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STABILIZATION OF HYBRID STOCHASTIC DIFFERENTIAL EQUATIONS BY DELAY FEEDBACK CONTROL BASED ON DISCRETE-TIME OBSERVATIONS

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Dedicated to Professor George Yin on the Occasion of his 70th Birthday

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Abstract. Response lags are necessary for most physical systems. For the sake of saving time and costs, the main aim of this paper is to design the feedback control term based on the response lags varying in a certain interval and the discrete-time observations of both the system states and the Markovian states to stabilize the controlled hybrid systems. The control principles are established, which permit the control function only depends on the partial information of the states and the modes. The upper bound on the sum of the upper bound $\bar{\tau}$ of response lags, and the duration τ between two consecutive observations is obtained. Some examples and numerical experiments are given to illustrate our theory.

1. **Introduction.** Hybrid stochastic differential equations (HSDEs) whose coefficients depend on the states of continuous-time Markov chains provide more realistic models to describe many systems in branches of science and industry. In the study of HSDEs, automatic control is one of the critical issues, with subsequent emphasis placed on the analysis of stability [14, 15, 23]. There are intensive literature on the stabilization theory, for example, [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

Consider an unstable HSDE described by

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dW(t), \ t \geq 0, \tag{1}$$

with the initial data

$$x(0) = x_0 \in \mathbb{R}^n, \ r(0) = i_0 \in \mathbb{S}, \tag{2}$$

where the state $x(t)$ takes values in $\mathbb{R}^n$, the mode $r(t)$ is a Markov chain taking values in a finite sates space $\mathbb{S}$, and $W(t)$ is an m-dimensional Brownian motion.

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Recently, for the sake of saving time and costs, Li *et al.* [10] established the delay feedback control principles for hybrid stochastic differential equations (HSDEs) based on the discrete-time observations of both the system states and the Markovian states. Namely, they designed the control term $u(x(\nu(t)-\tau_0), r(\nu(t)-\tau_0), t)$, where τ_0 is the response lag, and $\nu(t) = [t/\tau]\tau$ ($[t/\tau]$ is the integer part of t/τ) with $\tau > 0$ being the duration between two consecutive observations, such that the controlled HSDE

$$dx(t) = [f(x(t), r(t), t) + u(x(\nu(t)-\tau_0), r(\nu(t)-\tau_0), t)]dt \\ + g(x(t), r(t), t)dW(t)$$

becomes stable in pth moment, with probability one or in H_∞. However, the control function used in [10] has the form $u(x, i, t) = -\alpha(i)x$, where $\alpha(i)$'s are nonnegative constants, which implies that all components of the state $x(t)$ need to be controllable. Once any component of $x(t)$ is unobservable (then uncontrollable), the control function designed here fails.

Actually, it is hard to control some components of systems directly. For example, in finance, operators need to make decisions with a part of asymmetric information; in the industry, some component information of systems may be non-detectable, and then the state feedback information stems only from a part of the components. A question arises: Could we design a feedback control with such incomplete information to stabilize the controlled HSDEs?

Mao *et al.* [18] answered this question by designing the control function with the form $u(x, i) = D(i)x$ to stabilize the controlled HSDE, where $D(i) = F(i)G(i)$ with $F(i) \in \mathbb{R}^{n\times l}$ and $G(i) \in \mathbb{R}^{l\times n}$. They focused on designing one of $F(\cdot)$ and $G(\cdot)$ as the other one is known even degenerate. After that, sustained efforts are made to elaborate on the stabilization principle based on the discrete-time observations and enlarging the duration between two consecutive observations, for example, [8, 17, 19]. Due to the realistic requirement of both response lags and the discrete-time observations on the pair $(x(t), r(t))$, it is necessary to develop the input and output feedback control theory. In fact, due to the lack of continuity, even if $\tau+\tau_0$ is sufficiently small, $(x(\nu(t)-\tau_0), r(\nu(t)-\tau_0))$ may take different values from $(x(t), r(t))$ which brings essential difficulties for the stability analysis. Authors in [8, 10, 25, 28] tackled this trouble.

On the other hand, the response lag τ_0 in [10] is a constant. However, in practice, the response lags may take different values; for instance, when driving a car, the response time will be prolonged if the driver gets sidetracked. Meanwhile, designing a feedback control with a strict constant time lag is quite costly and burdensome. Dong and Mao [5] proposed the time-varying response lag within a determined interval in the feedback control, which is much easier to design and costs less. Therefore, it is more realistic to design the delay feedback control based on the discrete-time observations of both the system states and the Markovian states, where the response lag takes values in a determined interval.

Combined with the discrete-time observations, it is natural that the response lags happen after observations. Assume that $\{\tau_k\}_{k=0}^{\infty}$ is the sequence of response lags where τ_k ($k \in \{0, 1, 2, \cdots\}$) represents the response lag at the $(k+1)$th observation time taking values in $[0, \bar{\tau}]$ ($0 < \bar{\tau} \le \tau$) (it is reasonable to restrict $\bar{\tau} \le \tau$ since if $\tau < \bar{\tau}$, we can adjust the observation duration τ such that $\bar{\tau} \le \tau$). Therefore, the actuation duration of the feedback control corresponding to the $(k+1)$th observation for the states $(x(k\tau), r(k\tau))$ is $[k\tau + \tau_k, (k+1)\tau + \tau_{k+1})$. We give some definitions that express the feedback control function u based on such response lags and the discrete-time observations on $t \geq 0$. Define k_t be the unique nonnegative integer such that $k_t\tau + \tau_{k_t} \leq t < (k_t+1)\tau + \tau_{k_t+1}$, for $t \geq \tau_0$, and $k_t = -1$ for $t \in [0, \tau_0)$. Also set $r(-\tau) = r(0)$, $x(-\tau) = 0$ and $\tau_{-1} = 0$. Then define the delay function $\delta : R_+ \to [0, \tau + \bar{\tau}]$ by $\delta(t) = t - k_t\tau$, for $t \geq 0$. Thus for any $t \geq 0$, $u(x(t-\delta(t)), r(t-\delta(t)), t) = u(x(k_t\tau), r(k_t\tau), t)$. Thus, our main aim in this paper is to design the delay feedback control $u(x(t-\delta(t)), r(t-\delta(t)), t)$ so that the controlled HSDE

$$\begin{aligned} dx(t) = {} & [f(x(t), r(t), t) + u(r(t-\delta(t)), x(t-\delta(t)), t)]dt \\ & + g(x(t), r(t), t)dW(t), \quad t \geq 0 \end{aligned} \tag{3}$$

becomes stable in the mean square.

Mathematically speaking, this paper mainly utilizes the Lyapunov functional analysis, the linear matrix inequalities (LMIs), and the strong ergodicity theory of Markov chains to propose various criteria for uniform boundedness and the mean square exponential stability for linear and quasi-linear HSDEs. The main contributions of this paper are as follows:

- Targeted at the unstable HSDEs with incomplete information, we design proper feedback control function to stabilize the controlled HSDEs. To be precise, we define the feedback control function u depending on the matrix of the input information F and the output information G where F, G may be a degenerate matrix, such that the controlled HSDEs become stable.
- The feedback control function designed in this paper depends on the discrete-time observations and the response lag, a variable in a determined time interval but a fixed constant, which is much easier to design and costs less in practice. Therefore, it has the advantages of simple design and low cost in practical applications.
- Making use of the structure features of HSDEs, the upper bound of $\tau + \bar{\tau}$ is obtained explicitly such that the feedback control will stabilize the given system as long as $\tau + \bar{\tau}$ smaller than the upper bound. The Lyapunov exponential dependent on the value of $\tau + \bar{\tau}$ is also obtained.

The structure of the paper is as follows. Section 2 begins with notations and preliminaries on stabilization problems. Section 3 and Section 4 pay attention to the stability analysis of linear and quasi-linear HSDEs, respectively. The examples are used to illustrate the theoretical results. Section 5 concludes this paper.

2. Preliminary.

Throughout this paper, we use the following notations. Let $\mathbb{R}_+ = [0, +\infty)$. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a matrix or vector, its transpose is denoted by A^T. If A is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$, and its operator norm is denoted by $\|A\| = \max\{|Ax| : |x| = 1\}$. One notices that $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$. Especially, if A is symmetric positive definite, then $\|A\| = \lambda_{\max}(A)$. By $A \geq 0$ ($A > 0$), we mean A is non-negative (positive) definite. For two sequences of matrices $\{A(i)\}_{1\leq i\leq N}$ and $\{B(i)\}_{1\leq i\leq N}$ with appropriate dimensions, let $A_i = A(i)$, $A_iB_i = A(i) \times B(i)$, $M_A = \max_{1\leq i\leq N} \|A_i\|$ and $M_{AB} = \max_{1\leq i\leq N} \|A_iB_i\|$. For any $a, b \in \mathbb{R}$, let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. Let $W(t) = (W_1(t), \dots, W_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space, taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ ($N < \infty$) with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}(r(t+\Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $o(\Delta)$ satisfies $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $W(t)$. One notices that almost all sample paths of $r(t)$ are right continuous. To simplify the notation, we write $\check{\tau} := \tau + \bar{\tau}$.

In this paper, $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous and grow at most linearly (see, e.g. [23, p. 89, 91]). For easy operation, we use the feedback control function with a simple form $u(x, i, t) = D(i)x$ for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $D(i) \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{S}$. Then the controlled HSDE (3) becomes

$$dx(t) = [f(x(t), r(t), t) + D(r(t - \delta(t)))x(t - \delta(t))]dt + g(x(t), r(t), t)dW(t), \quad (4)$$

for $t \geq 0$ with the initial data

$$x(s) = x_0 \in \mathbb{R}^d \quad r(s) = r_0 \in \mathbb{S}, \quad s \in [-\check{\tau}, 0]. \quad (5)$$

Thus, by virtue of the results in [23], both system (1) and system (4) have a unique global solution. Next, we prepare some notations for the controlled HSDE (4). Define two segments $\overline{x}_t(s) := \{x(t+s) : -2\check{\tau} \leq s \leq 0\}$ and $\overline{r}_t(s) := \{r(t+s) : -2\check{\tau} \leq s \leq 0\}$ for $t \geq 0$. To make $\overline{x}_t$ and $\overline{r}_t$ well defined on $0 \leq t \leq 2\check{\tau}$, we set $x(s) = x_0$ and $r(s) = r_0$ for $s \in [-2\check{\tau}, -\check{\tau})$. Meanwhile, we enlarge the corresponding definition domains of f, g and u. For any $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times [-2\check{\tau}, 0)$, let $f(x, i, t) = f(x, i, 0)$, $g(x, i, t) = g(x, i, 0)$, $u(x, i, t) = u(x, i, 0)$. Now, define the Lyapunov functional as

$$V(\overline{x}_t, \overline{r}_t, t) = x^T(t)Q(r(t))x(t) + \eta I(t),$$

where for each $i \in \mathbb{S}$, $Q(i) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\eta > 0$ is a constant to be determined later, and

$$\begin{aligned} I(t) = \int_{-\check{\tau}}^{0}\int_{t+s}^{t}\Big[\check{\tau}\big|f(x(z), r(z), z) + D(r(z - \delta(z)))x(z - \delta(z))\big|^2 \\ + \big|g(x(z), r(z), z)\big|^2\Big]dzds. \end{aligned} \quad (6)$$

A direct calculation arrives at

$$dI(t) = J_1(t)dt - J_2(t)dt,$$

where

$$\begin{aligned} J_1(t) &= \check{\tau}\Big[\check{\tau}\big|f(x(t), r(t), t) + D(r(t - \delta(t)))x(t - \delta(t))\big|^2 + \big|g(x(t), r(t), t)\big|^2\Big], \\ J_2(t) &= \int_{-\check{\tau}}^{0}\Big[\check{\tau}\big|f(x(t+s), r(t+s), t+s) + D(r(t+s-\delta(t+s))) \\ &\quad \times x(t+s-\delta(t+s))\big|^2 + \big|g(x(t+s), r(t+s), t+s)\big|^2\Big]ds. \end{aligned} \quad (7)$$

Changing the integration order of (6) implies

$$I(t) \leq \check{\tau}J_2(t). \quad (8)$$

To control the deviation between $x(t)$ and $x(t-\delta(t))$ in (4), we analyze as follows. Using the Hölder inequality and the Itô isometry formula, we obtain from (4) that for $t-\delta(t)\geq 0$,

$$
\begin{aligned}
&\mathbb{E}|x(t)-x(t-\delta(t))|^2\\
&=\mathbb{E}\Big|\int_{t-\delta(t)}^{t}\big[f(x(s),r(s),s)+D(r(s-\delta(s)))x(s-\delta(s))\big]ds+\int_{-\delta(t)}^{t}g(x(s),r(s),s)dW(s)\Big|^2\\
&\leq 2\mathbb{E}\int_{t-\check{\tau}}^{t}\check{\tau}\big|f(x(s),r(s),s)+D(r(s-\delta(s)))x(s-\delta(s))\big|^2ds+2\mathbb{E}\int_{t-\check{\tau}}^{t}|g(x(s),r(s),s)|^2ds\\
&=2\mathbb{E}J_2(t),
\end{aligned}
$$

for $t\geq 0$ and $t-\delta(t)\leq 0$,

$$
\begin{aligned}
&\mathbb{E}|x(t)-x(t-\delta(t))|^2=\mathbb{E}|x(t)-x(0)|^2\\
&=\mathbb{E}\Big|\int_{0}^{t}\big[f(x(s),r(s),s)+D(r(s-\delta(s)))x(s-\delta(s)))\big]ds+\int_{0}^{t}g(x(s),r(s),s)dW(s)\Big|^2\\
&\leq 2\mathbb{E}\int_{0}^{t}\check{\tau}\big|f(x(s),r(s),s)+D(r(s-\delta(s)))x(s-\delta(s))\big|^2ds+2\mathbb{E}\int_{0}^{t}|g(x(s),r(s),s)|^2ds\\
&\leq 2\mathbb{E}\int_{t-\check{\tau}}^{t}\check{\tau}\big|f(x(s),r(s),s)+D(r(s-\delta(s)))x(s-\delta(s))\big|^2ds+2\mathbb{E}\int_{t-\check{\tau}}^{t}|g(x(s),r(s),s)|^2ds\\
&=2\mathbb{E}J_2(t),
\end{aligned}
$$

where the first inequality follows from $\check{\tau}\geq\delta(t)\geq t$. In consequence, for $t\geq 0$,

$$\mathbb{E}|x(t)-x(t-\delta(t))|^2\leq 2\mathbb{E}J_2(t). \tag{9}$$

On the other hand, a direct application of the generalized Itô formula (see, e.g., [23, p. 47-49]) derives

$$\mathbb{E}V(\overline{x}_t,\overline{r}_t,t)=\mathbb{E}V(x_0,i_0,0)+\mathbb{E}\int_0^t\mathcal{L}V(\overline{x}_s,\overline{r}_s,s)ds,$$

where

$$
\begin{aligned}
\mathcal{L}V(\overline{x}_t,\overline{r}_t,t)=&\,2x^T(t)Q(r(t))\big[f(x(t),r(t),t)+D(r(t-\delta(t)))x(t-\delta(t))\big]\\
&+\operatorname{trace}\big(g^T(x(t),r(t),t)Q(r(t))g(x(t),r(t),t)\big)\\
&+\sum_{j=1}^{n}\gamma_{r(t),j}x^T(t)Q(j)x(t)+\eta J_1(t)-\eta J_2(t).
\end{aligned}\tag{10}
$$

Here $\gamma_{r(t),j}=\gamma_{ij}$ when $r(t)=i$.

To close this section, we present a criterion on the mean square exponential stability of HSDEs.

Lemma 2.1. *If there are positive constants δ_1, δ_2 such that*

$$\mathbb{E}\mathcal{L}V(\overline{x}_t,\overline{r}_t,t)\leq-\delta_1\mathbb{E}|x(t)|^2-\delta_2\mathbb{E}J_2(t),\quad\forall t\geq 0, \tag{11}$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^2)\leq-\theta, \tag{12}$$

where

$$\theta=\min\Big\{\frac{\delta_1}{\lambda_M},\frac{\delta_2}{\eta\check{\tau}}\Big\},\quad\lambda_M=\max_{i\in\mathbb{S}}\lambda_{\max}(Q_i). \tag{13}$$

Proof. By the generalized Itô formula (see, e.g., [23, p. 47-49]), we have

$$e^{\theta t}\mathbb{E}V(\overline{x}_t,\overline{r}_t,t)=\mathbb{E}V(x_0,i_0,0)+\int_0^t e^{\theta s}\big[\theta\mathbb{E}V(\overline{x}_s,\overline{r}_s,s)+\mathbb{E}\mathcal{L}V(\overline{x}_s,\overline{r}_s,s)\big]ds, \quad (14)$$

where $\mathcal{L}V(\overline{x}_t,\overline{r}_t,t)$, θ are given by (10), (13) respectively. The matrix inequality together with the definition of $V(\overline{x}_t,\overline{r}_t,t)$ and (8) yields

$$\mathbb{E}V(\overline{x}_t,\overline{r}_t,t)\le\lambda_M\mathbb{E}|x(t)|^2+\eta\check{\tau}\mathbb{E}J_2(t),$$

where λ_M is defined by (13). Substituting the above inequality and (11) into (14), we get by the definition of θ that

$$\begin{aligned}&e^{\theta t}\mathbb{E}V(\overline{x}_t,\overline{r}_t,t)\\&\le\mathbb{E}V(x_0,i_0,0)-\big(\delta_1-\lambda_M\theta\big)\int_0^t e^{\theta s}\mathbb{E}|x(s)|^2ds-\big(\delta_2-\eta\check{\tau}\theta\big)\int_0^t e^{\theta s}\mathbb{E}J_2(s)ds\\&\le\mathbb{E}V(x_0,i_0,0)<\infty.\end{aligned}$$

We therefore derive

$$\lambda_m e^{\theta t}\mathbb{E}|x(t)|^2\le e^{\theta t}\mathbb{E}V(\overline{x}_t,\overline{r}_t,t)\le\mathbb{E}V(x_0,i_0,0),$$

where $\lambda_m=\min_{i\in S}\lambda_{\min}(Q_i)$. Then, the desired assertion (12) follows. □

3. **Stabilization of Linear HSDEs.** This section focuses on the stabilization of linear HSDEs. Consider an unstable linear HSDE described by

$$dx(t)=A(r(t))x(t)dt+\sum_{k=1}^m H_k(r(t))x(t)dW_k(t),\quad t\ge 0,$$

where for each $i\in\mathbb{S}$, $A(i),H_k(i)\in\mathbb{R}^{n\times n}, k=1,2,\dots,m$. Based on the discrete observations and the response lags, the controlled HSDE becomes

$$dx(t)=\big[A(r(t))x(t)+D(r(t-\delta(t)))x(t-\delta(t))\big]dt+\sum_{k=1}^m H_k(r(t))x(t)dW_k(t), \quad (15)$$

with the initial data given in (5). One notices that

$$f(x,i,t)=A(i)x,\ g(x,i,t)=(H_1(i)x,\cdots,H_m(i)x),$$

which are globally Lipschitz continuous.

Theorem 3.1. *Suppose that for each $i\in\mathbb{S}$, there exists a symmetric positive definite matrix $Q(i)=Q_i\in\mathbb{R}^{n\times n}$ such that*

$$\bar{Q}(i)=\bar{Q}_i:=Q_i(A_i+D_i)+(A_i+D_i)^TQ_i+\sum_{k=1}^m H_k^T(i)Q_iH_k(i)+\sum_{j=1}^N\gamma_{ij}Q_j<0.$$

Then, for any $0<\check{\tau}<\tau^*:=\big(1/(4M_D)\big)\wedge y^*$, the solution of HSDE (15)-(5) is exponentially stable in the mean square, where y^* is the positive root of $\beta(y)=\lambda$,

$$\begin{aligned}&\beta(y):=\sqrt{y}\Big[2\sqrt{2}M_{QD}N_H+3\lambda_M\Lambda+\Big(\frac{\sqrt{2}M_{QD}}{N_H}+8\lambda_M\Lambda y\Big)\Big(2y(M_A^2+2M_D^2)+N_H\Big)\Big],\\&\Lambda=\max_{i\in\mathbb{S}}\sqrt{-\gamma_{ii}}\max_{i,j\in\mathbb{S}}\|D_j-D_i\|,\ -\lambda=\max_{i\in\mathbb{S}}\lambda_{\max}(\bar{Q}_i),\ N_H=\max_{i\in\mathbb{S}}\sum_{k=1}^m\|H_k(i)\|^2,\end{aligned} \quad (16)$$

and λ_M is given by (13).

Proof. Fix $0 < \check{\tau} < \tau^*$. Using the elementary inequality, we derive from (10) that

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t)\\
&= \mathbb{E}\big[x^T(t)\bar{Q}(r(t))x(t) + 2x^T(t)Q(r(t))[(D(r(t-\delta(t))) - D(r(t)))x(t-\delta(t))\\
&\quad - D(r(t))(x(t) - x(t-\delta(t)))] + \eta J_1(t) - \eta J_2(t)\big]\\
&\le -\big(\lambda - 2\sqrt{2}M_{QD}N_H\sqrt{\check{\tau}} - \lambda_M\Lambda\sqrt{\check{\tau}}\big)\mathbb{E}|x(t)|^2 + I_1(t) + \eta\mathbb{E}J_1(t) - \eta\mathbb{E}J_2(t),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
I_1(t) = &\frac{M_{QD}}{2\sqrt{2}N_H\sqrt{\check{\tau}}}\mathbb{E}|x(t) - x(t-\delta(t))|^2\\
&+ \frac{\lambda_M}{\Lambda\sqrt{\check{\tau}}}\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|x(t-\delta(t))|^2\big].
\end{aligned}$$

We proceed to estimate the term $\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|x(t-\delta(t))|^2\big]$ in $I_1(t)$. The property of conditional expectation gives that for any t with $t-\delta(t) \ge 0$,

$$\begin{aligned}
&\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|x(t-\delta(t))|^2|\mathcal{F}_{t-\delta(t)}\big]\\
&= |x(t-\delta(t))|^2\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|\mathcal{F}_{t-\delta(t)}\big].
\end{aligned} \tag{18}$$

One further obtains that for any t with $t - \delta(t) \ge 0$,

$$\begin{aligned}
&\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|\mathcal{F}_{t-\delta(t)}\big]\\
&= \mathbb{E}\big[\sum_{i\in\mathbb{S}} I_{\{r(t-\delta(t))=i\}}\|D(r(t)) - D(i)\|^2|\mathcal{F}_{t-\delta(t)}\big]\\
&= \sum_{i\in\mathbb{S}} I_{\{r(t-\delta(t))=i\}}\mathbb{E}\big[I_{\{r(t)\ne i\}}\|D(r(t)) - D(i)\|^2|r(t-\delta(t)) = i\big]\\
&\le \max_{i,j\in\mathbb{S}}\|D_j - D_i\|^2\sum_{i\in\mathbb{S}} I_{\{r(t-\delta(t))=i\}}\mathbb{E}\big[I_{\{r(t)\ne i\}}|r(t-\delta(t)) = i\big]\\
&\le \max_{i,j\in\mathbb{S}}\|D_j - D_i\|^2\sum_{i\in\mathbb{S}} I_{\{r(t-\delta(t))=i\}}\mathbb{P}\big(\exists v \in (t-\delta(t), t], r(v) \ne i|r(t-\delta(t)) = i\big).
\end{aligned} \tag{19}$$

Note that the waiting time for the next jump of the Markov chain $r(\cdot)$ from current state j obeys the exponential distribution with parameter $-\gamma_{jj}$ (see, e.g., [23]) and $1 - e^{-x} \le x$ for $x \ge 0$. Thus the elementary inequality together with (18) and (19) yields that for any t with $t - \delta(t) \ge 0$,

$$\begin{aligned}
&\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|x(t-\delta(t))|^2\big]\\
&\le \mathbb{E}\big[|x(t-\delta(t))|^2\max_{i,j\in\mathbb{S}}\|D_j - D_i\|^2\sum_{i\in\mathbb{S}} I_{\{r(t-\delta(t))=i\}}(1 - e^{\gamma_{ii}\check{\tau}})\big]\\
&\le 2\Lambda^2\check{\tau}\mathbb{E}|x(t)|^2 + 2\Lambda^2\check{\tau}\mathbb{E}|x(t) - x(t-\delta(t))|^2,
\end{aligned} \tag{20}$$

where Λ is given in (16). By the initial value (5), as $t - \delta(t) < 0$, we have

$$\begin{aligned}
&\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2|x(t-\delta(t))|^2\big]\\
&\le \mathbb{E}\big[|x_0|^2\max_{j\in\mathbb{S}}\|D_j - D_{r_0}\|^2(1 - e^{\gamma_{r_0r_0}t})\big]\\
&\le 2\Lambda^2\check{\tau}\mathbb{E}|x(t)|^2 + 2\Lambda^2\check{\tau}\mathbb{E}|x(t) - x_0|^2\\
&= 2\Lambda^2\check{\tau}\mathbb{E}|x(t)|^2 + 2\Lambda^2\check{\tau}\mathbb{E}|x(t) - x(t-\delta(t))|^2.
\end{aligned} \tag{21}$$

Combining (20) and (21), one obtains that for $t \geq 0$,

$$\begin{aligned}&\mathbb{E}\big[\|D(r(t)) - D(r(t-\delta(t)))\|^2 |x(t-\delta(t))|^2\big]\\ &\leq 2\Lambda^2 \check{\tau}\mathbb{E}|x(t)|^2 + 2\Lambda^2\check{\tau}\mathbb{E}|x(t) - x(t-\delta(t))|^2.\end{aligned} \tag{22}$$

Inserting (22) into (17) yields

$$\begin{aligned}&\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t)\\ &\leq -\big(\lambda - 2\sqrt{2}M_{QD}N_H\sqrt{\check{\tau}} - 3\lambda_M\Lambda\sqrt{\check{\tau}}\big)\mathbb{E}|x(t)|^2 + \Big(\frac{M_{QD}}{2\sqrt{2}N_H\sqrt{\check{\tau}}}\\ &\quad +2\lambda_M\Lambda\sqrt{\check{\tau}}\Big)\mathbb{E}|x(t) - x(t-\delta(t))|^2 + \eta\mathbb{E}J_1(t) - \eta\mathbb{E}J_2(t).\end{aligned} \tag{23}$$

Now we deal with $\mathbb{E}J_1(t)$. Using the definition of $J_1(t)$ in (7) and the elementary inequality, we compute

$$\begin{aligned}\mathbb{E}J_1(t) &\leq \check{\tau}\mathbb{E}\big[2\check{\tau}\big(|A(r(t))x(t)|^2 + |D(r(t-\delta(t)))x(t-\delta(t))|^2\big) + N_H|x(t)|^2\big]\\ &\leq \check{\tau}(2M_A^2\check{\tau} + N_H)\mathbb{E}|x(t)|^2 + 2M_D^2\check{\tau}^2\mathbb{E}[|x(t)|^2 + |x(t) - x(t-\delta(t))|^2].\end{aligned}$$

Inserting the above inequality into (23) and using (9) imply that

$$\begin{aligned}&\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t)\\ &\leq -\Big(\lambda - 2\sqrt{2}M_{QD}N_H\sqrt{\check{\tau}} - 3\lambda_M\Lambda\sqrt{\check{\tau}} - \eta\check{\tau}\big(2\check{\tau}(M_A^2 + 2M_D^2) + N_H\big)\Big)\mathbb{E}|x(t)|^2\\ &\quad - \Big((1 - 8M_D^2\check{\tau}^2)\eta - \frac{M_{QD}}{\sqrt{2}N_H\sqrt{\check{\tau}}} - 4\lambda_M\Lambda\sqrt{\check{\tau}}\Big)\mathbb{E}J_2(t).\end{aligned}$$

Fix $\eta = \sqrt{2}M_{QD}/(N_H\sqrt{\check{\tau}}) + 8\lambda_M\Lambda\sqrt{\check{\tau}} > 0$. The above inequality becomes

$$\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t) \leq -\delta_1\mathbb{E}|x(t)|^2 - \delta_2\mathbb{E}J_2(t), \tag{24}$$

where $\delta_1 = \lambda - \beta(\check{\tau}), \delta_2 = \big(2(1 - 8M_D^2\check{\tau}^2) - 1\big)\big(M_{QD}/(\sqrt{2}N_H\sqrt{\check{\tau}}) + 4\lambda_M\Lambda\sqrt{\check{\tau}}\big)$. Due to the definition of τ^* and the increasing property of function $\beta(\cdot)$ defined by (16), one observes that

$$\lambda - \beta(\check{\tau}) > 0,\ 1 - 8M_D^2\check{\tau}^2 > \frac{1}{2},$$

which implies $\delta_1 > 0$ and $\delta_2 > 0$. Thus, by Lemma 2.1, the desired assertion follows from (24). □

Now we go a further step to consider the case $D(i) = F(i)G(i)$, where $F(i) \in \mathbb{R}^{n\times l}$ and $G(i) \in \mathbb{R}^{l\times n}$. Then the controlled system (15) becomes

$$\begin{aligned}dx(t) = &\big[A(r(t))x(t) + F(r(t-\delta(t)))G(r(t-\delta(t)))x(t-\delta(t))\big]dt\\ &+ \sum_{k=1}^{m} H_k(r(t))x(t)dW_k(t).\end{aligned} \tag{25}$$

Given F or G, we aim to design the other using LMIs (see, e.g., [23]). Both cases are known as:

- State feedback: design $F(\cdot)$ when $G(\cdot)$ is given
- Output injection: design $G(\cdot)$ when $F(\cdot)$ is given.

3.1. **State feedback.** This subsection investigates designing the mapping $F : \mathbb{S} \to \mathbb{R}^{n\times l}$ for stabilization as the mapping $G : \mathbb{S} \to \mathbb{R}^{l\times n}$ is given. By virtue of Theorem 3.1, for each $i \in \mathbb{S}$, we need to find a symmetric positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$ and a matrix $F_i \in \mathbb{R}^{n\times l}$ such that

$$\bar{Q}_i = Q_iA_i + A_i^TQ_i + Q_iF_iG_i + G_i^TF_i^TQ_i + \sum_{k=1}^{m} H_k^T(i)Q_iH_k(i) + \sum_{j=1}^{N}\gamma_{ij}Q_j < 0. \quad (26)$$

The above matrix inequality is not linear in Q_i and F_i. Let $Y_i = Q_iF_i$. Then (26) becomes the following LMI

$$\bar{Q}_i = Q_iA_i + A_i^TQ_i + Y_iG_i + G_i^TY_i^T + \sum_{k=1}^{m} H_k^T(i)Q_iH_k(i) + \sum_{j=1}^{N}\gamma_{ij}Q_j < 0. \quad (27)$$

If the above inequality has the solution Q_i with $Q_i = Q_i^T > 0$ and Y_i, then (26) holds with $F_i = Q_i^{-1}Y_i$. Thus, we yield the following corollary under Theorem 3.1.

Corollary 3.2. *Assume that for each $i \in \mathbb{S}$, (27) has the solution $Q_i \in \mathbb{R}^{n\times n}$ with $Q_i = Q_i^T > 0$ and $Y_i \in \mathbb{R}^{n\times l}$. Let $F_i = Q_i^{-1}Y_i$, $D_i = F_iG_i$. Then, for $0 < \check{\tau} < \tau^*$, the controlled HSDE (25)-(5) is exponentially stable in the mean square.*

To illustrate the result of Corollary 3.2, we give the following example where some system states are uncontrollable.

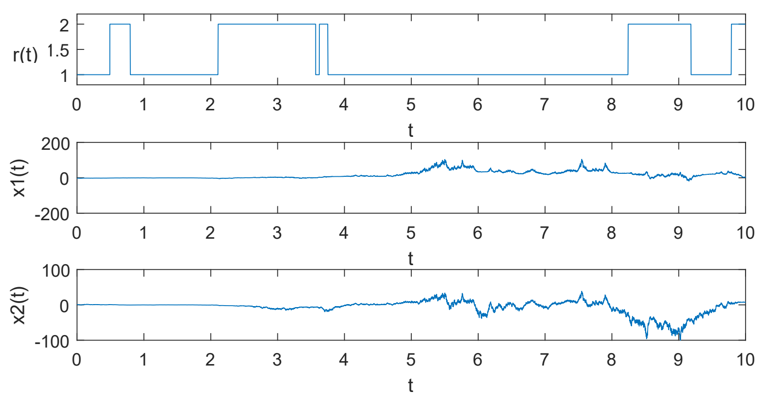

FIGURE 1. Paths of $r(t)$ and $x_1(t), x_2(t)$ of HSDE (28)-(29).

Example 3.1. Consider the 2-dimensional linear HSDE as the examples discussed in [8] and [18]

$$dx(t) = A(r(t))x(t)dt + H(r(t))x(t)dW(t), \ t \geq 0, \quad (28)$$

with initial data

$$x(0) = [x_1(0), x_2(0)]^T = [-2, 1]^T, \quad r(0) = 1. \quad (29)$$

Here $W(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator $\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; The parameter matrices are

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \ A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}, \ H(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ H(2) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Figure 1 depicts HSDE (28)-(29) is not exponentially stable in the mean square.

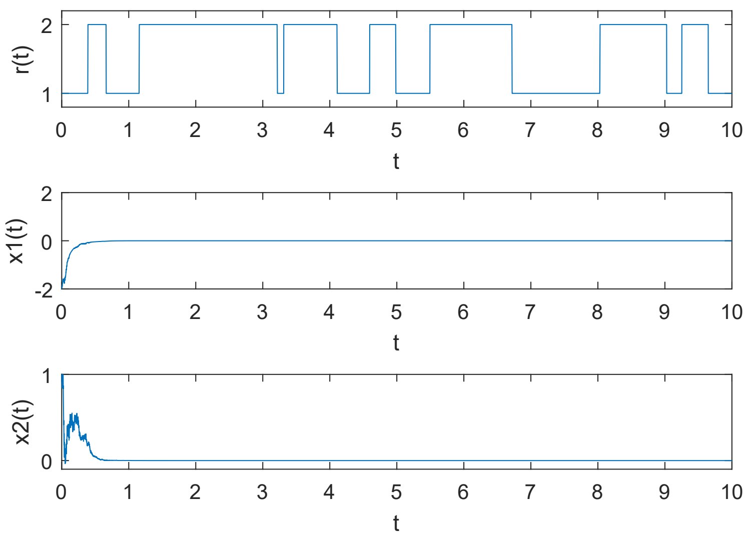

FIGURE 2. Paths of $r(t)$ and $x_1(t), x_2(t)$ of (30)-(31) in the state feedback case with $\tau = 0.005$ and $\bar{\tau} = 0.002$.

Based on both the delay response and the discrete-time observations of state and mode, the controlled HSDE is described by

$$\begin{aligned} dx(t) = &\big[A(r(t))x(t) + F(r(t-\delta(t)))G(r(t-\delta(t)))x(t-\delta(t))\big]dt \\ &+ H(r(t))x(t)dW(t), \end{aligned} \tag{30}$$

where $G_1 = (1,0)$ and $G_2 = (0,1)$ are known. Here, the response lag at each observation is a random value in $[0, \bar{\tau}]$. The corresponding initial data is

$$x(s) = x(0), \quad r(s) = r(0), \quad s \in [-\check{\tau}, 0]. \tag{31}$$

where $x(0)$ and $r(0)$ are defined in (29). In order for the exponential stability of (30), by virtue of Corollary 3.2, we require that

$$\bar{Q}_i := Q_iA_i + A_i^TQ_i + Y_iG_i + G_i^TY_i^T + H^T(i)Q_iH(i) + \sum_{j=1}^{2}\gamma_{ij}Q_j < 0 \tag{32}$$

holds for $i = 1, 2$. One notices that matrices

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ Y_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \text{ and } Q_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ Y_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix},$$

satisfy (32) with

$$\bar{Q}_1 = \begin{bmatrix} -14 & 0 \\ 0 & -18 \end{bmatrix}, \ \bar{Q}_2 = \begin{bmatrix} -18 & 0 \\ 0 & -14 \end{bmatrix}.$$

Let $F_1 = Q_1^{-1}Y_1, \ F_2 = Q_2^{-1}Y_2$. Then we compute

$$-\lambda = \max_{i=1,2} \lambda_{\max}(\bar{Q}_i) = -14, \ M_{QD} = M_{YG} = \max_{i=1,2} \|Y_iG_i\| = 10,$$

$$M_A = 5.2361,\ N_H = 2,\ M_D = 10,\ \Lambda = 10,\ \lambda_M = 2,\ \tau^* = 0.0075.$$

Choose $\tau = 0.005$, $\bar{\tau} = 0.002$. Then HSDE (28) is mean square exponentially stable by using Corollary 3.2. Figure 2 dispects the paths of $r(t)$ and $x_1(t)$, $x_2(t)$ of (30)-(31) in the state feedback case with $\tau = 0.005$ and $\bar{\tau} = 0.002$ by the Euler-Maruyama numerical method. One observes that by our result, $\tau < 0.0075$ is enough for the stabilization as $\bar{\tau} = 0$ while $\tau < 1.5 \times 10^{-5}$ is required by [8]. In other words, we obtain the wider range of τ.

3.2. **Output injection.** This subsection focuses on designing the mapping $G : \mathbb{S} \to \mathbb{R}^{l\times n}$ for stabilization as the mapping $F : \mathbb{S} \to \mathbb{R}^{n\times l}$ is given. Under Theorem 3.1, for each $i \in \mathbb{S}$, it is sufficient to find a symmetric positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$ and a matrix $G_i \in \mathbb{R}^{l\times n}$ such that (26) holds. Define $X_i = Q_i^{-1}$. By multiplying X_i from left and right, we derive from (26) that

$$\bar{Q}_i = A_iX_i + X_iA_i^T + F_iG_iX_i + X_iG_i^TF_i^T + \sum_{k=1}^{m} X_iH_k^T(i)X_i^{-1}H_k(i)X_i + \sum_{j=1}^{N} \gamma_{ij}X_iX_j^{-1}X_i < 0.$$

Let $Y_i := G_iX_i$. Then we have

$$\bar{Q}_i = A_iX_i + X_iA_i^T + F_iY_i + Y_i^TF_i^T + \sum_{k=1}^{m} X_iH_k^T(i)X_i^{-1}H_k(i)X_i + \sum_{j=1}^{N} \gamma_{ij}X_iX_j^{-1}X_i < 0.$$

By the Schur complements [23, Theorem 2.8, p. 64], the above inequality is equivalent to the following LMI

$$\begin{bmatrix} M_{i1} & M_{i2} & M_{i3} \\ M_{i2}^T & -M_{i4} & 0 \\ M_{i3}^T & 0 & -M_{i5} \end{bmatrix} < 0, \tag{33}$$

where

$$\begin{aligned}
M_{i1} &= A_iX_i + X_iA_i^T + F_iY_i + Y_i^TF_i^T + \gamma_{ii}X_i,\\
M_{i2} &= \left[X_iH_1^T(i), \cdots, X_iH_m^T(i)\right],\\
M_{i3} &= \left[\sqrt{\gamma_{i1}}X_i, \cdots, \sqrt{\gamma_{i(i-1)}}X_i, \sqrt{\gamma_{i(i+1)}}X_i, \cdots, \sqrt{\gamma_{iN}}X_i\right],\\
M_{i4} &= \mathrm{diag}[X_i, \cdots, X_i],\\
M_{i5} &= \mathrm{diag}[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_N].
\end{aligned}$$

If (33) has the solution X_i with $X_i = X_i^T > 0$ and Y_i, then (26) holds with $Q_i = X_i^{-1}$, $G_i = Y_iX_i^{-1}$. Thus, we gain the following corollary through Theorem 3.1.

Corollary 3.3. *Assume that for each $i \in \mathbb{S}$, (33) has the solution $X_i \in \mathbb{R}^{n\times n}$ with $X_i = X_i^T > 0$ and $Y_i \in \mathbb{R}^{l\times n}$. Let $Q_i = X_i^{-1}, G_i = Y_iX_i^{-1}, D_i = F_iG_i$. Then for $0 < \check{\tau} < \tau^*$, the controlled HSDE (25)-(5) is exponentially stable in the mean square.*

Example 3.2. Consider the linear HSDE (30) with the initial data (31), where $F_1 = [1, 0]^T$, $F_2 = [0, 1]^T$. Here, the response delays are random values in $[0, \bar{\tau}]$. In

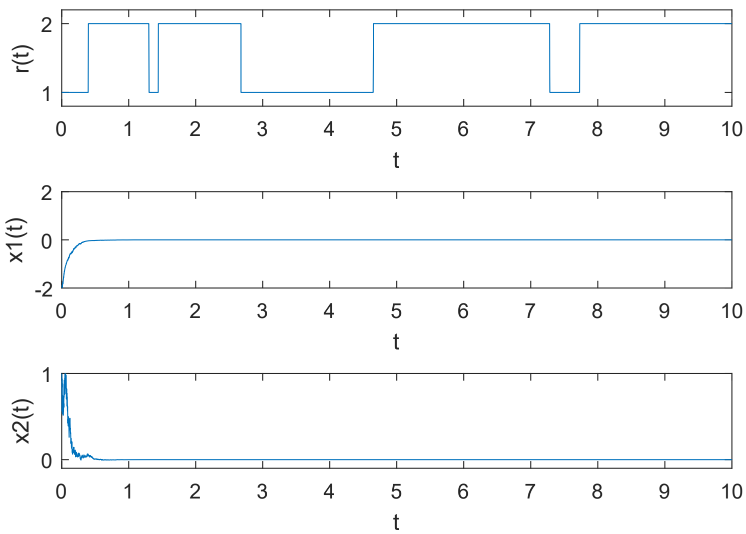

FIGURE 3. Paths of $r(t)$ and $x_1(t), x_2(t)$ of (30)-(31) in the state feedback case with $\tau = 0.0024$ and $\bar{\tau} = 0.002$.

order for the exponential stability of (28), using Corollary 3.3, we require that

$$\begin{aligned} \bar{Q}_i := & A_i X_i + X_i A_i^T + F_i Y_i + Y_i^T F_i^T + X_i H^T(i) X_i^{-1} H(i) X_i \\ & + \sum_{j=1}^{N} \gamma_{ij} X_i X_j^{-1} X_i < 0 \end{aligned} \tag{34}$$

holds for $i = 1, 2$. One notices that matrices

$$X_1 = X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; Y_1 = \begin{bmatrix} -10 & 0 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 0 & -10 \end{bmatrix},$$

satisfy (34) with

$$\bar{Q}_1 = \begin{bmatrix} -16 & 0 \\ 0 & -8 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -8 & 0 \\ 0 & -16 \end{bmatrix}.$$

Let $G_1 = Y_1 X_1^{-1}, G_2 = Y_2 X_2^{-1}$. Hence, we compute

$$\begin{aligned} & -\lambda = \max_{i=1,2} \lambda_{\max}(\bar{Q}_i) = -8, \quad M_{QD} = M_{FY} = 10, \quad M_A = 5.2361, \\ & N_H = 2, \; M_D = 10, \; \Lambda = 10, \; \lambda_M = 1, \; \tau^* = 0.0046. \end{aligned}$$

Choose $\tau = 0.0024$, $\bar{\tau} = 0.002$. Then, the controlled system (30)-(31) is mean square exponentially stable by Corollary 3.3. Figure 3 dispects the paths of $r(t)$ and $x_1(t)$, $x_2(t)$ of (30)-(31) in the state feedback case with $\tau = 0.0024$ and $\bar{\tau} = 0.002$ by the Euler-Maruyama numerical method. It is worth noting that with $\bar{\tau} = 0$, our result covers the results in [19].

4. **Stabilization of quasi-linear HSDEs.** This section discusses the stabilization of quasi-linear HSDEs. Consider an unstable quasi-linear HSDE described by (1) with initial data (2), where f and g are locally Lipschitz continuous and grow

linearly (see, e.g., [23, p. 89, 91]). Based on the discrete observations, the controlled delay HSDE becomes

$$dx(t) = \big(f(x(t), r(t), t) + D(r(t-\delta(t)))x(t-\delta(t))\big)dt + g(x(t), r(t), t)dW(t), \quad (35)$$

with the initial data (5).

For convenience, we impose some assumptions on f and g.

Assumption 4.1. For each $i \in \mathbb{S}$, there is a pair of symmetric matrices $Q_i, \hat{Q}_i \in \mathbb{R}^{n\times n}$ with $Q_i > 0$ such that

$$2x^T Q_i f(x,i,t) + \operatorname{trace}\big(g^T(x,i,t)Q_i g(x,i,t)\big) \le x^T \hat{Q}_i x$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Assumption 4.2. There is a pair of positive constants K_1 and K_2 such that

$$|f(x,i,t)|^2 \le K_1|x|^2 \quad \text{and} \quad |g(x,i,t)|^2 \le K_2|x|^2$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Theorem 4.3. *Let* Assumption 4.1 *and* Assumption 4.2 *hold. If for each* $i \in \mathbb{S}$,

$$U(i) := U_i = \hat{Q}_i + Q_i D_i + D_i^T Q_i^T + \sum_{j=1}^n \gamma_{ij} Q_j < 0. \quad (36)$$

Then, for $0 < \check{\tau} < \tilde{\tau}^* := \big(1/(4M_D)\big) \wedge y^*$, *the solution of HSDE* (35)-(5) *is exponential stable in the mean square, where* y^* *is the positive root of* $\iota(y) = \gamma$,

$$\iota(y) := \sqrt{y}\Big[2\sqrt{2}M_{QD}K_2 + 3\lambda_M\Lambda + \Big(\frac{\sqrt{2}M_{QD}}{K_2} + 8\lambda_M\Lambda y\Big)\Big(2y\big(K_1 + 2M_D^2\big) + K_2\Big)\Big]. \quad (37)$$

Here $\gamma = -\max_{i\in\mathbb{S}} \lambda_{\max}(U_i)$, *and* λ_M, Λ *are given by* (13) *and* (16), *respectively.*

Proof. Since the proof uses techniques similar to those in the proof of Theorem 3.1, we only give the outline to avoid duplication. Fix $0 < \check{\tau} < \tilde{\tau}^*$. By Assumption 4.1 and the elementary inequality, it follows from (10) that

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t)\\
&\le \mathbb{E}\big[x^T(t)U(r(t))x(t) + 2x^T(t)Q(r(t))\big((D(r(t-\delta(t))) - D(r(t)))x(t-\delta(t))\\
&\quad - D(r(t))(x(t) - x(t-\delta(t)))\big) + \eta J_1(t) - \eta J_2(t)\big]\\
&\le -\big(\gamma - 2\sqrt{2}M_{QD}K_2\sqrt{\check{\tau}} - \lambda_M\Lambda\sqrt{\check{\tau}}\big)\mathbb{E}|x(t)|^2 + I_2(t) + \eta\mathbb{E}J_1(t) - \eta\mathbb{E}J_2(t),
\end{aligned} \quad (38)$$

where

$$\begin{aligned}
I_2(t) = &\frac{M_{QD}}{2\sqrt{2}K_2\sqrt{\check{\tau}}}\mathbb{E}|x(t) - x(t-\delta(t))|^2\\
&+ \frac{\lambda_M}{\Lambda\sqrt{\check{\tau}}}\mathbb{E}[\|D(r(t)) - D(r(t-\delta(t)))\|^2 |x(t-\delta(t))|^2].
\end{aligned}$$

By the definition of $J_1(t)$ in (7) and Assumption 4.2, we compute

$$\mathbb{E}J_1(t) \le \check{\tau}(2(K_1 + 2M_D^2)\check{\tau} + K_2)\mathbb{E}|x(t)|^2 + 4M_D^2\check{\tau}^2\mathbb{E}|x(t) - x(t-\delta(t))|^2.$$

This together with (38) gives

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(\overline{x}_t, \overline{r}_t, t)\\
&\le -\Big(\lambda - 2\sqrt{2}M_{QD}K_2\sqrt{\check{\tau}} - 3\lambda_M\Lambda\sqrt{\check{\tau}} - \eta\check{\tau}\big(2\check{\tau}(K_1 + 2M_D^2) + K_2\big)\Big)\mathbb{E}|x(t)|^2
\end{aligned}$$

$$-\Big(\big(1-8M_D^2\check{\tau}^2\big)\eta-\frac{M_{QD}}{\sqrt{2}K_2\sqrt{\check{\tau}}}-4\lambda_M\Lambda\sqrt{\check{\tau}}\Big)\mathbb{E}J_2(t).$$

Fix $\eta=\sqrt{2}M_{QD}/(K_2\sqrt{\check{\tau}})+8\lambda_M\Lambda\sqrt{\check{\tau}}>0$. Then by (9) and (20) , we arrive at

$$\mathbb{E}\mathcal{L}V(\overline{x}_t,\overline{r}_t,t)\le-\delta_1\mathbb{E}|x(t)|^2-\delta_2\mathbb{E}J_2(t), \tag{39}$$

where

$$\delta_1=\gamma-\iota(\check{\tau}),\quad \delta_2=\Big(2(1-8M_D^2\check{\tau}^2)-1\Big)\Big(\frac{M_{QD}}{\sqrt{2}K_2\sqrt{\check{\tau}}}+4\lambda_M\sqrt{\Lambda\check{\tau}}\Big).$$

Due to the definition of $\tilde{\tau}^*$ and the increasing of function $\iota(\cdot)$ defined in (37), one observes that

$$\gamma-\iota(\check{\tau})>0,\ 1-8M_D^2\check{\tau}^2>\frac{1}{2},$$

which implies $\delta_1>0,\ \delta_2>0$. Thus, by virtue of Lemma 2.1, the desired assertion follows from (39). □

We continue to analysis the design of feedback control with $D(i)=F(i)G(i)$ ($i\in\mathbb{S}$), where $F(i)\in\mathbb{R}^{n\times l}$ and $G(i)\in\mathbb{R}^{l\times n}$. Thus (36) becomes

$$U(i):=U_i=\hat{Q}_i+Q_iF_iG_i+G_i^TF_i^TQ_i^T+\sum_{j=1}^n\gamma_{ij}Q_j<0. \tag{40}$$

When F_i or G_i is given, the aim is to design the other. If (40) has the solution F_i or G_i as G_i or F_i is given respectively. Thus, making use of Theorem 4.3, one obtains the following corollary.

Corollary 4.4. *Let* Assumption 4.1 *and* Assumption 4.2 *hold. Assume that for each $i\in\mathbb{S}$, (40) has the solution F_i or G_i as G_i or F_i is given. Let $D_i=F_iG_i$. Then, for $0<\check{\tau}<\tilde{\tau}^*$, the solution of HSDE (35)-(5) is exponential stable in the mean square.*

To apply this theory, we need two steps:

1. Seek suitable matrices Q_i and $\hat{Q}_i$ satisfying Assumption 4.1;
2. Solve (40) as F_i (or G_i) is given.

In the rest part of this section, we focus on developing the techniques to deal with step 1. Then, we proceed the step 2 by using Matlab software. We only discuss the state feedback case to avoid duplication, while the other one is similar. One notices that the flexible choice of $2N$ matrices Q_i and $\hat{Q}_i$ means more work involved in practice. Then, we introduce an extra assumption.

Assumption 4.5. There are $N+1$ symmetric matrices $Z,\ Z_i\in\mathbb{R}^{n\times n}$ with $Z>0$ such that

$$2x^TZf(x,i,t)+\text{trace}\Big(g^T(x,i,t)Zg(x,i,t)\Big)\le x^TZ_ix,$$

for all $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$.

Let $Q_i=q_iZ$ and $\hat{Q}_i=q_iZ_i$ for some $q_i>0$. Then (40) becomes

$$U_i=q_iZ_i+q_iZF_iG_i+q_iG_i^TF_i^TZ+\sum_{j=1}^n\gamma_{ij}q_jZ<0,\quad i\in\mathbb{S}.$$

This together with $Y_i = q_i F_i$ implies

$$U_i = q_i Z_i + Z Y_i G_i + G_i^T Y_i^T Z + \sum_{j=1}^{n} \gamma_{ij} q_j Z < 0, \quad i \in \mathbb{S}. \tag{41}$$

For each $i \in \mathbb{S}$, if (41) has the solution $q_i > 0$ and Y_i, then (40) holds with $Q_i = q_i Z$, $\hat{Q}_i = q_i Z_i$ and $F_i := q_i^{-1} Y_i$. Then, we obtain the following result by virtue of Corollary 4.4.

Corollary 4.6. *Let* Assumption 4.2 *and* Assumption 4.5 *hold. Assume that for each $i \in \mathbb{S}$, (41) has the solution $q_i > 0$ and Y_i. Let $Q_i = q_i Z$, $\hat{Q}_i = q_i Z_i$, $F_i = q_i^{-1} Y_i$ and $D_i = F_i G_i$. Then for $0 < \check{\tau} < \tilde{\tau}^*$, the controlled HSDE (35)-(5) is exponentially stable in the mean square.*

We further introduce a relatively simple (but stronger, in fact) assumption.

Assumption 4.7. For each $i \in \mathbb{S}$ there is a constant $z_i > 0$ such that

$$2x^T f(x,i,t) + |g(x,i,t)|^2 \le z_i |x|^2$$

holds for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Define $Q_i = q_i I$, $\hat{Q}_i = q_i z_i I$, for some $q_i > 0$, where I is an identity matrix in $\mathbb{R}^{n \times n}$. Then (40) becomes

$$U_i = q_i z_i I + q_i F_i G_i + q_i G_i^T F_i^T + \sum_{j=1}^{n} \gamma_{ij} q_j I < 0, \ \ i \in \mathbb{S}.$$

Let $Y_i = q_i F_i$. The above inequality directly becomes

$$U_i = q_i z_i I + Y_i G_i + G_i^T Y_i^T + \sum_{j=1}^{n} \gamma_{ij} q_j I < 0, \ \ i \in \mathbb{S}. \tag{42}$$

For each $i \in \mathbb{S}$, if (42) has the solution $q_i > 0$ and Y_i, then (40) holds with $Q_i = q_i I$, $\hat{Q}_i = q_i z_i I$ and $F_i = q_i^{-1} Y_i$. Then, we yield the following result by virtue of Corollary 4.4.

Corollary 4.8. *Let* Assumption 4.2 *and* Assumption 4.7 *hold. Assume that for each $i \in \mathbb{S}$, (42) has the solution $q_i > 0$ and Y_i. Let $Q_i = q_i I$, $\hat{Q}_i = q_i z_i I$, $F_i = q_i^{-1} Y_i$ and $D_i = F_i G_i$. Then, for $0 < \check{\tau} < \tilde{\tau}^*$, the controlled HSDE (35)-(5) is exponentially stable in the mean square.*

Finally, we illustrate our result with an example: some Markovian states are unobservable (and then uncontrollable).

Example 4.1. Consider a 2-dimensional unstable HSDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dW(t), \quad t \ge 0,$$

with initial data $x(0) = x_0, r(0) = i_0$. Here $f, \ g : \mathbb{R}^2 \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^2$ are locally Lipschitz continuous and satisfy Assumption 4.2 and Assumption 4.7, $W(t)$ is a scalar Brownian motion, and $\{r(t)\}_{t \ge 0}$ is a Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with generater

$$\Gamma = \begin{bmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{bmatrix}, \quad (\gamma_{12}, \gamma_{21} > 0).$$

Assume that mode 1 ($r(t) = 1$) is observable, mode 2 ($r(t) = 2$) is unobservable. Thus, for mode 1, we design a delay feedback control based on the discrete observations and the response lags, taking values in $[0, \bar{\tau}]$ randomly. Then, the controlled HSDE becomes

$$\begin{aligned} dx(t) =& \big(f(x(t), r(t), t) + F(r(t-\delta(t)))G(r(t-\delta(t)))x(t-\delta(t))\big)dt \\ &+ g(x(t), r(t), t)dW(t), \end{aligned} \tag{43}$$

with initial data (5), where $G_1 = I$ (Here I is an identity matrix in $\mathbb{R}^{2\times 2}$), $G_2 = \mathbf{0}$. Choose $F_2 = \mathbf{0}$ naturally. By virtue of Corollary 4.8, we require that

$$\begin{aligned} U_1 &= q_1 z_1 I + Y_1 + Y_1^T - \gamma_{12} q_1 I + \gamma_{12} q_2 I = -I, \\ U_2 &= q_2 z_2 I + \gamma_{21} q_1 I - \gamma_{21} q_2 I = -I, \end{aligned} \tag{44}$$

hold with $q_1,\ q_2 > 0$ and $Y_1 \in \mathbb{R}^{2\times 2}$. To solve (44), one needs $z_2 - \gamma_{21} < 0$ obviously, which implies that the switching rate from mode 2 to mode 1 should be large. Go a further step, one notices that

$$q_1 = 1, \quad q_2 = \frac{\gamma_{21} + 1}{\gamma_{21} - z_2}, \quad Y_1 = \frac{1}{2}\Big(-1 - z_1 + \gamma_{12} + \frac{\gamma_{12}(\gamma_{21}+1)}{z_2 - \gamma_{21}}\Big) I$$

satisfy (44). Thus we obtain that $Q_1 = q_1 I, Q_2 = q_2 I, D_1 = F_1 = Y_1, D_2 = 0$. We compute

$$\lambda_M = q_1 \vee q_2, \quad \Lambda = \frac{1}{2}\Big(\sqrt{\gamma_{12}} \vee \sqrt{\gamma_{21}}\Big)\Big| -1 - z_1 + \gamma_{12} + \frac{\gamma_{12}(\gamma_{21}+1)}{z_2 - \gamma_{21}}\Big|,$$

$$\gamma = 1, \quad M_{QD} = \frac{q_1}{2}\Big(-1 - z_1 + \gamma_{12} + \frac{\gamma_{12}(\gamma_{21}+1)}{z_2 - \gamma_{21}}\Big).$$

Therefore $\tilde{\tau}^*$ is obtained by the definition in Theorem 4.3. Choose τ and $\bar{\tau}$ sufficient small such that $\tau + \bar{\tau} < \tilde{\tau}^*$. Thus, by Corollary 4.6, the controlled system (43)-(5) is exponentially stable in the mean square.

5. **Conclusion.** In this paper, we design the delay feedback control function u based on discrete observations and partial information of both the system states and the Markovian states to stabilize the controlled system where the response lags vary in an interval. Making use of the structure features of linear and nonlinear HSDEs, we obtain the upper bound of $\tau + \bar{\tau}$ explicitly such that the feedback control stabilizes the given system as long as $\tau + \bar{\tau}$ smaller than the upper bound. Moreover, Example 3.1 shows that we can find the broader range of τ.

We have further advanced in two aspects: (a) The delay feedback designed here can stabilize the controlled HSDEs with incomplete information, namely, the feedback control function u depends on the degenerate matrix $F \times G$, which dealt with the case that Li et al. [10] can not cover. (b) The time lag in the feedback control takes value in a determined time interval $[0, \bar{\tau}]$ but a fixed constant, which is much easier to design and costs less in practice.

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