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Vortex interaction, chaos and quantum probabilities

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Abstract. – The motion of a single vortex is able to originate chaos in the quantum trajectories defined in Bohm’s interpretation of quantum mechanics. In this letter, we show that this is also the case in the general situation, in which many interacting vortices exist. This result gives support to those studies in which the Born’s probability rule is considered from the point of view of a thermal analogy.

Despite the impressive success of quantum mechanics along the past century, Born’s probability rule: $\rho = |\psi|^2$, one basic cornerstone in its standard formulation, still remains a postulate.

Due to its relevance, this fundamental issue has been revisited in the last years [1–3], in an effort to make probability an emergent phenomenon [4]. Notice, for example, that basic ingredients in the decoherence programme, such as reduced density matrix, are based on Born’s rule. In this respect, Zurek [1] introduced environment-assisted invariance (“envariance”), a causality-related symmetry of quantum entangled systems, to derive Born’s rule. Wallace and Deutsch [2] reported another approach based on classical decision theory in the context of Everett many-worlds interpretation. Another interesting point of view was started by a speculation contained in Bohm’s papers on the (also causal) de Broglie-Bohm (dBB) quantum formalism [5]. In it, it was suggested that probabilities have a dynamical origin, holding a status similar to that of thermal probabilities in ordinary statistical mechanics. Later, Valentini and Westman [3] numerically explored this possibility, showing that the standard distribution is obtained as the time evolution towards the equilibrium of initial non-equilibrium states, $\rho \neq |\psi|^2$, this taking place with a (exponential) decrease in the associated coarse-grained H -function. In this respect, another relevant issue is that of the typicality of $|\psi|^2$ (at equilibrium) which is overwhelmingly most probable; this point has been addressed by Dürr *et al.* [6].

Underlying Bohm, and Valentini and Westman arguments is the assumption that there is an effective chaotic dynamics in the dBB trajectories, something that should not be taken for granted. Unfortunately, most published results along this line are rather inconclusive. However, singularities in the wave function giving rise to vortices, have been proven very recently to play a prominent role in the problem. In ref. [7], the case of a single vortex

was considered, arriving at the conclusion that the motion of an isolated vortex is enough to originate chaos in Bohmian trajectories [8]. But nothing is known about the picture emerging in the general situation, in which many vortices exist. For this case, in addition to the vortex dynamics, a great deal of interaction should be expected, for example through a mechanism of creation/annihilation of pairs with opposite circulations. In this respect, Frisk [9] conjectured the importance of the number of nodes inducing mixing behavior in dBB trajectories. In view of the paramount importance of the problem under discussion, a deep understanding of this issue, similar to that in statistical mechanics, is highly desirable.

In this letter, we address the question of the complexity of Bohmian trajectories by presenting a systematic numerical study in which we show that the chaotic regime, due to the dynamics and interactions of the existing vortices, is the general rule in dBB trajectories even in the absence of nonlinear terms in the physical potential. This gives rise to a noticeable complexity that can be quantified in terms of common indicators, such as Lyapunov exponents. As an added bonus, the use of such indicator allows to explore some differences existing with the classical case.

In the dBB theory [5] the state of the system is described by a wave function, customarily expressed in polar form, $\psi(\mathbf{r}, t) = R(\mathbf{r}, t) e^{iS(\mathbf{r}, t)}$ ($\hbar = 1$ throughout the paper), and the position of the particles, \mathbf{r} [6]. The dynamical evolution of this two quantities (for spinless particles) is given by the time-dependent Schrödinger equation and the guidance equation, respectively:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\nabla S}{m} = \frac{i}{2m} \frac{\psi \nabla \psi^* - \psi^* \nabla \psi}{|\psi|^2}, \quad (1)$$

where m is the mass of the particle. Quantum trajectories, which make of dBB a true theory of quantum motion [10], can be obtained by numerical integration of this equation. The velocity field (1), guiding these trajectories, presents singularities giving rise to vortices in the associated probability fluid. This occurs only at points where the wave functions vanishes (isolated points in 2-dof systems, lines in 3-dof systems, etc.) and the phase S is singular. Moreover, and due to the single-valuedness of the wave function, the circulation, Γ , around a circuit, \mathcal{C} , encircling a vortex must be quantized [11, 12] according to

$$\Gamma = m \oint_{\mathcal{C}} \dot{\mathbf{r}} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \nabla S \cdot d\mathbf{r} = 2\pi n, \quad (2)$$

where n is a integer. This implies that the velocity must diverge at the vortex [13, 14].

The aim of our work is to study the behavior of the quantum trajectories of a system in the general situation in which a large number of interacting vortices exists. One of the simplest systems where this problem can be set up is the 2-dof rectangular billiard whose classical dynamics are integrable. In this way, any observed complexity is only due to quantum effects, without any contribution of chaos coming from forces derived from a physical potential. The dimensions of the rectangle are taken so that the smallest side is equal to 1, and the total area amounts to $A = 1 + \pi/2$, so that the corresponding eigenfunctions are $\phi_{n_x, n_y}(x, y) = (2/A^{1/2}) \sin(n_x \pi x) \sin(n_y \pi y/A)$, with wave numbers $k = (2mE)^{1/2} = \pi(n_x^2 + n_y^2/A^2)^{1/2}$. The initial pilot wave is constructed as a linear combination, $\psi(x, y) = \sum_{n_x, n_y}^N c_{n_x, n_y} \phi_{n_x, n_y}(x, y)$, of N such eigenfunctions with random coefficients. We will systematically vary this function by considering increasing values of the wave number mean value, $\langle k \rangle$, and N . For this purpose, the eigenstates entering into the linear combination, giving initially the pilot wave function, are selected in the following way. At any given value of the energy, a “central” state is determined by choosing the nearest integers, $n_x = n_y$, fulfilling the energy condition; the rest of intervening states are then chosen as the $N - 1$ states which are closest in energy to the central one.

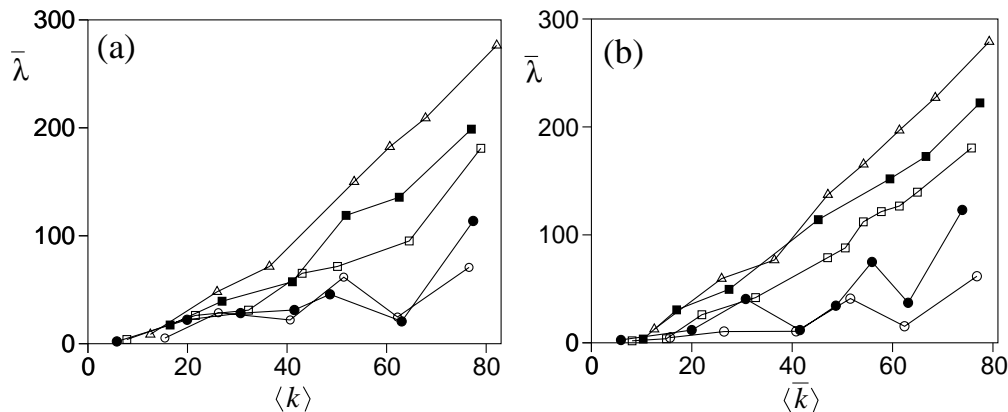


Fig. 1 – Averaged Lyapunov exponent as a function of the mean wave number for quantum trajectories on a rectangular billiard of smallest side length equal to 1 and area $1 + \pi/2$. The initial pilot wave function is taken as a linear combination of N rectangle eigenfunctions, being $N = 3$ (open circles), 5 (full circles), 11 (open squares), 20 (full squares), 30 (open triangles). In (a) the average is carried out over 50 randomly selected initial conditions, while in (b) the coefficients of the initial pilot wave function were also varied.

To gauge the complexity of our system we take, similarly to what it is well established in the usual chaos theory, the Lyapunov exponent of the quantum trajectories, $\bar{\lambda}$, statistically averaged over many initial random conditions. The corresponding results for $\bar{\lambda}$, as a function of $\langle k \rangle$ for different values of N , are shown in fig. 1. As can be seen, $\bar{\lambda}$ grows systematically both with $\langle k \rangle$ and N , showing a mean tendency which is approximately linear after a threshold at $\langle k \rangle_{\text{th}} \sim 30$. To check that this conclusion is not an artifact of the way in which the initial wave function (*i.e.* the coefficients entering in the linear combination) has been chosen, we have repeated the same calculation using a different averaging procedure for the Lyapunov function. In this second calculation we change not only the initial position of each quantum trajectory in the averaging ensemble, but we also change randomly the coefficients in the corresponding initial pilot wave. This double average defines a new mean wave number that will be denoted by $\langle \bar{k} \rangle$. The results are shown in fig. 1(b), where it is seen that they follow a behavior totally equivalent to that obtained in the previous case (curves in part (a) of the figure). This fact indicates that our conclusion is robust. Making it quantitative, the final increasing linear tendency of the mean Lyapunov exponent is given by the expression $\bar{\lambda} = 0.15N \langle k \rangle$.

Following Frisk [9], let us try now to explain these results in terms of the number of vortices existing in the system. This is a sensible assumption, since it is at these points where the complexity in the quantum trajectories is originated [7]. This is, however, not a straightforward task, since the number of vortices associated to each pilot wave function varies with time, significantly fluctuating around the mean value, by creation and annihilation of pairs of vortices with opposite circulations. Taking this into account, we compute the time average of the number of vortices (with circulation in a given sense), \bar{N}_v , as a numerical indicator characterizing the collective effect of the vortices.

The corresponding results, obtained for the same conditions of fig. 1(a), are shown in fig. 2. As can be seen, the mean number of vortices grows quadratically with $\langle k \rangle$, but it is completely independent of N , being the same at a given energy, regardless of the number of eigenfunctions contributing to the pilot wave function used in the computation of the quantum trajectories. This numerical calculation clearly indicates, contrary to Frisk expectations, that

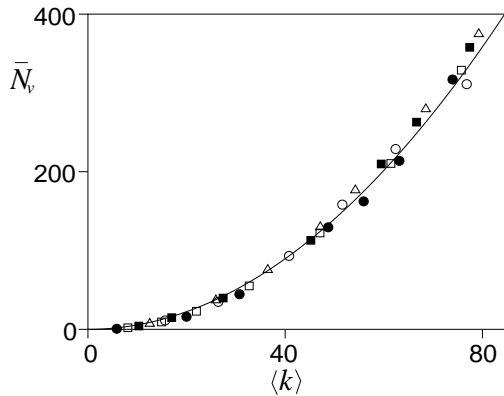


Fig. 2

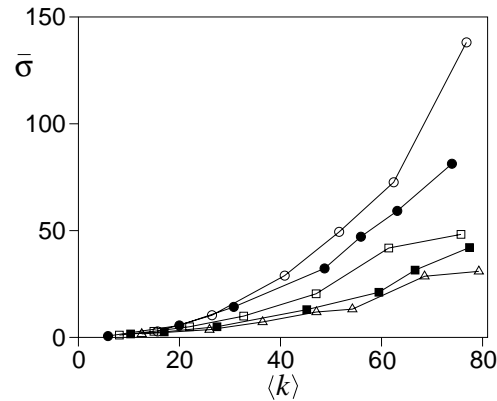


Fig. 3

Fig. 2 – Time average of the number of vortices with a given sense of the circulation for the different cases shown in fig. 1(a). The full line corresponds to the analytical expression $N_v^{\max} = (A/4\pi^2)k^2 - [(1+A)/2\pi]k$, giving a rough estimate of the maximum number of vortices fitting in our billiard (see text for details).

Fig. 3 – Time average of the mean root deviation on the number of vortices with a given sense of the circulation for the different cases shown in fig. 1(a).

the number of vortices alone is not enough to explain complexity found generically in Bohmian trajectories. Furthermore, the result in fig. 2 can be understood by considering the following rough estimate of the maximum possible number of vortices that can fill our billiard. Assuming that the minimum area in configuration space required for the existence of a vortex is given by the magnitude of the squared de Broglie wavelength, $\lambda_{\text{DB}}^2 = 4\pi^2/\langle k \rangle^2$, and taking into account the boundary effects at the walls, the maximum number of vortices in the billiard should be given by $N_v^{\max} = (A/4\pi^2)\langle k \rangle^2 - [(1+A)/2\pi]\langle k \rangle$, an expression that agrees perfectly well with the computed data, as shown by the full line in fig. 2.

Since \bar{N}_v is not enough to explain the behavior of $\bar{\lambda}$ in fig. 1(a), let us try now the second momentum of the corresponding temporal distribution. Notice that this is equivalent to assuming that the origin of the complexity of quantum trajectories is, in the general case, the interaction responsible for the vortex creation/annihilation mechanism. The results are plotted in fig. 3, where it can be seen that the time average of the mean root deviation on the number of vortices, $\bar{\sigma}$, grows monotonically with \bar{k} and decreases with N , *i.e.* the more complicated the function is the smallest dispersion is found. Moreover, this scaling dependence can be formulated in quantitative terms as $\bar{\sigma} = 0.035\langle k \rangle^2 N^{-1/2}$. The quadratical dependence on k is obvious, since it is a consequence of the functional form exhibited by \bar{N}_v , and accordingly the mean root deviation can be also expressed as $\bar{\sigma} = 0.54\bar{N}_v N^{-1/2}$. What it is interesting is that $\bar{\sigma}$ contains an additional inverse dependence on N . This result, if examined carefully (as will be discussed below), is in agreement with the systematic behavior previously found for the quantum trajectories Lyapunov exponent in fig. 1(a). Finally, by putting these two components together, the complexity of quantum trajectories as quantified by the averaged Lyapunov parameter can be solely expressed in terms of vortex properties, in the following way: $\bar{\lambda} = 0.17\bar{N}_v^{5/2}\bar{\sigma}^{-2}$.

Now, let us rationalize the dependence on N found in $\bar{\sigma}$. Fluctuations in the number of vortices, as a result of pair annihilations, leave areas of the billiard depopulated from

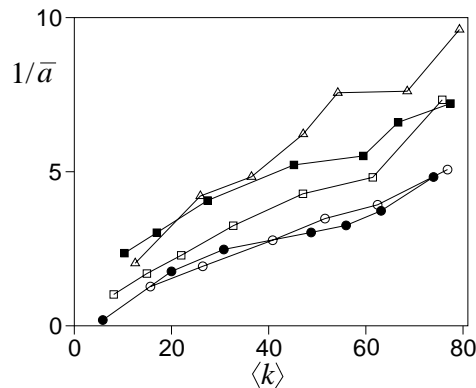


Fig. 4 – Inverse of the area of the largest compact vortex-free region in the rectangular billiard corresponding to the different cases shown in fig. 1(a).

them. It is expected that the increment of these areas decreases the value of the mean Lyapunov exponent, $\bar{\lambda}$, since, according to the results in ref. [7], the complexity in the Bohmian dynamics in these areas is null. That is, the corresponding flux of the quantum fluid is laminar in those regions, only getting turbulent close to the remaining vortices. The argument can be made quantitative by performing the following numerical calculation. At the same times used to compute the averages presented in fig. 1(a), and also using the same wave functions, the position of the corresponding vortices is calculated. Using a fine grid in the billiard configuration space, we mark differently those squares without any vortex inside and compute, at each value of $\langle k \rangle$, the size, a , of the largest compact region without any vortex. Then, the value of this largest vortex-free area is averaged on time as its value changes by subsequent creation-annihilations of vortices. Finally, we take the inverse of this quantity to get a magnitude comparable (from the point of view of the complexity) to $\bar{\lambda}$. The results are shown in fig. 4. As can be seen, $1/\bar{a}$ increases linearly with $\langle k \rangle$ for each value of N , and also it keeps the same dependence on N as found in fig. 1(a) for $\bar{\lambda}$. This is a numerical confirmation that chaos is controlled in this problem by two counteracting mechanisms. The first one has a local character, and corresponds to the randomization effect on the quantum trajectories induced by moving vortices [7]. The second one is global, and it is related to the appearance of vortex-free areas, reducing the complexity, due to annihilation of pairs with opposite circulations.

To close this discussion, it should be indicated that other calculations have been performed in order to gauge the influence of others effects in the complexity of quantum trajectories. Some obvious ones are the kinematical magnitudes. For this purpose, we have studied the velocity and acceleration distributions of the vortices that can be calculated from the expression

$$\mathbf{v} = \frac{1}{2mi} \frac{\mathbf{r} \times (\mathbf{w} \times \mathbf{w}^*)}{|\mathbf{w} \cdot \mathbf{r}|^2}, \quad (3)$$

where $\mathbf{w} \equiv \nabla\psi[\mathbf{r}(t)]$ [13]. Our results indicate that neither of them shows an obvious systematic, as is the case with the correlation between the Lyapunov exponent and the number of vortices.

Finally, another interesting point to be discussed is the classical limit of our results. The results in fig. 1 indicate that the averaged Lyapunov exponent do not seem to vanish as $\langle k \rangle \rightarrow \infty$. This would imply that in this semiclassical limit, the effect of vortices inducing

chaos and complexity in the quantum trajectories did not disappear. This behavior is not surprising, since it has also been found in similar problems. For example, in ref. [15] it was shown that the non-local character of the Bohmian trajectories (avoiding crossing, for example) survives when the quantum effects, represented by the quantum potential Q , were made to disappear by increasing the mass of the incident particle in a realistic model for rainbow diffraction in atom-surface scattering. In this case, the quantum trajectories simply mimic the classical distributions, without ever reaching strictly the classical limit. In this respect, Bowman [16] pointed out that the Bohmian classical limit can only be achieved by combination of narrow packets, mixing states, and environment decoherence. Certainly, further calculations, considering much higher values of $\langle k \rangle$, are needed in order to clarify this issue.

Summarizing, in this letter we have numerically shown that chaos is the generic scenario for quantum trajectories, in the situation in which many interacting vortices exist. In this way the picture started with ref. [7], in which the effect of only one vortex was studied, is completed. Moreover, we have quantified this assertion with the aid of the Lyapunov exponent as a numerical indicator of complexity. Our results show that the behavior of this quantity depends on the number of vortices of the pilot wave function, and only the first two momenta of the corresponding distribution are required to explain it satisfactorily. This result is interesting, since it shows the interplay between the quantum phase, which appears in the guiding equation (1), and quantum probabilities, these being the two components in which the wave function is separated in the Bohmian formulation of quantum mechanics.

* * *

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