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POWER-LOW EXPANSION IN K-ESSENCE COSMOLOGY

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We study spatially flat isotropic universes driven by k -essence. It is shown that Friedmann and k -field equations may be analytically integrated for arbitrary k -field potentials during evolution with a constant baryotropic index. It follows that there is an infinite number of dynamically different k -theories with equivalent kinematics of the gravitational field. We show that there is a large “window” of stable solutions, and that the dust-like behaviour separates stable from unstable expansion. Restricting to the family of power law solutions, it is argued that the linear scalar field model, with constant function F , is isomorphic to a model with divergent speed of sound and this makes them less suitable for cosmological modeling than the non-linear k -field solutions we find in this paper.

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To address the outstanding theoretical challenges of modern cosmology, especially the so-called coincidence problem, which questions as why is it exactly now, the universe driven by some sort of dark energy is accelerating, several authors have introduced and studied the so-called k -essence models^{1,2,3,4}.

Originally, the k -essence, or k -inflation, was introduced in⁵ in order to bridge phenomenologically the string theories with inflation (see Ref.⁶ for a recent review). The main ingredient of the k -essence is a scalar field, with non standard higher order kinetic terms. Interestingly enough, and contrary to what one could have expected, these non-standard terms do not necessarily lead to acausal propagation of the k -field⁷. Studying inflationary patterns with the k -fields the authors of⁵ were able to show that k -field may drive an accelerated expansion of the universe starting from

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a generic initial conditions without an assistance of the usual potential terms.

In a different development ^{1,2}, the k -essence was proposed as a dynamical solution to the coincidence problem. The basic idea of ^{1,2} is that k -essence could play a role of a dynamical attractor at the onset of matter domination period introducing cosmic acceleration at present time. Further study of k -essence was performed recently in ⁴. It was argued that in certain dynamical regimes the k -essence is equivalent to quintessence and it may prove difficult to distinguish between the two fields. In this Letter we make a step further and show that the dynamically different k -theories can produce kinematically equivalent cosmological models.

The construction of cosmological models with tracker-like, or the attractor behaviour ⁸, where the k -essence either mimics the equation of state of the matter-radiation component, or drives towards acceleration, is relied heavily on the existence of k -essence solutions which, re-written in terms of energy density and pressure, represent, hydrodynamically, fluids with a constant baryotropic index (BI). These, in turn, give rise to a power-law behaviour of the scale factor when the underlying geometry is that of a spatially flat isotropic universe.

In the k -essence models studied earlier ^{1,2,3,4} one would usually consider solutions where, during the k -field driven expansion with the constant BI, two things happen: i) The scalar field evolves linearly with time and ii) The k -field potential is an inverse square in terms of the k -field. The property ii) follows directly from i). The assumption i), triggered probably by the simplicity of finding solutions in the case of linear k -field, although permits to consider different k -theories, is too restrictive with the form of the k -potential and the evolution of the field itself.

The main purpose of this Letter is to show that in the case of the k -essence, one may find solutions with arbitrary potentials and non-linear scalar fields, but still have a constant BI. For the solutions we find, one can have a fixed evolution of the geometry, yet incredibly rich repertoire of scalar field behaviour and its k -potential. This kind of a degeneracy is quite problematic for the model building. Nevertheless, there seems to be a way to reduce this degeneracy. For solutions with constant BI we find that the linear k -field model is isomorphic to a divergent sound speed model. In fact, the former presents a superior type of degeneracy as compared to the solutions obtained from the non-linear k -field model, therefore fully justifying our quest for a different type of k -field solutions. The isomorphism between the linear k -field model and the divergent sound speed model looks especially interesting in the light of the results of the recent publication ¹¹ where the behaviour of the solutions near divergent sound speed was thoroughly investigated.

We start with a general Lagrangian

$$\mathcal{L} = -V(\phi) F(x), \quad x = g^{\mu\nu} \phi_\mu \phi_\nu, \quad (1)$$

where ϕ is the scalar field and $\phi_\mu = \partial\phi/\partial x^\mu$, and do not impose any conditions neither on V , nor F at this stage. One may easily figure out the energy-momentum

tensor for (1):

$$T_{\mu\nu} = V(\phi) [2F_x \phi_\mu \phi_\nu - g_{\mu\nu} F], \quad F_x = \frac{dF}{dx}. \quad (2)$$

Identifying (2) with the energy-momentum tensor of a perfect fluid we have

$$\rho_\phi = V(\phi) [F - 2xF_x], \quad p_\phi = \mathcal{L} = -V(\phi)F. \quad (3)$$

As usual in this setting we assume a spatially flat homogeneous and isotropic space-time with line element

$$ds^2 = -dt^2 + a^2(t) [dx^2 + dy^2 + dz^2], \quad (4)$$

where $a(t)$ is the scale factor and the expansion rate is defined as $H = \dot{a}/a$. The Einstein field equations then reduce to

$$3H^2 = \rho_\phi, \quad -2\dot{H} = \rho_\phi + p_\phi, \quad (5)$$

and the conservation equation reads

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0. \quad (6)$$

The field equation for the ϕ field may be either obtained by substituting expressions (3) into the conservation equation (6), or varying directly the Lagrangian (1). Doing so, we get:

$$[F_x + 2xF_{xx}]\ddot{\phi} + 3HF_x\dot{\phi} + \frac{V'}{2V}[F - 2xF_x] = 0, \quad (7)$$

where $V' = dV/V\phi$. On the other hand, assuming a “formal” equation of state of the form $p_\phi = (\gamma - 1)\rho_\phi$ for the k-essence and using Eqs. (3), (5) we obtain the BI γ

$$\gamma = -\frac{2\dot{H}}{3H^2} = -\frac{2xF_x}{F - 2xF_x}. \quad (8)$$

We now assume that the BI is a constant. This kinematically leads to a power-law scale factor $a = a_0 t^{2/3\gamma}$.

The first question we ask is, how stable are the solutions with the constant BI $\gamma = \gamma_0$?. To answer this question, we allow γ to vary with time. Differentiating the equation of state and using the conservation equation we find

$$\dot{\gamma} = 3H\gamma(\gamma - 1) + \frac{\dot{p}_\phi}{\rho_\phi}, \quad (9)$$

which together with (3) and (8) lead to

$$\dot{\gamma} + \left[3H\gamma + \frac{\dot{V}}{V} + \frac{\dot{F}}{F} \right] (1 - \gamma) = 0. \quad (10)$$

We further check as to whether $\gamma = \gamma_0$ are solutions to this equation at all. Obviously, there are two different ways for this to happen: either $\gamma_0 = 1$, or generically, the following stationary condition holds

$$\frac{\dot{V}}{V} + \frac{\dot{F}}{F} = -3H\gamma_0, \quad (11)$$

When the stationarity condition (11) holds, the potential V and the function F are related by:

$$VF = \frac{4(1-\gamma_0)a_0^{3\gamma_0}}{3\gamma_0^2 a^{3\gamma_0}}. \quad (12)$$

Here, we have integrated (11) and inserted the solution into the Einstein equation (5) to fix the integration constant. For a positive potential V , the constrain (12) gives rise to two different theories depending on whether $\gamma_0 < 1$ or $\gamma_0 > 1$. In the case $\gamma_0 < 1$ we take the function F to be positive, whereas in the case $\gamma_0 > 1$ we take it negative. We denote these as F^+ and F^- respectively.

We now assume that the stationarity condition (11) holds. So, the equation (10) reads

$$\dot{\gamma} + 3H(\gamma - \gamma_0)(1 - \gamma) = 0. \quad (13)$$

Integrating, we find:

$$\gamma = \frac{\gamma_0 + c a^{-3(1-\gamma_0)}}{1 + c a^{-3(1-\gamma_0)}}. \quad (14)$$

Here c is an integration constant. For the expanding universe and $\gamma_0 < 1$ we see that the solutions of (13) have the asymptotic limit γ_0 . Therefore, the solutions with constant BI γ_0 are attractors in the case $\gamma_0 < 1$. This attractor behaviour holds even for superaccelerated universes⁹ with $\gamma_0 < 0$.

The limit $\gamma_0 \rightarrow 1$, should be considered apart, and the solution of the equation (13) is

$$\gamma = 1 - \frac{1}{c + \ln a^3}, \quad (15)$$

where c is an integration constant. Hence, for an expanding universe the solution with $\gamma_0 = 1$ is stable as well. The $\gamma_0 = 1$ solutions separate stable from unstable regions in the phase space (for a positive expansion rate) as can be easily seen from the equation (13), and since $\gamma_0 = 1$ corresponds to dust, we conclude that the dust-like solutions define the border line between stable and unstable behavior. It is probably worthwhile to mention that the above stability analysis is simple and direct as compared to the study performed directly in the field variables using the solution $\phi \propto t$ as an input.

As from now we stick to the solutions with the constant BI γ . It follows then that the Einstein and the field equations (5), (7) have *two* different classes of solutions:

- 1) The solutions with constant $x = x_0 = -\dot{\phi}^2$.

In this case for $\gamma = \text{const} \neq 0$ we have $a = a_0 t^{2/3\gamma}$, the first term of the l.h.s. of the eq. (7) vanishes, and the consistent solution of eqs. (5), (7) becomes $\phi = \pm\sqrt{-x_0} t$ and

$$V = -\frac{4x_0}{3\gamma^2[F - 2xF_x]} \frac{1}{\phi^2}, \quad (16)$$

with an arbitrary F evaluated at $x = x_0$. We will not discuss these solutions further, since these were thoroughly investigated and exploited in model building in ^{1,7}. The particular case with $x_0 = 0$ ($\phi = \phi_0$) must be solved apart and gives a de Sitter solution $a = a_0 e^{\sqrt{VF/3}t}$ for arbitrary F evaluated at $x = 0$ and constant potential V .

2) Solutions with $x \neq \text{const}$.

In this case the conservation equation (6) can be readily integrated to find the first integral of the field equation (7)

$$VF_\gamma = \frac{\rho_0}{a^{3\gamma}}. \quad (17)$$

Comparing this expression with the constrain equation (5) we are lead to the relation (12) between the potential V and the function F . Hence, the integration constants a_0 and ρ_0 are left fixed to $\rho_0 = 4(1 - \gamma)a_0^{3\gamma}/3\gamma^2$.

We now look at (8) as a differential equation for $F(x)$. Its immediate general solution is

$$F_\gamma(x) = c(-x)^{\frac{\gamma}{2(\gamma-1)}}. \quad (18)$$

Without any loss of generality, one may fix the integration constant $c = \pm 1$. The two corresponding families of solutions are then F_γ^+ and F_γ^- respectively. Inserting the last equation into (3) we get two possibilities

$$\rho_\phi^+ = \frac{VF_\gamma^+}{1-\gamma}, \quad p_\phi^+ = -VF_\gamma^+, \quad \gamma < 1, \quad (19)$$

and

$$\rho_\phi^- = \frac{VF_\gamma^-}{1-\gamma}, \quad p_\phi^- = -VF_\gamma^-, \quad \gamma > 1, \quad (20)$$

where we have assumed that both the k -potential and the energy density are positive definite.

Inserting (18) into (12) one gets a relation of the form $t^2V \propto \dot{\phi}^{\gamma/1-\gamma}$. Finally, the general relations connecting the field ϕ and the potential V follows:

$$t^{\frac{2-\gamma}{\gamma}} = \frac{2-\gamma}{\gamma} \left[\pm \frac{3\gamma^2}{4(1-\gamma)} \right] \int V^{\frac{\gamma-1}{\gamma}} d\phi, \quad \gamma \neq 2, \quad (21)$$

$$\ln t = \sqrt{3} \int \sqrt{V} d\phi, \quad \gamma = 2. \quad (22)$$

The (+) branch in equation (21) corresponds to $\gamma < 1$ while the (-) branch to $\gamma > 1$. For linear ϕ the integral (21) is not defined and this situation corresponds to the first class (i) of the solutions. The relations (21) and (22) should be read as follows: given $V(\phi)$, one may integrate and obtain $t = t(\phi)$, invert and find $\phi = \phi(t)$. Then $F(x)$ is still given by (18). Note, that for a fixed γ (fixed power of the scale factor) one has different potentials and different field evolutions, and consequently different k -theory. It looks as the k -essence theories have a considerable amount of freedom in choosing the theory, the potential and the scalar field behaviour, all describing the same kinematics of the universe. This sounds somewhat “fantastic” for these are not just simple field redefinitions, and all the theories with the different ϕ and V are dynamically different.

We now show how the power law solutions with the linear scalar field and the inverse square potential are related to the family of solutions with the divergent velocity of sound. We do so by constructing a one-to-one mapping between these solutions. We suggest that this might be the reason as to why the solutions with the linear scalar field run into trouble as discussed in a recent paper by Malquarty et al ¹¹.

To do so we introduce what we call a “divergent” k - essence Lagrangian with the kinetic energy proportional to the velocity. For such a theory one may take the function F^∞ as

$$F^\infty = c + b\sqrt{-x}. \quad (23)$$

It is easy to see that the above function leads immediately to a divergent sound velocity $C_s^2 = -F_x^\infty / (F_x^\infty + 2xF_{xx}^\infty)$ and to an inverse square potential by using the k -field equation (7). This does not constitute a major problem in itself, for one could have just avoided using this sort of a model. It follows, however, that the solutions of the models with linear scalar field and the inverse square potential discussed in ^{1,7} are isomorphic to those obtained in the divergent models.

To see the relation between the models we work with the power law solutions. Consider a typical model cosmology given by:

$$a = t^n, \quad V = \frac{\beta}{\phi^2}, \quad \phi = \phi_0 t, \quad (24)$$

obtained by evaluating F and F_x at $x = x_0 = -\phi_0^2$. We further use $f = F(-\phi_0^2)$ and $f' = F_x(-\phi_0^2)$. Substituting these constants into the Friedmann and k -field equations (5), (7) we find that the index n and the slope of the potential β are given by

$$n = \frac{1}{3} \frac{f + 2\phi_0^2 f'}{\phi_0^2 f'} = \frac{2}{3\gamma}, \quad \beta = \frac{n}{f'}, \quad (25)$$

On the other hand, if in the divergent model we choose the constants $c = 3n^2\phi_0^2/\beta$ and $b = -2n\phi_0/\beta$ we obtain the same solution. Therefore all the power-law solutions obtained from the model with the linear scalar field and the inverse square potential map into the solutions of the divergent model with the same potential.

Moreover, the following reasoning underlines the highly degenerate character of the linear k -field solutions. Consider the series expansion of the function $F(x)$ around $x = x_0$. The background cosmology is completely determined by just the first two co-efficients in the expansion (f, f') and the value of ϕ_0 as seen from (25). Put in different words, the model is insensitive to keeping the first two co-efficients in the expansion of the function F and the same value of ϕ_0 , but varying the rest of the higher order terms. Since, given the same value of ϕ_0 , the first two terms in the F expansion for the linear and divergent models co-incide, they should be thought of as equivalent. Therefore, the models with the linear k -field and the inverse square potential possess a symmetry, or rather a degeneracy in the sense that all the solutions for which the first two terms of the expansion of the function F around $x = x_0$ co-incide are equivalent among themselves and also equivalent to the divergent model. This does not happen with the solutions with the non-linearly behaved scalar field, and suggests that physically the later are more acceptable, thus partially removing the degeneracy of the solutions.

A more subtle distinction between the models, to completely remove the degeneracy, would be probably seen by perturbing these solutions. This, however, is beyond the scope of the present Letter.

In this Letter we have studied particular solutions to the Einstein Equations coupled to k -essence. Imposing spatially flat isotropic geometry we have shown that different k -theory Lagrangians may lead to the same kinematical evolution of the universe.

We have seen, however, that the linear k -field power-law solutions possess an odd property of being isomorphic to a family of solutions with a divergent speed of sound generated by the function F^∞ . This relation of isomorphism induces, in fact, problems with the power-law solutions regardless of the model (function F) as long as the potential is inverse square and the field is linear, leading to consider different solutions. It has been recently argued¹¹ that the cosmological models based on linear k field lead to serious problems. These problems are associated with the behaviour of the models in the divergent sound speed region. We believe that our findings relating the linear k -field models with the divergent models sheds new light on the reasons of the peculiar behavior of those models in the region of the divergent speed of sound.

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