exterior ball condition)**.** We say a point $\mathbf{x}_0$ in a hypersurface M satisfies an interior/exterior ball condition at $\mathbf{x}_0$ with radius r if

$$\overline{B}(\mathbf{x}_0 \pm r\nu_M(\mathbf{x}_0), r) \cap M = \mathbf{x}_0$$

where $\nu_M(\mathbf{x}_0)$ denotes the unit normal to M at $\mathbf{x}_0$. We say a hypersurface $M \subset \mathbb{R}^{n+1}$ satisfies an interior/exterior ball condition with radius r if the interior/exterior ball condition holds for every $\mathbf{x}_0 \in M$ with radius r.

Lemma 4.B.4. *Let $\mathcal{C}$ be a smooth cone and Σ be a smooth expander with $\mathcal{C}(\Sigma) = \mathcal{C}$.*

Let M_0 satisfy Assumption 4.2.1 and suppose $\mathcal{M}^1$ is a smooth flow on some time interval starting from M_0 that satisfies Assumption 4.2.2. There exists a radius $\mu > 0$ and a time $T_0 > 0$ such that for $t \in (0, T_0)$, M_t^1 satisfies the interior/exterior ball condition with radius $r = \mu\sqrt{t}$

Proof. This is a consequence of local smooth convergence of the rescaled flow $\mathcal{RM}^1$ to Σ and pseudolocality everywhere else.

In Proposition 4.3.8, it was shown there exists T such that for $0 < t < T$, M_t^1 is a smooth hypersurface. Since M_t^1 is compact, there certainly exists some $\mu(t)$ such that M_t^1 satisfies an interior/exterior ball condition with $r = \mu(t)$.

We first examine the expander region. Σ is a smooth, asymptotically conical hypersurface. Since the asymptotic geometry is controlled, we see there is uniform $\mu_\Sigma > 0$ such that Σ satisfies an interior ball condition with $r = \mu_\Sigma$. As noted in the proof of Lemma 4.3.11, the flow $\mathcal{RM}^1$ converges locally smoothly to Σ as $\tau \to -\infty$. We deduce, for every $\theta \in (0, 1)$, there exists some time $\tau_0 = \tau_0(\theta, R)$ such that for $\tau \le \tau_0$, the interior/exterior ball condition is satisfied at $\mathbf{x} \in RM_\tau^1 \cap B(0, 2R)$ with radius $\theta\mu_\Sigma$. Taking $\theta = \frac{1}{2}$ and switching back to the non-rescaled flow, we see that at $(\mathbf{x}, t) \in \mathcal{M}^1 \lfloor \Omega(R, \exp(\tau_0(\frac{1}{2}, R)))$, the interior/exterior ball condition is satisfied with radius $\frac{\mu_\Sigma}{2}\sqrt{t}$.

Turning our attention to the pseudolocal region, we consider a point $(\mathbf{x}, t) \in \mathcal{M}^1 \lfloor \mathcal{G}(R, T)$. By our definition of the pseudolocal region, there exists a point $\mathbf{x}_0 \in M_0$ such that,

$$\mathcal{D}_{(C_{\text{scale}}|\mathbf{x}_0|)^{-1}}(\mathcal{M} - \mathbf{x}_0) \cap C(0, \delta)$$

can be parameterised as a smooth graph of the function $w_{\mathbf{x}_0}$ over $B^n(0, \delta)$. In [INS19], pseudolocality is proven by using the monotonicity formula to verify that there is only one 'sheet' of M_t^1 contained in the cylinder $C(0, \delta)$. Indeed, the same reasoning shows $\mathcal{D}_{(C_{\text{scale}}|\mathbf{x}_0|)^{-1}}(M_t^1)$ satisfies an interior/exterior ball condition at $\mathbf{x}'$ (the image of $\mathbf{x}$ under the above transformation) with some radius μ_{PL}, depending only on the closeness to the plane in the ball $B(0, 1)$ of the (dilated) initial condition. Importantly, μ_{PL} is independent of $\mathbf{x}$ and $\mathbf{x}_0$. Reversing the dilation and noting $|\mathbf{x}_0| > \frac{R}{2}\sqrt{t}$ (Proposition 4.3.4), we deduce at $(\mathbf{x}, t) \in \mathcal{M}^1 \lfloor \mathcal{G}(R, T)$, the interior/exterior ball condition is satisfied with radius $\frac{\mu_{\text{PL}} C_{\text{scale}} R}{2}\sqrt{t}$.

Finally, we set $\mu = \min\left\{\frac{\mu_\Sigma}{2}, \frac{\mu_{\text{PL}} C_{\text{scale}} R}{2}\right\}$. We conclude, for $t < T_0 = \exp(\tau_0)$, the interior/exterior ball condition is satisfied at every $\mathbf{x} \in M_t^1$ with radius $\mu\sqrt{t}$.

QED

Definition 4.B.5 (Common Graphical Atlas)**.** Let $M^1, M^2 \subset \mathbb{R}^{n+1}$ be two hypersurfaces. Let $\mathbf{x}_j \in \mathbb{R}^{n+1}$ for $j \in I$, where I is some indexing set. Suppose there exists an n-ball $B^n(\mathbf{x}_j, r_j)$ of radius r_j in some n-plane through $\mathbf{x}_j$, open (in the topology on M^i) sets

$$U_{j,i} := C(\mathbf{x}_j, r_j) \cap M_i,$$

and functions $w^i_{\mathbf{x}_j}$ such that

$$\begin{aligned} U_{j,1} &= \mathrm{graph}_{B^n(\mathbf{x}_j,r_j)} w^1_{\mathbf{x}_j}, \\ U_{j,2} &= \mathrm{graph}_{B^n(\mathbf{x}_j,r_j)} w^2_{\mathbf{x}_j}. \end{aligned}$$

We say

$$\bigcup_{i\in\{1,2\}} \bigcup_{j\in I} \Big\{ \Big(w^i_{\mathbf{x}_j}, B^n(\mathbf{x}_j, r_j), U_{j,i} \Big) \Big\}$$

forms a *common graphical atlas* for M_1, M_2 if the open sets $U_{i,j}$ form a cover for M_i. i.e.

$$M_i = \cup_{j\in I} U_{j,i}$$

for $i \in \{1, 2\}$.

Lemma 4.B.6. *Let $\mathcal{C}$ be a smooth cone and Σ be a smooth expander with $\mathcal{C}(\Sigma) = \mathcal{C}$. Let M_0 satisfy Assumption 4.2.1 and suppose $\mathcal{M}^1, \mathcal{M}^2$ are smooth flows satisfying Assumption 4.2.2 B with respect to Σ.*

There exists $T > 0$ such that, for each $t \in (0, T)$ there exists a common graphical atlas for the hypersurfaces $M^1_t, M^2_t \subset \mathbb{R}^{n+1}$.

Moreover, the above atlas yields a common (space-time) graphical atlas for the space-time hypersurfaces $\mathcal{M}^1, \mathcal{M}^2 \subset \mathbb{R}^{n+1,1}$ restricted to the time interval $(0, T)$.

Proof. This follows from pseudolocality and local smooth convergence of the rescaled flow to Σ.

Fix $\varepsilon > 0$ small and take $R_{\min}$ as in Proposition 4.3.4 and fix $R > 2R_{\min}$. We find $T_0 > 0$ such that $\mathcal{G}(R, T_0)$ is a pseudolocal region and $\Omega(4R, T_0)$ is a $(\Sigma, \varepsilon, 4R)$-expander region.

We claim $T = T_0$ satisfies the claim made in the statement of the lemma. To see this, we fix $t_0 \in (0, T)$, and split into the pseudolocal and expander regions.

Claim 4.B.2. There is a common graphical atlas for $M^1_{t_0} \backslash \overline{B}(0, R\sqrt{t_0})$ and $M^2_{t_0} \backslash \overline{B}(0, R\sqrt{t_0})$.

Proof. For $\mathbf{x}_0 \in M_0 \backslash \{0\}$, let $B^n(\mathbf{x}_0, r)$ denote the n-ball of radius $r > 0$ in the tangent plane $T_{\mathbf{x}_0} M_0$ and $C(\mathbf{x}_0, r)$ denote the cylinder at x_0 over $B^n(\mathbf{x}_0, r)$.

Recalling Definition 4.3.3, we write $\hat{w}^i_{\mathbf{x}_0}$ for the parabolic dilation of the function constructed via pseudolocality at $\mathbf{x}_0 \in M_0$ by $C_{\mathrm{scale}}|\mathbf{x}_0|$.

Proposition 4.3.4 hence shows

$$\bigcup_{i\in\{1,2\}} \bigcup_{\mathbf{x}_0\in M_{R_{\min}}} \left\{\left(\hat{w}^i_{\mathbf{x}_0}(\cdot,t_0), B^n(\mathbf{x}_0, \delta C_{\text{scale}}|\mathbf{x}_0|), C(\mathbf{x}_0, \delta C_{\text{scale}}|\mathbf{x}_0|)\cap M^i_{t_0}\right)\right\},$$

forms a common graphical atlas for $M^1_{t_0}\backslash\overline{B}(0, R\sqrt{t_0})$ and $M^2_{t_0}\backslash\overline{B}(0, R\sqrt{t_0})$, where $M_{R_{\min}} := M_0\backslash\overline{B}(0, R_{\min}\sqrt{t_0}))$

QED

Claim 4.B.3. There exists a common graphical atlas for $M^1_{t_0}\cap B(0, 4R\sqrt{t_0})$ and $M^2_{t_0}\cap B(0, 4R\sqrt{t_0})$.

Proof. This is achieved by finding suitable charts on a portion of the expanding flow $\mathcal{M}_\Sigma$ and using convergence. We recall Remark 4.3.10: in the expander region $\Omega(4R, T)$, each time-slice of the flows $\mathcal{RM}^j\lfloor\tilde{\Omega}(4R, T)$ can be parameterised as graphs over a non-explicit, but controlled, subset of the expander Σ. We write $\Xi = \Sigma_{4R+\varepsilon}$. Further, our parameterising function is defined on Σ_{8R}, so we can be sure the flow is well defined as a graph over Ξ.

On the time interval $(0, \infty)$, $\mathcal{M}_\Sigma$ is a smooth flow with well understood asymptotic geometry. We can find a uniform radius $r > 0$ and functions

$$w_{x_0} : P^n((\mathbf{x_0}, 1), r) \to \mathbb{R},$$

where $P^n((\mathbf{x_0}, 1), r) := B^n(\mathbf{x}_0, r) \times (1 - r^2, 1 + r^2)$[§], such that we may write, for every $(\mathbf{x}_0, 1) \in \Sigma\times\{1\}$,

$$\mathcal{M}_\Sigma\lfloor C(\mathbf{x}_0, r) \times (1 - r^2, 1 + r^2) = \text{graph}_{P^n((\mathbf{x_0},1),r)} w_{\mathbf{x}_0}.$$

As a consequence of local smooth convergence, we may presume T_0 has been chosen small enough that we may write, for $t_0 \in (0, T_0)$, $\mathbf{x}_0 \in \Xi$,

$$\hat{\mathcal{M}}^i\lfloor C(\mathbf{x}_0, r) \times (1 - r^2, 1 + r^2) = \text{graph}_{P^n((\mathbf{x}_0,1),r)} w^i_{\mathbf{x}_0,t_0}$$

where $\hat{\mathcal{M}}^i = \mathcal{D}_{\sqrt{t_0}}(\mathcal{M}^i)$ and $w^i_{\mathbf{x}_0,t_0} : P^n((\mathbf{x}_0, 1), r) \to \mathbb{R}$.

Remark 4.B.7. Note, by scaling we may presume $r > 0$ has been taken sufficiently small that, for $\tau \in (1 - r^2, 1 + r^2)$,

$$\begin{aligned}\|w_{\mathbf{x}_0}(\cdot,\tau)\|_{C^0(B^n(\mathbf{x}_0,r))} &< \frac{\eta_1 r}{2},\\ \|\nabla w_{\mathbf{x}_0}(\cdot,\tau)\|_{C^0(B^n(\mathbf{x}_0,r))} &< \frac{\eta_1}{2},\end{aligned}$$

where η_1 is the constant from Proposition 4.B.2 and ∇ denotes the spatial derivatives. Such a scaling argument works as Σ is smooth.

[§]We call P^n a parabolic n-ball

We return to the proof. By taking $T_0 > 0$ sufficiently small, we may presume for $\mathbf{x}_0 \in \Xi$ and $\tau \in (1 - r^2, 1 + r^2)$

$$\begin{aligned} \left\| w^i_{\mathbf{x}_0,t_0}(\cdot,\tau) \right\|_{C^0(B^n(\mathbf{x}_0,r))} &< \eta_1 r, \\ \left\| \nabla w^i_{\mathbf{x}_0,t_0}(\cdot,\tau) \right\|_{C^0(B^n(\mathbf{x}_0,r))} &< \eta_1. \end{aligned}$$

Reversing the dilation, we have

$$\bigcup_{i\in\{1,2\}} \bigcup_{\mathbf{x}_0\in\sqrt{t_0}\Xi} \left\{ \left(\hat{w}^i_{\mathbf{x}_0,t_0}(\cdot,t_0), B^n(\mathbf{x}_0,\sqrt{t_0}r), C(\mathbf{x}_0,\sqrt{t_0}r)\cap M^i_{t_0}\right)\right\}$$

forms a common graphical atlas for $M^1_{t_0} \cap B(0, 4R\sqrt{t_0})$ and $M^2_{t_0} \cap B(0, 4R\sqrt{t_0})$. As in the pseudolocal case, $\hat{w}$ denotes the parabolic dilation of the function w. QED

Since $\mathcal{G}(R,T)$ and $\Omega(4R,T)$ overlap in $\mathrm{Col}(R,4R,T)$, we have constructed a common graphical atlas for $M^1_{t_0}$ and $M^2_{t_0}$. This holds for all $t_0 \in (0,T)$.

To construct the space-time atlas, we note that the functions constructed above were time-slices of functions defined over parabolic n-balls. Choose $\{t_j\}_{j=1}^{\infty} \subset (0,T)$ to be a set of times dense in $[0,T]$. Repeating the above construction for each t_j, and taking the union over j, yields a space-time graphical atlas for $\mathcal{M}^1$, $\mathcal{M}^2$ restricted to the time interval $(0,T)$. QED

4.B.2 Transforming between coordinate systems

Our barriers are defined by considering graphs over the expander, whilst the separation estimate describes the behaviour of graphs over the smooth flow $\mathcal{M}^1$. We therefore need to be able to translate the 'height' information between these two coordinate systems. This is used in Lemma 4.6.3, where we first use the 'barrier height' to get a 'height' of $\mathcal{RM}^2$ as a graph over $\mathcal{RM}^1$. This is then propagated out over the rest of the flow by the maximum principle, Theorem 4.5.1. To complete the argument, we turn this height back into the 'height over the expander' in order to take 'lower' barriers on the same time interval.

For the following results, we will use the setup as stated in Section 4.6. To recall, we let $\mathcal{M}^1, \mathcal{M}^2$ be two mean curvature flows from M_0, smooth on the time interval $(0,T), T > 0$. We suppose M_0 satisfies Assumption 4.2.1 and the flows $\mathcal{M}^i$ satisfy Assumption 4.2.2 B with respect to the chosen expander Σ. Fix $R \in (1,\infty), \mathfrak{R} \in (2C_{\mathrm{len}}R, \infty)$ and $\alpha \in (0,\infty)$ and let s_0, ε_0 be as in Proposition 4.4.4. Let T be such that $\Omega(C_{\mathrm{len}}R, T)$ is a $(\Sigma, \varepsilon_0, C_{\mathrm{len}}R)$-expander region for both $\mathcal{M}^1, \mathcal{M}^2$. We suppose the functions

$$u^{\pm}_{s,\alpha,R,\mathfrak{R}} : \Sigma_R \times (-\infty, \log(T)) \to \mathbb{R}$$

satisfy the conclusion of Proposition 4.4.4 for $s \in [0, s_0]$.

Lemma 4.B.8. *For every $\vartheta \in (1,2)$, there exists a $\varepsilon_1 = \varepsilon_1(\vartheta) \in (0,\varepsilon_0)$ such that if $\Omega(C_{\mathrm{len}}R, T')$ is a $(\Sigma, \varepsilon_1, C_{\mathrm{len}}R)$-expander region for $\mathcal{M}$, we have*

$$|d(\mathbf{x}, \nu_{\mathcal{RM}^1}(\mathbf{x},\tau), \Gamma_s^{\pm}(\tau))| \leq \vartheta s f_\alpha^{\mathfrak{R}}(u_1^{-1}(\mathbf{x},\tau)),$$

where $(\mathbf{x},\tau) \in \mathcal{RM}^1$, $|\mathbf{x}| = R$, $\tau \in (-\infty, \log(T'))$ and $s \in (0, s_0)$.

Remark 4.B.9. The above result is motivated by the Euclidean model case. Consider the n-plane:

$$\Pi := \{\mathbf{x} \in \mathbb{R}^{n+1} \text{ s.t. } \langle e_{n+1}, \mathbf{x}\rangle = 0\}$$

and the cone

$$\mathcal{C} := \mathrm{graph}_\Pi(a + b|\mathbf{x}|).$$

Let $V : \Pi \to \mathbb{R}^{n+1}$ be a unit-vector field on Π. By elementary trigonometry, for every $\varepsilon > 0$ there exists a $\delta = \delta(b,\varepsilon) > 0$ such that if $|V - e_{n+1}| < \delta$, we have,

$$a \leq d(0, V(0), \mathcal{C}) < (1+\varepsilon)a.$$

Proof. Fix $\vartheta \in (1,2)$. We show the claim for $\Gamma_s^+(\tau)$, the claim for the other barrier follows with the relevant signs and orientations reversed. The idea of the proof is to exploit the bounds on the gradient to reduce the problem to the above remark.

We define the 'truncated cone'

$$V^+(\gamma,\sigma) := \left\{ v \in \mathbb{R}^{n+1}, |v| \leq \gamma, \angle(v, e_{n+1}) = \sigma \right\},$$

where $\angle(v_1, v_2)$ denotes the angle between the vectors v_1, v_2. For every $s \in [0, s_0]$, u_s^+ is a smooth function on $\Sigma_{C_{\mathrm{len}}R} \times (-\infty, \log(T))$. By the definition of u_s^+, we have gradient bounds on $(-\infty, \log(T))$. In particular, the Lipschitz constant for each $u_s^+(\cdot,\tau)$ is bounded by a constant uniform in s and τ. We deduce[¶] there exist $\gamma_0 \in (0,\infty), \sigma_0 \in (0,1)$ such that

$$V^+(\mathbf{y},\tau,s,\gamma_0) := \left\{ \mathbf{y} + u_s^+(\mathbf{y},\tau)\nu_\Sigma(\mathbf{y}) + v,\ v \in A(e_{n+1}, \nu_\Sigma(\mathbf{y}))V^+(\gamma_0,\sigma_0) \right\}$$

lies strictly outside $\Gamma_s^+(\tau)$ for each $\mathbf{y} \in \Sigma_{C_{\mathrm{len}}R}$, and $s \in [0,s_0]$, $\tau \in (-\infty, \log(T))$. Here, $A(e_{n+1}, \nu_\Sigma(\mathbf{y}))$ denotes the rotation taking e_{n+1} to $\nu_\Sigma(\mathbf{y})$.

[¶]This follows by noting the intrinsic distance of Σ is 'close' to the Euclidean distance in some small ball around each point. Note, the radius of this ball can actually be taken uniform across Σ, since the asymptotic geometry is controlled.

We consider the following set of unit vectors:

$$W^+(\mathbf{y},\tau,s_0,\gamma) :=$$
$$\{w\in\mathbb{R}^{n=1}, w := \frac{s_0 f_\alpha^{\mathfrak{R}}(\mathbf{y})\nu_\Sigma(\mathbf{y})+v}{|s_0 f_\alpha^{\mathfrak{R}}(\mathbf{y})\nu_\Sigma(\mathbf{y})+v|}, v\in A(e_{n+1},\nu_\Sigma(\mathbf{y}))V^+(\gamma_0,\sigma_0)\},$$

these are precisely the directions of the line segments joining $\mathbf{x}$ to points in $V^+(\mathbf{y},\tau,s_0,\gamma_0)$.

For $\gamma_0=0$, we have

$$V^+(\mathbf{y},\tau,s_0,0)=\{u_{s_0}^+(\mathbf{y},\tau)\nu_\Sigma(\mathbf{y})+\mathbf{y}\},$$

and thus

$$W^+(\mathbf{y},\tau,s_0,0)=\{\nu_\Sigma(\mathbf{y})\}.$$

Thus, by continuity, there exists $\gamma_1=\gamma_1(\vartheta,s_0)\in(0,\gamma_0)$ such that for $w\in W^+(\mathbf{x},\tau,s_0,\gamma_1)$ we have

$$d(\mathbf{x},w,V^+(\mathbf{y},\tau,s_0,\gamma_1))\le\vartheta s_0 f_\alpha^{\mathfrak{R}}(\mathbf{y}).$$

Moreover, for $s\in[0,s_0]$, a straightforward scaling argument yields

$$d(\mathbf{x},w,V^+(\mathbf{y},\tau,s,\gamma_1))\le\vartheta s f_\alpha^{\mathfrak{R}}(\mathbf{y}).$$

From Lemma 4.3.11 there exists a T' such that for $\tau\in(-\infty,\log(T'))$, we have

$$\nu_{\mathcal{RM}^1}(\mathbf{x},\tau)\in W^+(\mathbf{y},\tau,s_0,\gamma_1),$$

since $\nu_\Sigma(\mathbf{y})\in W^+(\mathbf{y},\tau,s_0,\gamma_1)$.

The lemma follows since the line segment joining $\mathbf{x}$ to $V^+(\mathbf{y},\tau,s,\gamma_1)$ must pass through $\Gamma_s^+(\tau)$, since $V^+(\mathbf{y},\tau,s,\gamma_1)$ and $\mathbf{x}$ are on opposite sides of $\Gamma_s^+(\tau)$. QED

Proposition 4.B.10. *For every $\vartheta\in(1,2)$, there exists an $\varepsilon_1=\varepsilon_1(\vartheta)\in(0,\varepsilon_0)$ such that, with the same assumptions as Lemma 4.B.8, we have*

$$|d(\mathbf{x},\nu_{\mathcal{RM}^1}(\mathbf{x},\tau),\Gamma_s^\pm(\tau))|\le\vartheta\max_{\mathbf{y}\in\Sigma,|\mathbf{y}|=R} s f_\alpha^{\mathfrak{R}}(\mathbf{y}),$$

where $(\mathbf{x},\tau)\in\mathcal{RM}^1$, $|\mathbf{x}|=R$, $\tau\in(-\infty,\log(T'))$ and $s\in(0,s_0)$.

Proof. Fix $\vartheta\in(1,2)$ and $(\mathbf{x},\tau)\in\mathcal{RM}^1$ as in the statement. The function $f_\alpha^{\mathfrak{R}}$ is smooth, and thus Lipschitz, i.e.

$$f_\alpha^{\mathfrak{R}}(\mathbf{y})\le f_\alpha^{\mathfrak{R}}(\mathbf{z})+\omega_{f_\alpha^{\mathfrak{R}}} d_\Sigma(\mathbf{y},\mathbf{z}),$$

for $\mathbf{y}, \mathbf{z} \in \Sigma_{\mathfrak{R}}$, where $\omega_{f^{\mathfrak{R}}_{\alpha}}$ denotes the Lipschitz constant of $f^{\mathfrak{R}}_{\alpha}$ and d_{Σ} is the intrinsic distance function for Σ.

Let $Z := \{\mathbf{z} \in \Sigma, |\mathbf{z}| = R\}$. Note, if $\Omega(C_{\text{len}}R, T)$ is a $(\Sigma, \varepsilon_1, C_{\text{len}}R)$-expander region, then by Remark 4.3.10, $||u_1^{-1}(\mathbf{x}, \tau)| - R| \leq \varepsilon$. Thus, it is clear we can take ε_1 sufficiently small that

$$d_{\Sigma}(u_1^{-1}(\mathbf{x}, \tau), Z) < (\sqrt{\vartheta} - 1)\frac{\max_{\mathbf{y} \in \Sigma, |\mathbf{y}|=R} f^{\mathfrak{R}}_{\alpha}}{\omega_{f^{\mathfrak{R}}_{\alpha}}}.$$

With this choice of ε_1, we have

$$f^{\mathfrak{R}}_{\alpha}(u_1^{-1}(\mathbf{x}, \tau)) \leq \sqrt{\vartheta} \max_{\mathbf{y} \in \Sigma, |\mathbf{y}|=R} f^{\mathfrak{R}}_{\alpha}(y).$$

If we additionally take ε_1 sufficiently small that Lemma 4.B.8 holds for $\sqrt{\vartheta}$, then the proposition follows. QED

The second part of the welding argument requires we go from a bound on the 'height' over $\mathcal{M}^1$ to a 'height' over Σ.

Proposition 4.B.11. *Let R For every $\vartheta \in (1, 2)$, there exists $\varepsilon_2 \in (0, \varepsilon_0)$ such that if $\varepsilon < \varepsilon_2$ and $\Omega(C_{\text{len}}R, T')$ is a $(\Sigma, \varepsilon, C_{\text{len}}R)$-expander region, and*

$$|u|(\mathbf{x}, t) \leq h\sqrt{t},\ \mathbf{x} \in M^1_t \backslash \overline{B}(0, R\sqrt{t}),\ t \in (0, T').$$

Then,

$$|u_2 - u_1|(\mathbf{x}, \tau) \leq \vartheta h,\ \mathbf{x} \in \Sigma, |\mathbf{x}| = C_{\text{len}}R,\ \tau \in (-\infty, \log(T')).$$

Proof. This again follows from essentially the same trigonometric argument as Lemma 4.B.8, as our choice of ε_2 controls (at least) the C^2-norm of $u_1(\mathbf{x}, \tau)$, and the angle between $\nu_{\Sigma}(\mathbf{x})$ and $\nu_{\mathcal{RM}^1}(u_1(\mathbf{x}, \tau))$. Note, the trigonometry is more straightforward as we now consider barriers that are a constant height over the flow $\mathcal{RM}^1$. QED

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