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Projective resolutions of associative algebras and ambiguities ☆

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The aim of this article is to give a method to construct bimodule resolutions of associative algebras, generalizing Bardzell's well-known resolution of monomial algebras. We stress that this method leads to concrete computations, providing thus a useful tool for computing invariants associated to the considered algebras. We illustrate how to use it by giving several examples in the last section of the article. In particular we give necessary and sufficient conditions for noetherian down–up algebras to be 3-Calabi–Yau.

1. Introduction

The invariants attached to associative algebras and in particular to finite dimensional algebras have been widely studied during the last decades. Among others, Hochschild homology and cohomology of diverse families of algebras have been computed.

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The first problem one faces when computing Hochschild (co)homology is to find a convenient projective resolution of the algebra as a bimodule over itself. Of course, the bar resolution is always available but it is almost impossible to perform computations using it.

M. Bardzell provided in [4] a bimodule resolution for monomial algebras, that is, algebras $A = kQ/I$ with k a field, Q a finite quiver and I a two-sided ideal which can be generated by monomial relations; in this situation, the set of classes in A of paths in Q which are not zero is a basis of A. Moreover, this resolution is minimal. A simple proof of the exactness of Bardzell's complex has been given by E. Sköldberg in [29], where he provided a contracting homotopy. Of course, having such a resolution does not solve the whole problem, it is just a starting point.

The non-monomial case is more difficult, since it involves rewriting the paths in terms of a basis of A. Different kinds of resolutions for diverse families of algebras have been provided in the literature. For augmented k-algebras, Anick constructed in [1] a projective resolution of the ground field k. The projective modules in this resolution are constructed in terms of ambiguities (or n-chains), and the differentials are not given explicitly. In practice, it is hard to make this construction explicit enough in order to compute cohomology. For quotients of path algebras over a quiver Q with a finite number of vertices, Anick and Green exhibited in [2] a resolution for the simple module associated to each vertex, generalizing the result of [1], which deals with the case where the quiver Q has only one vertex. Also, Y. Kobayashi in [23] proposes a method to construct a resolution which is difficult to use in concrete examples.

One may think that the case of binomial algebras is easier than others, but in fact it is not quite true since it is necessary to keep track of all reductions performed when writing an element in terms of a chosen basis of the algebra as a vector space.

In this article we construct in an inductive way, given an algebra A, a projective bimodule resolution of A, which is a kind of deformation of Bardzell's resolution of a monomial algebra associated to A. For this, we use ideas coming from Bergman's Diamond Lemma and from the theory of Gröbner bases. The resolution we give is not always minimal, but we prove minimality for various families of algebras.

In the context of quotients of path algebras corresponding to a quiver with a finite number of vertices, our method consists in constructing a resolution whose projective bimodules come from ambiguities present in the rewriting system. Of course there are many different ways of choosing a basis, so we must state conditions that assure that the rewriting process ends and that it is efficient.

One of the advantages of doing this is that, once a bimodule resolution is obtained, it is easy to construct starting from it a resolution of any module on one side and, in particular, to recover those constructed in [1] and [2] for the case of the simple modules associated to the vertices of the quiver.

To deal with the problem of effective computation of these resolutions, Theorem 4.1 below gives sufficient conditions for a complex defined over these projective bimodules to be exact. We will be, in consequence, able to prove that some complexes are resolutions without following the procedure prescribed in the proof of the existence theorem.

Briefly, we do the following: given an algebra $A = kQ/I$ we compute a bimodule resolution of A from a reduction system $\mathcal{R}$ for I which satisfies a condition we denote $(\lozenge)$. We prove that such a reduction system always exists, but we also show in an example that it may not be the most convenient one. In particular the resolution obtained may not be minimal.

Applying our method we recover a well-known resolution of quantum complete intersections, see for example [7] and [9]. We also construct a short resolution for down–up algebras which allows us to prove that a noetherian down–up algebra $A(\alpha, \beta, \gamma)$ is 3-Calabi–Yau if and only if $\beta = -1$.

The contents of the article are as follows. In Section 2 we fix notations and prove some preliminary results. In Section 3 we deal with ambiguities. In Section 4 we state the main theorems of this article, namely Theorem 4.1 and Theorem 4.2, after proving some results on orders and differentials. Section 5 is devoted to the proofs of these theorems; it contains several technical lemmas. In Section 6 we construct explicitly the differentials in low degrees and in Section 7 we give several applications of our results.

Finally, in Section 8 we give sufficient conditions on the reduction system for minimality of any resolution obtained from it. We also prove that in case A is graded by the length of paths, and it has a reduction system satisfying the conditions required for minimality of the resolution, then A is N-Koszul if and only if the associated monomial algebra A_S is N-Koszul.

We have just seen a recent preprint by Guiraud, Hoffbeck and Malbos [20] where they construct a resolution that may be related to ours.

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2. Preliminaries

In this section we give some definitions, present some basic constructions and we also prove results that are necessary in the sequel.

Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of nonnegative integers.

Let k be a field and Q a quiver with a finite set of vertices. Given $n \in \mathbb{N}$, Q_n denotes the set of paths of length n in Q and $Q_{\geq n}$ the set of paths of length at least n, that is, $Q_{\geq n} = \bigcup_{i \geq n} Q_i$. Whenever $c \in Q_n$, we will write $|c| = n$. If $a, b, p, q \in Q_{\geq 0}$ are such that $q = apb$, we say that p is a *divisor* of q; if, moreover, $a \in Q_0$, we say that p is a *left divisor* of q and analogously for $b \in Q_0$ a *right divisor*. We denote $\mathsf{t}, \mathsf{s} : Q_1 \to Q_0$ the usual source and target functions. Given $s \in Q_{\geq 0}$ and a finite sum $f = \sum_i \lambda_i c_i \in kQ$ such that $c_i \in Q_{\geq 0}$ and $\mathsf{t}(s) = \mathsf{t}(c_i)$, $\mathsf{s}(s) = \mathsf{s}(c_i)$ for all i, we say that f is *parallel* to s. Let $E := kQ_0$ be the subalgebra of the path algebra generated by the vertices of Q.

Given a set X and a ring R, we denote $\langle X\rangle_R$ the left R-module freely spanned by X.

Let I be a two sided ideal of kQ, $A = kQ/I$ and $\pi : kQ \to A$ the canonical projection. We assume that $\pi(Q_0 \cup Q_1)$ is linearly independent.

We recall some terminology from [3] that we will use. A set of pairs $\mathcal{R} = \{(s_i, f_i)\}_{i\in\Gamma}$ where $s_i \in Q_{\geq 0}$, $f_i \in kQ$ is called a *reduction system*. We will always assume that a reduction system $\mathcal{R} = \{(s_i, f_i)\}_{i\in\Gamma}$ satisfies the following conditions:

- for all i, f_i is parallel to s_i and $f_i \neq s_i$,
- s_i does not divide s_j for $i \neq j$.

Given $(s, f) \in \mathcal{R}$ and $a, c \in Q_{\geq 0}$ such that $asc \neq 0$ in kQ, we will call the triple (a, s, c) a *basic reduction* and write it $r_{a,s,c}$. Note that $r_{a,s,c}$ determines an E-bimodule endomorphism $r_{a,s,c} : kQ \to kQ$ such that $r_{a,s,c}(asc) = afc$ and $r_{a,s,c}(q) = q$ for all $q \neq asc$.

A *reduction* is an n-tuple $(r_n, \dots, r_1)$ where $n \in \mathbb{N}$ and r_i is a basic reduction for $1 \leq i \leq n$. As before, a reduction $r = (r_n, \dots, r_1)$ determines an E-bimodule endomorphism of kQ, the composition of the endomorphisms corresponding to the basic reductions $r_n, \dots, r_1$.

An element $x \in kQ$ is said to be *irreducible* for $\mathcal{R}$ if $r(x) = x$ for all basic reductions r. We will omit mentioning the reduction system whenever it is clear from the context. A path $p \in Q_{\geq 0}$ will be called *reduction-finite* if for any infinite sequence of basic reductions $(r_i)_{i\in\mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $r_n \circ \cdots \circ r_1(p) = r_{n_0} \circ \cdots \circ r_1(p)$. Moreover, the path p will be called *reduction-unique* if it is reduction-finite and for any two reductions r and r' such that $r(p)$ and $r'(p)$ are both irreducible, the equality $r(p) = r'(p)$ holds.

Definition 2.1. We say that a reduction system $\mathcal{R}$ satisfies condition $(\Diamond)$ for I if

- the ideal I is equal to the two sided ideal generated by the set $\{s - f\}_{(s,f)\in\mathcal{R}}$,
- every path is reduction-unique, and
- for each $(s, f) \in \mathcal{R}$, f is irreducible.

Definition 2.2. If $\mathcal{R}$ is a reduction system satisfying $(\Diamond)$ for I, we define $S := \{s \in Q_{\geq 0} : (s, f) \in \mathcal{R}$ for some $f \in kQ\}$.

Remark 2.2.1. Notice that:

(1) $S = \{p \in Q_{\geq 0} : p \notin \mathcal{B}$ and $p' \in \mathcal{B}$ for all proper divisors p' of $p\}$, where $\mathcal{B}$ is the set of irreducible paths.
(2) If s and s' are elements of S such that s divides s', then $s = s'$.
(3) Given $q \in Q_{\geq 0}$, q is irreducible if and only if there exists no $p \in S$ such that p divides q.

Definition 2.3. Given a path p and $q = \sum_{i=1}^{n} \lambda_i c_i \in kQ$ with $\lambda_1, \dots, \lambda_n \in k^\times$ and $c_1, \dots, c_n \in Q_{\geq 0}$, we write $p \in q$ if $p = c_i$ for some i, or, in other words, when p is in the support of q.

Given $p, q \in Q_{\geq 0}$ we write $q \rightsquigarrow p$ if there exist $n \in \mathbb{N}$, basic reductions $r_1, \dots, r_n$ and paths $p_1, \dots p_n$ such that $p_1 = q$, $p_n = p$, and for all $i = 1, \dots, n-1$, $p_{i+1} \in r_i(p_i)$.

Lemma 2.4. *Suppose that every path is reduction-finite with respect to $\mathcal{R}$.*

(1) If p is a path and r a basic reduction such that $p \in r(p)$, then $r(p) = p$.
(2) The binary relation $\rightsquigarrow$ is an order on the set $Q_{\geq 0}$ which is compatible with concatenation, that is, $\rightsquigarrow$ satisfies that $q \rightsquigarrow p$ implies $aqc \rightsquigarrow apc$ for all $a, c \in Q_{\geq 0}$ such that $apc \neq 0$ in kQ.
(3) The binary relation $\rightsquigarrow$ satisfies the descending chain condition.

Proof. *(1)* The hypothesis means that $r(p) = \lambda p + x$ with $\lambda \in k^\times$ and $p \notin x$. If $x \neq 0$ or $\lambda \neq 1$, then r acts nontrivially on p and so it acts trivially on x. Since the sequence of reductions $(r, r, \cdots)$ stabilizes when acting on p, there exists $k \in \mathbb{N}$ such that $\lambda^k p + (\lambda^{k-1} + \cdots + \lambda + 1)x = r^k(p) = r^{k+1}(p) = \lambda^{k+1} p + (\lambda^k + \cdots + \lambda + 1)x$. As a consequence, $\lambda = 1$ and $x = 0$.

(2) It is clear that $\rightsquigarrow$ is a transitive and reflexive relation and that it is compatible with concatenation. Let us suppose that it is not antisymmetric, so that there exist $n \in \mathbb{N}$, paths $p_1, \dots, p_{n+1}$ and basic reductions $r_1, \dots, r_n$ such that $p_{i+1} \in r_i(p_i)$ for $1 \leq i \leq n$ and $p_{n+1} = p_1$. Suppose that n is minimal. There exist $x_1, \dots, x_n \in kQ$ and $\lambda_1, \dots, \lambda_n \in k^\times$ such that $r_i(p_i) = \lambda_i p_{i+1} + x_i$ with $p_{i+1} \notin x_i$. Notice that since n is minimal, $r_i(p_i) \neq p_i$ and then r_i acts trivially on every path different from p_i, for all i.

Let us see that

$$p_i \notin x_j \text{ for all } i \neq j.$$

Since the sequence $p_1, \dots, p_{n+1} = p_1$ is cyclic, it is enough to prove that $p_1 \notin x_j$ for all j. Suppose that $p_1 \in x_j$ for some $j \in \{1, \dots, n\}$. Since $p_{i+1} \notin x_i$ for all i and $p_{n+1} = p_1$, it follows that $j \neq n$, and by part (1), $j \neq 1$. Let $u_k = p_k$ and $t_k = r_k$ for $1 \leq k \leq j$ and $u_{j+1} = p_1$. Notice that $u_{k+1} \in t_k(u_k)$ for $1 \leq k \leq j$ and $u_{j+1} = u_1$. Since $j < n$ this contradicts the choice of n. It follows that

$$p_i \notin x_j \text{ for all } i, j.$$

One can easily check that this implies $r_n \circ \cdots \circ r_1(p_1) = \lambda p_1 + x$ with $p_i \notin x$ for all i. Now, define inductively for $i > n$, $r_i := r_{i-n}$. The sequence $(r_i)_{i \in \mathbb{N}}$ acting on p_1 never stabilizes, which contradicts the reduction-finiteness of the reduction system $\mathcal{R}$.

(3) Suppose not, so that there is a sequence $(p_i)_{i\in\mathbb{N}}$ of paths and a sequence of basic reductions $(t_i)_{i\in\mathbb{N}}$ such that $p_{i+1} \in t_i(p_i)$. Since $\rightsquigarrow$ is an antisymmetric relation, $p_i \neq p_j$ if $i \neq j$.

Let $i_1 = 1$. Suppose that we have constructed $i_1, \dots, i_k$ such that $i_1 < \dots < i_k$, $p_{i_k} \in t_{i_{k-1}} \circ \dots \circ t_1(p_1)$ and $p_j \notin t_{i_{k-1}} \circ \dots \circ t_1(p_1)$ for all $j > i_k$. Set $X_k = \{i > i_k : p_i \in t_{i_k} \circ \dots \circ t_{i_1}(p_1)\}$. By the inductive hypothesis, there is $x \in kQ$ and $\lambda \in k^\times$ such that $t_{i_{k-1}} \circ \dots \circ t_{i_1}(p_1) = \lambda p_{i_k} + x$ with $p_{i_k} \notin x$. Since we also know that $p_{i_k+1} \in t_{i_k}(p_{i_k})$, and $p_{i_k+1} \notin t_{i_{k-1}} \circ \dots \circ t_{i_1}(p_1)$ it follows that $p_{i_k+1} \in t_{i_k}(p_{i_k}) + x$. Also, $t_{i_k} \circ \dots \circ t_{i_1}(p_1) = \lambda t_{i_k}(p_{i_k}) + t_{i_k}(x) = \lambda t_{i_k}(p_{i_k}) + x$, so $p_{i_k+1} \in t_{i_k} \circ \dots \circ t_{i_1}(p_1)$. Therefore X_k is not empty. We may define $i_{k+1} = \max X_k$, because X_k is a finite set.

This procedure constructs inductively a strictly increasing sequence of indices $(i_k)_{k\in\mathbb{N}}$ with $p_{i_k} \in \tilde{p}_{i_k} := t_{i_{k-1}} \circ \dots \circ t_{i_1}(p_1)$ for all $k \in \mathbb{N}$. The set $\{t_{i_{k-1}} \circ \dots \circ t_{i_1}(p_1) : k \in \mathbb{N}\}$ is therefore infinite. This contradicts the reduction-finiteness of $\mathcal{R}$. □

The converse to Lemma 2.4 also holds, that is, if $\mathcal{R}$ is a reduction system for which $\rightsquigarrow$ is a partial order satisfying the descending chain condition, then every path is reduction-finite. In other words, the order $\rightsquigarrow$ captures most of the properties we require $\mathcal{R}$ to verify, and it will be important in the next sections.

The reason why we are interested in these reduction systems is the following lemma. Its proof is just a consequence of Bergman's Diamond Lemma, and it is not given here.

Lemma 2.5. *If the reduction system $\mathcal{R}$ satisfies $(\Diamond)$ for I, then the set $\mathcal{B}$ of irreducible paths satisfies the following properties:*

(i) $\mathcal{B}$ is closed under divisors, that is, if $b \in \mathcal{B}$ and b' divides b, then b' belongs to $\mathcal{B}$.
(ii) $\pi(b) \neq \pi(b')$ for all $b, b' \in \mathcal{B}$ with $b \neq b'$.
(iii) $\{\pi(b) : b \in \mathcal{B}\}$ is a basis of $A = kQ/I$.

Remark 2.5.1. In view of Lemma 2.5, we can define a k-linear map $i : A \to kQ$ such that $i(\pi(b))) = b$ for all $b \in \mathcal{B}$. We denote by $\beta : kQ \to kQ$ the composition $i \circ \pi$. Notice that if p is a path and r is a reduction such that $r(p)$ is irreducible, then $r(p) = \beta(p)$. In the bibliography, $\beta(p)$ is sometimes called the normal form of p.

The following characterization of the relation $\rightsquigarrow$ is very useful in practice.

Lemma 2.6. *If p, q are paths, then $q \rightsquigarrow p$ if and only if $p = q$ or there exists a reduction t such that $p \in t(q)$.*

Proof. First we prove the necessity of the condition. Let $n \in \mathbb{N}$, $r_1, \dots, r_n$ and $p_1, \dots, p_n$ be as in the definition of $\rightsquigarrow$, and suppose that n is minimal. Let $\tilde{p}_1 = p_1$ and for each

$i = 1, \ldots, n-1$ put $\tilde{p}_{i+1} = r_i(\tilde{p}_i)$. Notice that the minimality implies that $r_i(p_i) \neq p_i$. Let us first show that

$$\text{if } i > j \text{ then } p_i \notin \tilde{p}_j. \tag{2.1}$$

Suppose otherwise and let (i, j) be a counterexample with j minimal. We will prove that in this situation, $p_l \in \tilde{p}_l$ for all $l < j$. We proceed by induction on l. By definition, $p_1 \in \tilde{p}_1$. Suppose $1 \leq l < j-1$ and $p_l \in \tilde{p}_l$. Then we have $p_{l+1} \in r_l(p_l)$ and, since $l < j$, $p_{l+1} \notin \tilde{p}_l$. Write $\tilde{p}_l = \lambda p_l + x$ with $x \in kQ$ and $p_l \notin x$. Since r_l acts nontrivially on p_l, it acts trivially on x; it follows that $r_l(\tilde{p}_l) = \lambda r_l(p_l) + x$ and so $p_{l+1} \in r_l(\tilde{p}_l) = \tilde{p}_{l+1}$. In particular $p_{j-1} \in \tilde{p}_{j-1}$. Since $p_i \notin \tilde{p}_{j-1}$ and $p_i \in \tilde{p}_j$, we must have $p_i \in r_{j-1}(p_{j-1})$.

Now, let $m = n + j - i$, $t_k = r_k$ and $u_k = p_k$ if $k \leq j-1$, and $t_k = r_{i+k-j}$ and $u_k = p_{i+k-j}$ if $j \leq k \leq m$. One can check that $u_1 = q$, $u_{n+j-i} = p$ and that $u_{k+1} \in t_k(u_k)$ for all $k = 1, \ldots, m-1$. Since $m < n$ this contradicts the choice of n. We thus conclude that (2.1) holds.

We can use the same inductive argument as before to prove that $p_i \in \tilde{p}_i$ for all $1 \leq i \leq n$. Denoting $t = (r_n, \ldots, r_1)$, observe that $p \in t(q)$.

Let us now prove the converse. Let $t = (t_m, \ldots, t_1)$ be a reduction such that $p \in t(q)$ and m is minimal, and let us proceed by induction on m. Notice that if $m = 1$ there is nothing to prove. If t_i is the basic reduction r_{a_i,s_i,c_i}, let $p_i = a_i s_i c_i$. Using the same ideas as above one can show that

$$\begin{aligned} &\text{if } u \neq q \text{ and } u \notin t_i(p_i) \text{ for each } 1 \leq i \leq m, \\ &\text{then } u \notin t_l \circ \cdots \circ t_1(q) \text{ for each } 0 \leq l \leq m. \end{aligned}$$

Since $p \in t(q)$ either $p = q$ or there exists $i \in \{1, \ldots, m\}$ such that $p \in t_i(p_i)$. In the first case $q \rightsquigarrow p$. In the second case, we know that $p_i \rightsquigarrow p$ and we need to prove that $q \rightsquigarrow p_i$. Since m is minimal, $t_i(t_{i-1} \circ \cdots \circ t_1(q)) \neq t_{i-1} \circ \cdots \circ t_1(q)$ and then $p_i \in t_{i-1} \circ \cdots \circ t_1(q)$. The result now follows by induction because $i - 1 < m$. $\square$

Proposition 2.7. *If $I \subseteq kQ$ is an ideal, then there exists a reduction system $\mathcal{R}$ which satisfies condition ($\Diamond$) for I.*

We will prove this result by putting together a series of lemmas.

Let $\leq$ be a well-order on the set $Q_0 \cup Q_1$ such that $e < \alpha$ for all $e \in Q_0$ and $\alpha \in Q_1$. Let $\omega : Q_1 \to \mathbb{N}$ be a function and extend it to $Q_{\geq 0}$ defining $\omega(e) = 0$ for all $e \in Q_0$ and $\omega(c_n \cdots c_1) = \sum_{i=1}^{n} \omega(c_i)$ if $c_i \in Q_1$ and $c_n \cdots c_1$ is a path. Given $c, d \in Q_{\geq 0}$ we write that $c \leq_\omega d$ if

- $\omega(c) < \omega(d)$, or
- $c, d \in Q_0$ and $c \leq d$, or
- $\omega(c) = \omega(d)$, $c = c_n \cdots c_1, d = d_m \cdots d_1 \in Q_{\geq 1}$ and there exists $j \leq \min(|c|, |d|)$ such that $c_i = d_i$ for all $\in \{1, \ldots, j-1\}$ and $c_j < d_j$.

Notice that the order $\leq_\omega$ is in fact the *deglex* order with weight ω, and it has the following two properties:

(1) If $p, q \in Q_{\geq 0}$ and $p \leq_\omega q$, then $cpd \leq_\omega cqd$ for all $c, d \in Q_{\geq 0}$ such that $cpd \neq 0$ and $cqd \neq 0$ in kQ.
(2) For all $q \in Q_{\geq 0}$ the set $\{p \in Q_{\geq 0} : p \leq_\omega q\}$ is finite.

It is straightforward to prove the first claim. For the second one, let $\{c^i\}_{i\in\mathbb{N}}$ be a sequence in $Q_{\geq 0}$ such that $c^{i+1} \leq_\omega c^i$ for all i. If $c^i \in Q_0$ for some i, then it is evident that the sequence stabilizes, so let us suppose that $\{c^i\}_{i\in\mathbb{N}}$ is contained in $Q_{\geq 1}$ and $c^{i+1} <_\omega c^i$ for all $i \in \mathbb{N}$. Since $(\omega(c^i))_{i\in\mathbb{N}}$ is a decreasing sequence of natural numbers, it must stabilize, so we may also suppose that $\omega(c^i) = \omega(c^j)$ for all i, j and that the lengths of the paths are bounded above by some $M \in \mathbb{N}$. By definition of $\leq_\omega$, we know that the sequence of first arrows of elements of $\{c^i\}_{i\in\mathbb{N}}$ forms a decreasing sequence in $(Q_1, \leq)$, which must stabilize because $(Q_1, \leq)$ is well-ordered. Let $N \in \mathbb{N}$ be such that the first arrow of c^i equals the first arrow of c^j for all $i, j \geq N$. If $c^i = c^i_{n_i} \cdots c^i_1$, and we denote $c'^{\,i} = c^i_{n_i} \cdots c^i_2$, then $\{c'^{\,i}\}_{i\geq N}$ is a decreasing sequence in $(Q_{\geq 0}, \leq_\omega)$ with $|c'^{\,i}| = M - 1$ for all i. Iterating this process we arrive to a contradiction.

Definition 2.8. Consider as before a well-order $\leq$ on $Q_0 \cup Q_1$ and $\omega : Q_1 \to \mathbb{N}$, and $\leq_\omega$ be constructed from them. If $p \in kQ$ and $p = \sum_{i=1}^n \lambda_i c_i$ with $\lambda_i \in k^\times$, $c_i \in Q_{\geq 0}$ and $c_i <_\omega c_1$ for all $i \neq 1$, we write $\mathsf{tip}(p)$ for c_1. If $X \subseteq kQ$, we let $\mathsf{tip}(X) := \{\mathsf{tip}(x) : x \in X \setminus \{0\}\}$.

Consider the set

$$S := \mathsf{Mintip}(I) = \{p \in \mathsf{tip}(I) : p' \notin \mathsf{tip}(I) \text{ for all proper divisors } p' \text{ of } p\}.$$

Notice that if s and s' both belong to S and $s \neq s'$, then s does not divide s'. For each $s \in S$, choose $f_s \in kQ$ such that $s - f_s \in I$, $f_s <_\omega s$ and f_s is parallel to s.

Describing the set $\mathsf{tip}(I)$ is not easy in general. We comment on this problem at the beginning of the last section, where we compute examples.

Lemma 2.9. *Let $\leq_\omega$ and S be as before. The ideal I equals the two sided ideal generated by the set $\{s - f_s\}_{s\in S}$, which we will denote by $\langle s - f_s\rangle_{s\in S}$.*

Proof. It is clear that $\langle s - f_s\rangle_{s\in S}$ is contained in I. Choose $x = \sum_{i=1}^n \lambda_i c_i \in I$ with $\lambda_i \in k^\times$ and $c_i \in Q_{\geq 0}$. We may suppose that $c_1 = \mathsf{tip}(x)$, so that $c_1 \in \mathsf{tip}(I)$. There is a divisor s of c_1 such that $s \in \mathsf{tip}(I)$ and $s' \notin \mathsf{tip}(I)$ for all proper divisor s' of s and $s \in S$ by definition of S. Let $a, c \in Q_{\geq 0}$ with $asc = c_1$.

Define $x' := af_sc + \sum_{i=2}^n \lambda_i c_i$. We have $x = \lambda_1 c_1 + \sum_{i=2}^n \lambda_i c_i = \lambda_1 a(s - f_s)c + x'$, so that $x' \in I$ and, by property (1) of the order $\leq_\omega$, we see that $c_1 > \mathsf{tip}(x')$. We can apply this procedure again to x' and iterate: the process will stop by property (2) and we conclude that $x \in \langle s - f_s\rangle_{s\in S}$. $\square$

Lemma 2.10. *Let $\leq_\omega$ and S be as before. The set $\mathcal{R} := \{(s, f_s)\}_{s\in S}$ is a reduction system such that every path is reduction-unique.*

Proof. Since $s >_\omega \mathsf{tip}(f_s)$ for all $s \in S$, properties (1) and (2) guarantee that every path is reduction-finite. We need to prove that every path is reduction-unique. Recall that π is the canonical projection $kQ \to kQ/I$. Let p be a path. Since $I = \langle s - f_s\rangle_{s\in S}$, we see that $\pi(r(p)) = \pi(p)$ for any reduction r. Let r and t be reductions such that $r(p)$ and $t(p)$ are both irreducible. Clearly, $\pi(r(p) - t(p)) = \pi(p) - \pi(p) = 0$, so that $r(p) - t(p) \in I$. If this difference is not zero, then the path $d = \mathsf{tip}(r(p) - t(p))$ can be written as $d = asc$ with a, c paths and $s \in S$. It follows that the reduction $r_{a,s,c}$ acts nontrivially either on $r(p)$ or on $t(p)$, and this is a contradiction. □

This lemma implies that for each $s \in S$, there exists a reduction r and an irreducible element f'_s such that $r(f_s) = f'_s$. Consider the reduction system $\mathcal{R}' := \{(s, f'_s) : s \in S\}$. The set of irreducible paths for $\mathcal{R}$ clearly coincides with the set of irreducible paths for $\mathcal{R}'$ and, since $\pi(s - f'_s) = \pi(s - f_s) = 0$, we have that $\langle s - f'_s\rangle_{s\in S} \subseteq I$. From Bergman's Diamond Lemma it follows that $I = \langle s - f'_s\rangle_{s\in S}$. We can conclude that the reduction system $\mathcal{R}'$ satisfies condition ($\Diamond$), thereby proving Proposition 2.7.

It is important to emphasize that different choices of orders on $Q_0 \cup Q_1$ and of weights ω will give very different reduction systems, some of which will better suit our purposes than others. Moreover, there are reduction systems which cannot be obtained by this procedure, as the following example shows.

Example 2.10.1. Consider the algebra

$$A = k\langle x, y, z\rangle/(x^3 + y^3 + z^3 - xyz)$$

and let $\mathcal{R} = \{(xyz, x^3 + y^3 + z^3)\}$. Clearly this reduction system does not come from a monomial order and neither from a monomial order with weights. It is not entirely evident but this reduction system satisfies ($\Diamond$). See also Example 3.4.7 in [20].

Finally, we define a relation $\preceq$ on the set $k^\times Q_{\geq 0} := \{\lambda p : \lambda \in k^\times, p \in Q_{\geq 0}\} \cup \{0\}$ as the least reflexive and transitive relation such that $\lambda p \preceq \mu q$ whenever there exists a reduction r such that $r(\mu q) = \lambda p + x$ with $p \notin x$. We state $0 \preceq \lambda p$ for all $\lambda p \in k^\times Q_{\geq 0}$.

Lemma 2.11. *The binary relation $\preceq$ is an order satisfying the descending chain condition and it is compatible with concatenation.*

Proof. The second claim is clear. In order to prove the first claim, let us first prove that if $p \in Q_{\geq 0}$ is such that there exists a reduction r with $r(p) = \lambda p + x$ and $p \notin x$, then $\lambda = 1$ and $x = 0$. Suppose not. For r a basic reduction, this has already been done in Lemma 2.4. If r is not basic, then $r = (r_n, \ldots, r_1)$ with r_i basic and $n \geq 2$. Let $r' = (r_n, \ldots, r_2)$. Since $p \in r(p) = r'(r_1(p))$, there exists $p_1 \in r_1(p)$ such that

$p \in r'(p_1)$. By the previous case, we obtain that $p \notin r_1(p)$, so $p \neq p_1$. As a consequence of Lemma 2.6, we know that $p \rightsquigarrow p_1$ since $p_1 \in r_1(p)$ and that $p_1 \rightsquigarrow p$ since $p \in r'(p_1)$. This contradicts the antisymmetry of $\rightsquigarrow$.

It is an immediate consequence of the previous fact that given a path p and a reduction t,

$$\text{if } t(\lambda_1 p) = \lambda_2 p + x \text{ with } p \notin x, \text{ then } \lambda_1 = \lambda_2. \tag{2.2}$$

Let $\lambda_1, \dots, \lambda_{n+1} \in k^\times$, $p_1, \dots, p_{n+1} \in Q_{\geq 0}$, $x_1, \dots, x_n \in kQ$ and reductions $t_1, \dots, t_n$ be such that $t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i$, $p_{i+1} \notin x_i$ and $\lambda_{n+1} p_{n+1} = \lambda_1 p_1$. This implies that $p_i \rightsquigarrow p_{i+1}$ for each $1 \leq i \leq n$ and $p_{n+1} = p_1$. Since $\rightsquigarrow$ is antisymmetric, it follows that $p_i = p_1$ for all i and (2.2) implies that $\lambda_i = \lambda_1$ for all i. We thus see that $\preceq$ is antisymmetric.

Let now $(\lambda_i p_i)_{i \in \mathbb{N}}$ be a sequence in $k^\times Q_{\geq 0}$ and $(t_i)_{i \in \mathbb{N}}$ a sequence of reductions such that $t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i$ with $p_{i+1} \notin x_i$. Then $p_i \rightsquigarrow p_{i+1}$ for all i and since $\rightsquigarrow$ satisfies the descending chain condition there exists i_0 such that $p_i = p_{i_0}$ for all $i \geq i_0$. Observation (2.2) implies then that $\lambda_i = \lambda_{i_0}$ for all $i \geq i_0$, so that the sequence $(\lambda_i p_i)_{i \in \mathbb{N}}$ stabilizes. □

If $x = \sum_{i=1}^n \lambda_i p_i \in kQ$ with $\lambda_i \in k^\times$ and λp belongs to $k^\times Q_{\geq 0}$, we write $x \preceq \lambda p$ if $\lambda_i p_i \preceq \lambda p$ for all i. If in addition $x \neq \lambda p$ we also write $x \prec p$. The following simple fact is the key to proving everything that follows.

Corollary 2.12. *Given a path p, its normal form $\beta(p)$ is such that $\beta(p) \preceq p$. Moreover, $\beta(p) \prec p$ if and only if $p \notin \mathcal{B}$.*

Proof. There is a reduction r such that $\beta(p) = r(p) = \sum_{i=1}^n \lambda_i p_i$. It is clear that $\lambda_i p_i \preceq p$ for all i, so that $\beta(p) \preceq p$. The last claim follows from the fact that $\beta(p) = p$ if and only if $p \in \mathcal{B}$. □

3. Ambiguities

Given an algebra $A = kQ/I$ and a reduction system $\mathcal{R}$ satisfying $(\lozenge)$ for I, there is a monomial algebra associated to A defined as $A_S := kQ/\langle S \rangle$ and equipped with the canonical projection $\pi' : kQ \to A_S$. The set $\pi'(\mathcal{B})$ is a k-basis of A_S. The algebra A_S is a generalization of the algebra A_{mon} defined in [18]: in that article, the order is necessarily monomial.

From now on we fix the reduction system $\mathcal{R}$ satisfying condition $(\lozenge)$. Notice that in this situation we can suppose without loss of generality, that $S \subseteq Q_{\geq 2}$.

The family of modules $\{\mathcal{P}_i\}_{i \geq 0}$ appearing in the resolution of A as A-bimodule will be in bijection with those appearing in Bardzell's resolution of the monomial algebra A_S. More precisely, we will define E-bimodules $k\mathcal{A}_i$ for $i \geq -1$, such that the former will be

$\{A \otimes_E k\mathcal{A}_i \otimes_E A\}_{i \geq -1}$ while the latter will be $\{A_S \otimes_E k\mathcal{A}_i \otimes_E A_S\}_{i \geq -1}$. The resolution will start as usual: $\mathcal{A}_{-1} = Q_0$, $\mathcal{A}_0 = Q_1$ and $\mathcal{A}_1 = S$.

For $n \geq 2$, $\mathcal{A}_n$ will be the set of n-ambiguities of $\mathcal{R}$. We will next recall the definition of n-ambiguity – or n-chain according to the terminology used in [29,1,2] and to Bardzell's [4] *associated sequences of paths*, and we will take into account that the sets of left n-ambiguities and right n-ambiguities coincide. This fact is proved in [4] and also in [29]. See [19] too.

Definition 3.1. Given $n \geq 2$ and $p \in Q_{\geq 0}$,

(1) the path p is a *left n-ambiguity* if there exist $u_0 \in Q_1$, $u_1, \dots, u_n$ irreducible paths such that
  (a) $p = u_0 u_1 \cdots u_n$,
  (b) for all i, $u_i u_{i+1}$ is reducible but $u_i d$ is irreducible for any proper left divisor d of u_{i+1};
(2) the path p is a *right n-ambiguity* if there exist $v_0 \in Q_1$ and $v_1, \dots, v_n$ irreducible paths such that
  (a) $p = v_n \cdots v_0$,
  (b) for all i, $v_{i+1} v_i$ is reducible but $d v_i$ is irreducible for any proper right divisor of v_{i+1}.

Proposition 3.2. *Let $n, m \in \mathbb{N}$, $p \in Q_{\geq 1}$. If $u_0, \hat{u}_0 \in Q_1$ and $u_1, \dots, u_n, \hat{u}_1, \dots, \hat{u}_n$ are paths in Q such that both $u_0, \dots, u_n$ and $\hat{u}_0, \dots, \hat{u}_n$ satisfy conditions (1a) and (1b) of the previous definition for p, then $n = m$ and $u_i = \hat{u}_i$ for all i, $0 \leq i \leq n$.*

Proof. Suppose $n \leq m$. It is obvious that $u_0 = \hat{u}_0$, since both of them are arrows. Notice that $kQ = T_{kQ_0} kQ_1$, that is the free algebra generated by kQ_1 over kQ_0, which implies that either $u_0 u_1$ divides $\hat{u}_0 \hat{u}_1$ or $\hat{u}_0 \hat{u}_1$ divides $u_0 u_1$, and moreover $u_0 u_1, \hat{u}_0 \hat{u}_1 \in \mathcal{A}_1 = S$. Remark 2.2.1 says that $u_0 u_1 = \hat{u}_0 \hat{u}_1$. Since $u_0 = \hat{u}_0$, we must have $u_1 = \hat{u}_1$. By induction on i, let us suppose that $u_j = \hat{u}_j$ for $j \leq i$. As a consequence, $u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m$.

If $i + 1 = n$, this reads $u_n = \hat{u}_n \cdots \hat{u}_m$, and the fact that u_n is irreducible and $\hat{u}_j \hat{u}_{j+1}$ is reducible for all $j < m$ implies that $m = n$ and $u_n = \hat{u}_n$. Instead, suppose that $i + 1 < n$. From the equality $u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m$ we deduce that there exists a path d such that $u_{i+1} = \hat{u}_{i+1} d$ or $\hat{u}_{i+1} = u_{i+1} d$. If $u_{i+1} = \hat{u}_{i+1} d$ and $d \in Q_{\geq 1}$, we can write $d = d_2 d_1$ with $d_1 \in Q_1$. The path $\hat{u}_{i+1} d_2$ is a proper left divisor of u_{i+1} and by condition (1b) we obtain that $u_i \hat{u}_{i+1} d_2$ is irreducible. This is absurd since $u_i \hat{u}_{i+1} d_2 = \hat{u}_i \hat{u}_{i+1} d_2$ by inductive hypothesis, and the right hand term is reducible by condition (1b). It follows that $d \in Q_0$ and then $u_{i+1} = \hat{u}_{i+1}$. The case where $\hat{u}_{i+1} = u_{i+1} d$ is analogous. □

Corollary 3.3. *Given $n, m \geq -1$, $\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ if n and m are different.*

Just to get a flavor of what $\mathcal{A}_n$ is, one may think about an element of $\mathcal{A}_n$ as a minimal proper superposition of n elements of S.

We end this section with a proposition that indicates how to compute ambiguities for a particular family of algebras.

Proposition 3.4. *Suppose $S \subset Q_2$. For all $n \geq 1$,*

$$\mathcal{A}_n = \{\alpha_0 \dots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1 \text{ for all } i \text{ and } \alpha_{i-1}\alpha_i \in S\}$$

Moreover, given $p = \alpha_0 \dots \alpha_n \in \mathcal{A}_n$, we can write p as a left ambiguity choosing $u_i = \alpha_i$, for all i, and as a right ambiguity choosing $v_i = \alpha_{n-i}$

Proof. We proceed by induction on n. If $n = 1$ we know that $\mathcal{A}_1 = S$ in which case there is nothing to prove. Let $u_0 \cdots u_n u_{n+1} \in \mathcal{A}_{n+1}$ and suppose that the result holds for all $p \in \mathcal{A}_n$. Since $u_0 \cdots u_n$ belongs to $\mathcal{A}_n$ we only have to prove that $u_{n+1} \in Q_1$ and that $u_n u_{n+1} \in S$. We know that $u_n \in Q_1$, that u_{n+1} is irreducible and that $u_n u_{n+1}$ is reducible. As a consequence, there exist $s \in S$ and $v \in Q_{\geq 0}$ such that $u_n u_{n+1} = sv$. Moreover, $u_n d$ is irreducible for any proper left divisor d of u_{n+1}, so the only possibility is $v \in Q_0$. We conclude that $u_n u_{n+1}$ belongs to S. Since $S \subseteq Q_2$ and $u_n \in Q_1$, we deduce that $u_{n+1} \in Q_1$. This proves that $\mathcal{A}_{n+1} \subseteq \{\alpha_0 \cdots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1 \text{ for all } i \text{ and } \alpha_{i-1}\alpha_i \in S\}$.

The other inclusion is clear. □

4. The resolution

In this section our purpose is to construct bimodule resolutions of the algebra A. We achieve this in Theorems 4.1 and 4.2: in the first one we construct homotopy maps to prove that a given complex is exact, while in the second one we define differentials inductively.

We will make use of differentials of Bardzell's resolution for monomial algebras, so we begin this section by recalling them. Keeping the notations of the previous section, note that the kQ-bimodule $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ is a k-vector space with basis $\{a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, apc \neq 0 \text{ in } kQ\}$.

As we have already done for A, we define a k-linear map $i' : A_S \to kQ$ such that $i'(\pi'(b)) = b$ for all $b \in \mathcal{B}$, and we denote by $\beta' : kQ \to kQ$ the composition $i' \circ \pi'$.

Given $n \geq -1$, let us fix notation for the following k-linear maps:

$$\pi_n := \pi \otimes id_{k\mathcal{A}_n} \otimes \pi, \qquad \pi'_n := \pi' \otimes id_{k\mathcal{A}_n} \otimes \pi',$$

$$i_n := i \otimes id_{k\mathcal{A}_n} \otimes i, \qquad i'_n := i' \otimes id_{k\mathcal{A}_n} \otimes i',$$

$$\beta_n := i_n \circ \pi_n, \qquad \beta'_n := i'_n \circ \pi'_n.$$

Consider the following sequence of kQ-bimodules,

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{f_1} & kQ \otimes_E k\mathcal{A}_0 \otimes_E kQ & \xrightarrow{f_0} & kQ \otimes_E kQ & \xrightarrow{f_{-1}} & kQ & \longrightarrow & 0 \\ & & & & \downarrow \cong & & & & \\ & & & & kQ \otimes_E k\mathcal{A}_{-1} \otimes_E kQ & & & & \end{array}$$

where

(1) $f_n : kQ \otimes_E k\mathcal{A}_n \otimes_E kQ \to kQ \otimes_E k\mathcal{A}_{n-1} \otimes_E kQ$ for $n \geq 0$,
(2) $f_{-1}(a \otimes b) = ab$,
(3) if n is even, $q \in \mathcal{A}_n$ and $q = u_0 \cdots u_n = v_n \cdots v_0$ are respectively the factorizations of q as left and right n-ambiguity,

$$f_n(1 \otimes q \otimes 1) = v_n \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes u_n,$$

(4) if n is odd and $q \in \mathcal{A}_n$,

$$f_n(1 \otimes q \otimes 1) = \sum_{\substack{apc=q \\ p \in \mathcal{A}_{n-1}}} a \otimes p \otimes c.$$

Also, for each $n \geq -1$, let

$$\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$$

be the morphism of A-bimodules defined by

(1) $\delta_{-1}(\pi(a) \otimes \pi(b)) = \pi(ab)$,
(2) if n is even, $q \in \mathcal{A}_n$ and $q = u_0 \cdots u_n = v_n \cdots v_0$ are respectively the factorizations of q as left and right n-ambiguity,

$$\delta_n(1 \otimes q \otimes 1) = \pi(v_n) \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes \pi(u_n),$$

(3) if n is odd and $q \in \mathcal{A}_n$,

$$\delta_n(1 \otimes q \otimes 1) = \sum_{\substack{apc=q \\ p \in \mathcal{A}_{n-1}}} \pi(a) \otimes p \otimes \pi(c).$$

In the same way, define

$$\delta'_n : A_S \otimes_E k\mathcal{A}_n \otimes_E A_S \to A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S$$

by replacing A and π by A_S and π' in the definition of δ_n.

Notice that δ_{-1} and δ'_{-1} are respectively multiplication in A and in A_S, and that

$$\delta_n := \pi_{n-1} \circ f_n \circ i_n,$$
$$\delta'_n := \pi'_{n-1} \circ f_n \circ i'_n.$$

The algebra A_S is monomial. The following complex provides a projective resolution of A_S as A_S-bimodule [4]:

$$\cdots \xrightarrow{\delta'_2} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E A_S \xrightarrow{\delta'_{-1}} A_S \longrightarrow 0.$$

We will make use of the homotopy that Sköldberg defined in [29] when proving that this complex is exact. We recall it, but we must stress that our signs differ from the ones in [29] due to the fact that he considers right modules, while we always work with left modules.

Given $n \geq -1$, the morphism of $kQ - E$-bimodules S_n is defined as follows.

For $n = -1$, $S_{-1} : kQ \to kQ \otimes_E k\mathcal{A}_{-1} \otimes_E kQ$ is the $kQ - E$-bimodule map given by $S_{-1}(a) = a \otimes 1$, for $a \in kQ$.

For $n \in \mathbb{N}_0$, $S_n : kQ \otimes_E k\mathcal{A}_{n-1} \otimes_E kQ \to kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ is given by

$$S_n(1 \otimes q \otimes b) = (-1)^{n+1} \sum_{\substack{apc=qb \\ p\in\mathcal{A}_n}} a \otimes p \otimes c.$$

Let $s'_n := \pi'_n \circ S_n \circ i'_{n-1}$. The family of maps $\{s'_n\}_{n\geq -1}$ verifies the equalities

$$s'_n \circ \delta'_n + \delta'_{n-1} \circ s'_{n-1} = id_{A_S \otimes_E k\mathcal{A}_n \otimes_E A_S} \text{ for } n \geq 0 \text{ and } s'_{-1} \circ \delta'_{-1} = id_{A_S \otimes_E k\mathcal{A}_{-1} \otimes_E A_S}.$$

Also, define $s_n := \pi_n \circ S_n \circ i_{n-1}$.

Next we define some sets that will be useful in the sequel. For any $n \geq -1$ and $\mu q \in k^\times Q_{\geq 0}$, consider the following subsets of $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$:

- $\mathcal{L}_n^{\preceq}(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda apc \preceq \mu q\}$,
- $\mathcal{L}_n^{\prec}(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda apc \prec \mu q\}$,

and the following subsets of $A \otimes_E k\mathcal{A}_n \otimes_E A$:

- $\overline{\mathcal{L}}_n^{\preceq}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \preceq \mu q\}$,
- $\overline{\mathcal{L}}_n^{\prec}(\mu q) := \{\lambda\pi(b) \otimes p \otimes \pi(b') : b, b' \in \mathcal{B}, p \in \mathcal{A}_n, \lambda bpb' \prec \mu q\}$.

Remark 4.0.1. We observe that

$$f_{n+1}(x) \in \langle \mathcal{L}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}, \qquad \text{for all } x \in \mathcal{L}_{n+1}^{\preceq}(\mu q), \text{ and}$$
$$S_n(x) \in \langle \mathcal{L}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}, \qquad \text{for all } x \in \mathcal{L}_{n-1}^{\preceq}(\mu q).$$

Moreover, the only possible coefficients appearing in the linear combinations are +1 and −1.

We will now state the main theorems. Recall that our aim is to construct, for non-necessarily monomial algebras, a bimodule resolution starting from a related monomial algebra. The first theorem says that if the difference between its differentials and the monomial differentials can be "controlled", then we will actually obtain an exact complex. The second theorem says that it is possible to construct the differentials.

Theorem 4.1. *Set $d_{-1} := \delta_{-1}$ and $d_0 := \delta_0$. Given $N \in \mathbb{N}_0$ and morphisms of A-bimodules $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \to A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$ for $1 \le i \le N$. If*

(1) $d_{i-1} \circ d_i = 0$ for all i, $1 \le i \le N$,
(2) $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q)\rangle_k$ for all $i \in \{1, \dots, N\}$ and for all $q \in \mathcal{A}_i$,

then the complex

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

is exact.

Theorem 4.2. *There exist A-bimodule morphisms $d_i : A\otimes_E k\mathcal{A}_i \otimes_E A \to A\otimes_E k\mathcal{A}_{i-1}\otimes_E A$ for $i \in \mathbb{N}_0$ and $d_{-1} : A \otimes_E A \to A$ such that*

(1) $d_{i-1} \circ d_i = 0$, for all $i \in \mathbb{N}_0$,
(2) $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q)\rangle_{\mathbb{Z}}$ for all $i \ge -1$ and $q \in \mathcal{A}_i$.

We will carry out the proofs of these theorems in the following section.

5. Proofs of the theorems

We keep the same notations and conditions of the previous section. We start by proving some technical lemmas.

Lemma 5.1. *Given $n \ge 0$, the following equalities hold*

(1) $\delta_n \circ \pi_n = \pi_{n-1} \circ f_n$,
(2) $\delta'_n \circ \pi'_n = \pi'_{n-1} \circ f_n$.

The proof is straightforward after the definitions.

Next we prove three lemmas where we study how various maps defined in Section 4 behave with respect to the order.

Lemma 5.2. *For all $n \in \mathbb{N}_0$ and $\mu q \in k^\times Q_{\geq 0}$, the images by π_n of $\mathcal{L}_n^{\preceq}(\mu q)$ and of $\mathcal{L}_n^{\prec}(\mu q)$ are respectively contained in $\langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$ and in $\langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.*

Proof. Given $n \in \mathbb{N}_0$, $\mu q \in k^\times Q_{\geq 0}$ and $x = \lambda a \otimes p \otimes c \in \mathcal{L}_n^{\preceq}(\mu q)$, where $a, c \in Q_{\geq 0}$ and $p \in \mathcal{A}_n$, suppose $\beta(a) = \sum_i \lambda_i b_i$ and $\beta(c) = \sum_j \lambda'_j b'_j$. Since $\beta(a) \preceq a$ and $\beta(c) \preceq c$, then $\lambda_i b_i \preceq a$ and $\lambda'_j b'_j \preceq c$ for all i, j. This implies

$$\lambda\lambda_i\lambda_j b_i p b'_j \preceq \lambda apc \preceq \mu q$$

and so $\lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j)$ belong to $\overline{\mathcal{L}}_n^{\preceq}(\mu q)$ for all i, j. The result follows from the equalities

$$\pi_n(x) = \lambda\pi(a) \otimes p \otimes \pi(c) = \lambda\pi(\beta(a)) \otimes p \otimes \pi(\beta(c)) = \sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j).$$

The proof of the second part is analogous. □

Corollary 5.3. *Let $n \geq -1$ and $\mu q \in k^\times Q_{\geq 0}$. Keeping the same notations of the proof of the previous lemma, we conclude that*

(1) if $x \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$, then $\lambda\pi(a)x\pi(c) \in \langle \overline{\mathcal{L}}_n^{\preceq}(\lambda\mu aqc)\rangle_{\mathbb{Z}}$,
(2) if $x \in \overline{\mathcal{L}}_n^{\prec}(\mu q)$, then $\lambda\pi(a)x\pi(c) \in \langle \overline{\mathcal{L}}_n^{\prec}(\lambda\mu aqc)\rangle_{\mathbb{Z}}$.

Lemma 5.4. *Given $n \in \mathbb{N}_0$ and $\mu q \in k^\times Q_{\geq 0}$, there are inclusions*

(1) $\delta_n(\overline{\mathcal{L}}_n^{\preceq}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$,
(2) $\delta_n(\overline{\mathcal{L}}_n^{\prec}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$,
(3) $s_n(\overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_n^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$,
(4) $s_n(\overline{\mathcal{L}}_{n-1}^{\prec}(\mu q)) \subseteq \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

Proof. From $x = \lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$, with $b, b' \in \mathcal{B}$ and $p \in \mathcal{A}_n$, we get $i_n(x) = \lambda b \otimes p \otimes b'$. This element belongs to $\mathcal{L}_n^{\preceq}(\mu q)$ and this implies that $f_n(\lambda b \otimes p \otimes b')$ belongs to $\langle \mathcal{L}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$, by Remark 4.0.1. As a consequence of Lemma 5.2 we obtain that $\delta_n(x) = \pi_{n-1}(f_n(\lambda b \otimes p \otimes b'))$ belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$. The proofs of the other statements are similar. □

Lemma 5.5. *Given $n \geq -1$ and $\mu q \in k^\times Q_{\geq 0}$, if $x = \lambda a \otimes p \otimes c \in \mathcal{L}_n^{\preceq}(\mu q)$ is such that $\pi'_n(x) = 0$, then*

$$\pi_n(x) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}.$$

Proof. By hypothesis we get that $0 = \pi'_n(x) = \pi'(a) \otimes p \otimes \pi'(c)$. The only possibilities are $\pi'(a) = 0$ or $\pi'(c) = 0$, this is, $a \notin \mathcal{B}$ or $c \notin \mathcal{B}$, namely $\beta(a) \prec a$ or $\beta(c) \prec c$.

Writing $\beta(a) = \sum_i \lambda_i b_i$ and $\beta(c) = \sum_j \lambda'_j b'_j$, we deduce that $\lambda\lambda_i\lambda'_j b_i p b_j \prec \mu q$ for all i, j. As a consequence, $\sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

The proof ends by computing

$$\pi_n(x) = \pi_n(\beta(x)) = \pi_n(\sum_{i,j} \lambda\lambda_i\lambda'_j b_i \otimes p \otimes b'_j) = \sum_{i,j} \lambda\lambda_i\lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j). \qquad \square$$

The importance of the preceding lemmas is that they guarantee how differentials and morphisms used for the homotopy behave with respect to the order. This is stated explicitly in the following corollary.

Corollary 5.6. *Given $n \geq 1$, $\mu q \in k^\times Q_{\geq 0}$ and $x \in \overline{\mathcal{L}}_n^{\preceq}(\mu q)$, the following facts hold:*

(1) $\delta_{n-1} \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_{n-2}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$,
(2) $x - \delta_{n+1} \circ s_{n+1}(x) - s_n \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

Proof. Let us first write $x = \lambda\pi(b) \otimes p \otimes \pi(b')$ with $b, b' \in \mathcal{B}$ and $x' := i_n(x) = \lambda b \otimes p \otimes b'$. Lemma 5.1 implies that

$$\delta_{n-1} \circ \delta_n(x) = \delta_{n-1} \circ \delta_n \circ \pi_n(x') = \delta_{n-1} \circ \pi_{n-1} \circ f_n(x') = \pi_{n-2} \circ f_{n-1} \circ f_n(x').$$

By Remark 4.0.1, $f_{n-1} \circ f_n(x') \in \mathcal{L}_{n-2}^{\preceq}(\mu q)$. Next, by Lemma 5.5, in order to prove that $\delta_{n-1} \circ \delta_n(x) \in \langle \overline{\mathcal{L}}_{n-2}^{\prec}(\mu q)\rangle_{\mathbb{Z}}$, it suffices to verify that $\pi'_{n-2} \circ f_{n-1} \circ f_n(x') = 0$, which is in fact true using Lemma 5.1, and the fact that $(A_S \otimes_E k\mathcal{A}_\bullet \otimes_E A_S, \delta'_\bullet)$ is a complex.

In order to prove (2), we first remark that if $k \in \mathbb{N}_0$ and $y \in \langle \mathcal{L}_k^{\preceq}(\mu q)\rangle_{\mathbb{Z}}$, then $i'_k \circ \pi'_k(y) - i_k \circ \pi_k(y) \in \langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$. Indeed, let us write $y = \lambda a \otimes p \otimes c \in \mathcal{L}_k^{\preceq}(\mu q)$. In case $a \in \mathcal{B}$ and $c \in \mathcal{B}$, there are equalities $i'_k \circ \pi'_k(y) = y = i_k \circ \pi_k(y)$, and so the difference is zero. If either $a \notin \mathcal{B}$ or $c \notin \mathcal{B}$, then $\pi'_k(y) = 0$ and in this case Lemma 5.5 implies that $\pi_k(y) \in \langle \overline{\mathcal{L}}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$. So, $i_k \circ \pi_k(y) \in \langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$ and the difference we are considering belongs to $\langle \mathcal{L}_k^{\prec}(\mu q)\rangle_{\mathbb{Z}}$.

Fix now $x = \lambda\pi(b) \otimes p \otimes \pi(b')$ and $x' = i_n(x) = \lambda b \otimes p \otimes b'$, with $b, b' \in \mathcal{B}$.

Since $x' = i'_n \circ \pi'_n(x')$,

$$\begin{aligned} x - \delta_{n+1} \circ s_{n+1}(x) - s_n \circ \delta_n(x) = \pi_n(x') &- \pi_n(f_{n+1} \circ i_{n+1} \circ \pi_{n+1} \circ S_{n+1}(x')) \\ &- \pi_n(S_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x')). \end{aligned}$$

The previous comments and Remark 4.0.1 allow us to write that

$$\pi_n \circ f_{n+1} \circ (i'_{n+1} \circ \pi'_{n+1} - i_{n+1} \circ \pi_{n+1}) \circ S_{n+1}(x') \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}},$$
$$\pi_n \circ S_n \circ (i'_{n-1} \circ \pi'_{n-1} - i_{n-1} \circ \pi_{n-1}) \circ f_n(x') \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}}.$$

It is then enough to prove that

$$\pi_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x')) \in \langle \overline{\mathcal{L}}_n^{\prec}(\mu q)\rangle_{\mathbb{Z}},$$

but

$$\begin{aligned}&\pi'_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x'))\\ &= \pi'_n(x') - \delta'_{n+1} \circ s'_{n+1}(\pi'_n(x')) - s'_n \circ \delta'_n(\pi'_n(x'))\\ &= 0.\end{aligned}$$

Finally, we deduce from Lemma 5.5 that

$$\pi_n(x' - f_{n+1} \circ i'_{n+1} \circ \pi'_{n+1} \circ S_{n+1}(x') - S_n \circ i'_{n-1} \circ \pi'_{n-1} \circ f_n(x')) \in \langle \overline{\mathcal{L}}^{\prec}_n(\mu q)\rangle_{\mathbb{Z}}. \qquad \square$$

Next we prove another technical lemma that shows how to control the differentials.

Lemma 5.7. *Fix $n \in \mathbb{N}_0$, let R be either k or $\mathbb{Z}$.*

(1) If $d : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ is a morphism of A-bimodules such that $(d - \delta_n)(1 \otimes p \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{n-1}(p)\rangle_R$ for all $p \in \mathcal{A}_n$, then given $x \in \langle \overline{\mathcal{L}}^{\preceq}_n(\mu q)\rangle_R$, $(d - \delta_n)(x) \in \langle \overline{\mathcal{L}}^{\prec}_{n-1}(\mu q)\rangle_R$ for all $\mu q \in k^\times Q_{\geq 0}$.

(2) If $\rho : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n+1} \otimes_E A$ is a morphism of $A - E$-bimodules such that $(\rho - s_n)(1 \otimes p \otimes \pi(b)) \in \langle \overline{\mathcal{L}}^{\prec}_{n+1}(pb)\rangle_R$, for all $p \in \mathcal{A}_n$ and $b \in \mathcal{B}$, then for all $x \in \langle \overline{\mathcal{L}}^{\preceq}_n(\mu q)\rangle_R$, $(\rho - s_n)(x)$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n+1}(\mu q)\rangle_R$ for all $\mu q \in k^\times Q_{\geq 0}$.

Proof. Given $\mu q \in k^\times Q_{\geq 0}$ and $x \in \langle \overline{\mathcal{L}}^{\preceq}_n(\mu q)\rangle_R$, let us see that $(d-\delta_n)(x) \in \langle \overline{\mathcal{L}}^{\prec}_{n-1}(\mu q)\rangle_R$. It suffices to prove the statement for $x = \lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}^{\preceq}_n(\mu q)$.

By hypothesis, $(d - \delta_n)(1 \otimes p \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n-1}(p)\rangle_R$, so $(d - \delta_n)(x)$ equals $\lambda\pi(b)(d - \delta_n)(1 \otimes p \otimes 1)\pi(b')$ and it belongs to $\langle \overline{\mathcal{L}}^{\prec}_{n-1}(\lambda bpb')\rangle_R \subseteq \langle \overline{\mathcal{L}}^{\prec}_{n-1}(\mu q)\rangle_R$, using Corollary 5.3.

The second part is analogous. $\square$

Next proposition will provide the remaining necessary tools for the proofs of Theorem 4.1 and Theorem 4.2.

Proposition 5.8. *Fix $n \in \mathbb{N}_0$ and let R be either k or $\mathbb{Z}$. Suppose that for each $i \in \{0, \dots, n\}$ there are morphisms of A-bimodules $d_i : A \otimes_E k\mathcal{A}_i \otimes_E A \to A \otimes_E k\mathcal{A}_{i-1} \otimes_E A$, and morphisms of $A - E$-bimodules $\rho_i : A \otimes_E k\mathcal{A}_{i-1} \otimes_E A \to A \otimes_E k\mathcal{A}_i \otimes_E A$. Denote $d_{-1} = \delta_{-1}$ and define $\rho_{-1} : A \to A \otimes_E A$ as $\rho(a) = a \otimes 1$.*

If the following conditions hold,

(i) $d_{i-1} \circ d_i = 0$ for all $i \in \{0, \dots, n\}$,
(ii) $(d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{i-1}(q)\rangle_R$ for all $i \in \{0, \dots, n\}$ and for all $q \in \mathcal{A}_i$,
(iii) for all $i \in \{-1, \dots, n-1\}$ and for all $x \in A \otimes_E k\mathcal{A}_i \otimes_E A$, $x = d_{i+1} \circ \rho_{i+1}(x) + \rho_i \circ d_i(x)$,

(iv) $(\rho_i - s_i)(1 \otimes q \otimes \pi(b)) \in \langle \overline{\mathcal{L}}_i^{\prec}(qb) \rangle_R$ *for all* $i \in \{0, \dots, n\}$*, for all* $q \in \mathcal{A}_i$ *and for all* $b \in \mathcal{B}$*,*

then:

(1) If $d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \to A \otimes_E k\mathcal{A}_n \otimes_E A$ *is a map satisfying the following conditions:*
 (i) $d_n \circ d_{n+1} = 0$*,*
 (ii) $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}_n^{\prec}(q) \rangle_R$*,*
 then there exists a morphism $\rho_{n+1} : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n+1} \otimes_E A$ *of* $A-E$ *bimodules such that*
 (a) for all $x \in A \otimes_E k\mathcal{A}_n \otimes_E A$*,* $x = d_{n+1} \circ s_{n+1}(x) + s_n \circ d_n(x)$*,*
 (b) for all $q \in \mathcal{A}_n$ *and for all* $b \in \mathcal{B}$*,* $(\rho_{n+1} - s_{n+1})(1 \otimes q \otimes \pi(b)) \in \langle \mathcal{L}_{n+1}^{\prec}(qb) \rangle_R$*.*

(2) There exists a morphism of A-bimodules $d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \to A \otimes_E k\mathcal{A}_n \otimes_E A$ *such that*
 (i) $d_n \circ d_{n+1} = 0$*,*
 (ii) $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in \langle \overline{\mathcal{L}}_n^{\prec}(q) \rangle_R$*.*

Proof. In order to prove (2), fix $q \in \mathcal{A}_{n+1}$. By Lemma 5.4, $\delta_{n+1}(1 \otimes q \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}_n^{\preceq}(q) \rangle_{\mathbb{Z}}$ and using Lemma 5.7, $(d_n - \delta_n)(\delta_{n+1}(1 \otimes q \otimes 1))$ belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\prec}(q) \rangle_R$. Corollary 5.6 tells us that $\delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$ is in $\langle \overline{\mathcal{L}}_{n-1}^{\prec}(q) \rangle_{\mathbb{Z}}$. We deduce from the equality

$$d_n(\delta_{n+1}(1 \otimes q \otimes 1)) = \delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1) + (d_n - \delta_n)(\delta_{n+1}(1 \otimes q \otimes 1))$$

that $d_n(\delta_{n+1}(1 \otimes q \otimes 1))$ belongs to $\langle \overline{\mathcal{L}}_{n-1}^{\prec}(q) \rangle_R$.

Let us define $\tilde{d}_{n+1} : A \times k\mathcal{A}_{n+1} \times A \to A \otimes_E k\mathcal{A}_n \otimes_E A$ by

$$\tilde{d}_{n+1}(a, q, c) = a\delta_{n+1}(1 \otimes q \otimes 1)c - a\rho_n(d_n(\delta_{n+1}(1 \otimes q \otimes 1)))c,$$

for $a, c \in A$, $q \in \mathcal{A}_{n+1}$. The map $\tilde{d}_{n+1}$ is E-multilinear and balanced, and it induces a unique map

$$d_{n+1} : A \otimes_E k\mathcal{A}_{n+1} \otimes_E A \to A \otimes_E k\mathcal{A}_n \otimes_E A.$$

It is easy to verify that d_{n+1} is in fact a morphism of A-bimodules.

Putting together the equality $\rho_n = s_n + (\rho_n - s_n)$ and Lemmas 5.4 and 5.7, we obtain that $(d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) = -\rho_n \circ d_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}_n^{\prec}(q) \rangle_R$. Moreover, given $x \in A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$, $x = d_n \circ \rho_n(x) + \rho_{n-1} \circ d_{n-1}(x)$, choosing $x = d_n(\delta_{n+1}(1 \otimes q \otimes 1))$ yields the equality

$$d_n \circ \delta_{n+1}(1 \otimes q \otimes 1) = d_n \circ \rho_n \circ d_n \circ \delta_{n+1}(1 \otimes q \otimes 1)$$

which proves that $d_n \circ d_{n+1} = 0$.

For the proof of (1), fix $q \in \mathcal{A}_n$ and $b \in \mathcal{B}$. Using Lemmas 5.4 and 5.7, we deduce that the element

$$\begin{aligned}&1 \otimes q \otimes \pi(b) - \rho_n \circ d_n(1 \otimes q \otimes \pi(b)) \\ &\quad = 1 \otimes q \otimes \pi(b) - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b)) - \rho_n \circ (d_n - \delta_n)(1 \otimes q \otimes \pi(b))\end{aligned}$$

differs from $1 \otimes q \otimes \pi(b) - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b))$ by elements in $\langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R$. We will write that

$$\begin{aligned}&(id - \rho_n \circ \delta_n + \rho_n \circ (d_n - \delta_n))(1 \otimes q \otimes \pi(b)) \\ &\quad \equiv id - \rho_n \circ \delta_n(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R.\end{aligned}$$

Also,

$$\begin{aligned}(id - \rho_n \circ \delta_n)(1 \otimes q \otimes \pi(b)) &\equiv (id - s_n \circ \delta_n)(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R \\ &\equiv \delta_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R \\ &\equiv d_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) \bmod \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R.\end{aligned}$$

We deduce from this that there exists a unique $\xi \in \langle \overline{\mathcal{L}}_n^{\prec}(qb)\rangle_R$ such that

$$(id - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) + \xi.$$

It is evident that ξ belongs to the kernel of d_n.

The order $\preceq$ satisfies the descending chain condition, so we can use induction on $(k^\times Q_{\geq 0}, \preceq)$. If there is no $\lambda p \in k^\times Q_{\geq 0}$ such that $\lambda p \prec qb$, then $\xi = 0$ and we define $\rho_{n+1}(1 \otimes q \otimes \pi(b)) = s_{n+1}(1 \otimes q \otimes \pi(b))$. Inductively, suppose that $\rho_{n+1}(\xi)$ is defined. The equality $d_n(\xi) = 0$ implies that $\xi = d_{n+1} \circ \rho_{n+1}(\xi)$ and

$$(id - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1}(s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi)).$$

We define $\rho_{n+1}(1 \otimes q \otimes \pi(b)) := s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi)$.

Lemmas 5.4 and 5.7 assure that $\rho_{n+1}(\xi)$ belongs to $\langle \overline{\mathcal{L}}_{n+1}^{\prec}(qb)\rangle_R$, and as a consequence

$$\rho_{n+1}(1 \otimes q \otimes \pi(b)) - s_{n+1}(1 \otimes q \otimes \pi(b)) \in \langle \overline{\mathcal{L}}_{n+1}^{\prec}(qb)\rangle_R. \qquad \square$$

We are now ready to prove the theorems.

Proof of Theorem 4.1. We will prove the existence of an $A-E$-bimodule map $\rho_0 : A \otimes_E k\mathcal{A}_{-1} \otimes_E A \to A \otimes_E k\mathcal{A}_0 \otimes_E A$ satisfying $d_0 \circ \rho_0 + \rho_{-1} \circ d_{-1} = id$, where $d_{-1} = \mu$ and $\rho_{-1}(a) = s_{-1}(a) = a \otimes 1$ for all $a \in A$. Once this achieved, we apply Proposition 5.8 inductively with $R = k$, for all n such that $0 \leq n \leq N-1$, obtaining this way a homotopy retraction of the complex

$$A \otimes_E k\mathcal{A}_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0$$

proving thus that it is exact.

Given $b = b_k \cdots b_1 \in \mathcal{B}$, with $b_i \in Q_1$, $1 \leq i \leq k$,

$$s_0(1 \otimes \pi(b)) = -\sum_i \pi(b_k \cdots b_{k-i+1}) \otimes b_{k-i} \otimes \pi(b_{k-i-1} \cdots b_1).$$

On one hand $1 \otimes \pi(b) - \pi(b) \otimes 1 = 1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b)))$ and on the other hand the left hand term equals $\delta_0(s_0(1 \otimes \pi(b)))$, yielding $1 \otimes \pi(b) - s_{-1}(1 \otimes \pi(b)) = \delta_0(s_0(1 \otimes \pi(b))$. By hypothesis, $(d_0 - \delta_0)(1 \otimes \pi(b))$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{-1}(b) \rangle_k$, and so there exists $\xi \in \langle \overline{\mathcal{L}}^{\prec}_{-1}(b) \rangle_k$ such that

$$1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b))) = d_0(s_0(1 \otimes \pi(b))) + \xi.$$

It follows that $d_{-1}(\xi) = 0$. Suppose first that there exists no $\lambda p \in k^{\times} Q_{\geq 0}$ such that $\lambda p \prec b$.

In this case $\xi = 0$ and we define $\rho_0(1 \otimes \pi(b)) = s_0(1 \otimes \pi(b))$. Inductively, suppose that $\rho_0(\xi)$ is defined for any ξ such that $d_{-1}(\xi) = 0$. Since in this case $\xi = d_0(\rho_0(\xi))$, we set $\rho_0(1 \otimes \pi(b)) := s_0(1 \otimes \pi(b)) + \rho_0(\xi)$. □

Proof of Theorem 4.2. It follows from the proof of Theorem 4.1 that

$$1 \otimes \pi(b) = (s_{-1} \circ d_{-1} + \delta_0 \circ s_0)(1 \otimes \pi(b))$$

and so $s_{-1} \circ d_{-1} + \delta_0 \circ s_0 = id_{A \otimes_E A}$. Setting $d_0 := \delta_0$, the theorem follows applying Proposition 5.8 for $R = \mathbb{Z}$. □

We end this section by showing that this construction is a generalization of Bardzell's resolution for monomial algebras.

Proposition 5.9. *Given an algebra A, let $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ be a resolution of A as A-bimodule such that $d_\bullet$ satisfies the hypotheses of Theorem 4.1. If $p \in \mathcal{A}_n$ is such that $r(p) = 0$ or $r(p) = p$ for every reduction r, then for all $a, c \in kQ$,*

$$d_n(\pi(a) \otimes p \otimes \pi(c)) = \delta_n(\pi(a) \otimes p \otimes \pi(c)).$$

Proof. By hypothesis, we know that there exists no $\lambda' p' \in k^{\times} Q_{\geq 0}$ such that $\lambda' p' \prec p$, so $\mathcal{L}^{\prec}_{n-1}(p) = \{0\}$ and $d_n(1 \otimes p \otimes 1) = \delta_n(1 \otimes p \otimes 1)$. Given $a, c \in kQ$ we deduce from the previous equality that

$$\begin{aligned} &d_n(\pi(a) \otimes p \otimes \pi(c)) - \delta_n(\pi(a) \otimes p \otimes \pi(c)) \\ &\quad = \pi(a)(d_n(1 \otimes p \otimes 1) - \delta_n(1 \otimes p \otimes 1))\pi(c) = 0. \end{aligned}$$ □

Corollary 5.10. *Suppose the algebra $A = kQ/I$ has a monomial presentation. Choose a reduction system $\mathcal{R}$ whose pairs have the monomial relations generating the ideal I as first coordinate and 0 as second coordinate. In this case, the only maps d verifying the hypotheses of Theorem 4.2 are those of Bardzell's resolution.*

6. Morphisms in low degrees

In this section we describe the morphisms appearing in lower degrees of the resolution.

Let us consider the following data: an algebra $A = kQ/I$ and a reduction system $\mathcal{R}$ satisfying condition $\Diamond$.

We start by recalling the definition of δ_0 and δ_{-1}. For $a, c \in kQ$, $\alpha \in Q_1$,

$$\begin{aligned} &\delta_{-1} : A \otimes_E A \to A, && \delta_0 : A \otimes_E k\mathcal{A}_0 \otimes_E A \to A \otimes_E A, \\ &\delta_{-1}(\pi(a) \otimes \pi(c)) = \pi(ac), && \delta_0(\pi(a) \otimes \alpha \otimes \pi(c)) = \pi(a\alpha) \otimes \pi(c) - \pi(a) \otimes \pi(\alpha c). \end{aligned}$$

Definition 6.1. We state some definitions.

- Let $\phi_0 : kQ \to A \otimes_E k\mathcal{A}_0 \otimes_E A$ be the unique k-linear map such that

$$\phi_0(c) = \sum_{i=1}^{n} \pi(c_n \cdots c_{i+1}) \otimes c_i \otimes \pi(c_{i-1} \cdots c_1)$$

  for $c \in Q_{\geq 0}$, $c = c_n \cdots c_1$ with $c_i \in Q_1$ for all i, $1 \leq i \leq n$.
- Given a basic reduction $r = r_{a,s,c}$, let $\phi_1(r, -) : kQ \to A \otimes_E k\mathcal{A}_1 \otimes_E A$ be the unique k-linear map such that, given $p \in Q_{\geq 0}$

$$\phi_1(r, p) = \begin{cases} \pi(a) \otimes s \otimes \pi(c), & \text{if } p = asc, \\ 0 & \text{if not.} \end{cases} \tag{6.1}$$

  In case $r = (r_n, \ldots, r_1)$ is a reduction, where r_i is a basic reduction for all i, $1 \leq i \leq n$, we denote $r' = (r_n, \ldots, r_2)$ and we define in a recursive way the map $\phi_1(r, -)$ as the unique k-linear map from kQ to $A \otimes_E k\mathcal{A}_1 \otimes_E A$ such that

$$\phi_1(r, p) = \phi_1(r_1, p) + \phi_1(r', r_1(p)).$$

- Finally, we define an A-bimodule morphism $d_1 : A \otimes_E k\mathcal{A}_1 \otimes_E A \to A \otimes_E k\mathcal{A}_0 \otimes_E A$ by the equality

$$d_1(1 \otimes s \otimes 1) = \phi_0(s) - \phi_0(\beta(s)), \text{ for all } s \in \mathcal{A}_1.$$

Next we prove four lemmas necessary to the description of the complex in low degrees.

Lemma 6.2. *Let us consider $p \in Q_{\geq 0}$ and $x \in kQ$ such that $x \prec p$. For any reduction r the element $\phi_1(r, x)$ belongs to $\langle \overline{\mathcal{L}}_1^{\prec}(p) \rangle_{\mathbb{Z}}$.*

Proof. We will first prove the result for $x = \mu q \in k^\times Q_{\geq 0}$. The general case will then follow by linearity. Fix $x = \mu q \in k^\times Q_{\geq 0}$. We will use an inductive argument on $(k^\times Q_{\geq 0}, \preceq)$.

To start the induction, suppose first that there exists no $\mu' q' \in k^\times Q_{\geq 0}$ and that $\mu' q' \prec \mu q = x$. In this case, every basic reduction $r_{a,s,c}$ satisfies either $r_{a,s,c}(x) = x$ or $r_{a,s,c} = 0$. In the first case, $asc \neq q$ and so $\phi_1(r_{a,s,c}, x) = 0$. In the second case, $asc = q$, so $\phi_1(r_{a,s,c}, x) = \mu\pi(a) \otimes s \otimes \pi(c)$.

Given an arbitrary reduction $r = (r_n, \ldots, r_1)$ with r_i basic for all i, there are three possible cases.

(1) $r_1(x) = x$ and $n > 1$,
(2) $r_1(x) = x$ and $n = 1$,
(3) $r_1(x) = 0$.

Denote $r' = (r_n, \ldots, r_2)$ as before and $r_1 = r_{a,s,c}$. In case 1), $\phi_1(r, x) = \phi_1(r', x)$. In case 3), $\phi_1(r, x) = \phi_1(r_1, x) = 0$. Finally, in case 2), $\phi_1(r, x) = \phi_1(r_1, x) = \mu\pi(a) \otimes s \otimes \pi(c)$. Using Lemma 5.2, we obtain that in all three cases $\phi_1(r, x) \in \langle \overline{\mathcal{L}}_1^{\prec}(p)\rangle_{\mathbb{Z}}$.

Next, suppose that $x = \mu q$ and that the result holds for $\mu' q' \in k^\times Q_{\geq 0}$ such that $\mu' q' \prec \mu q = x$. Let us consider r, r_1 and r' as before. Again, there are three possible cases:

(1) $asc = q$,
(2) $asc \neq q$ and $n > 1$,
(3) $asc \neq q$ and $n = 1$.

Case 3) is immediate, since in this situation $\phi_1(r, x) = 0$. The second case reduces to the other ones, since $\phi_1(r, x) = \phi_1(r', x)$. In the first case,

$$\phi_1(r, x) = \mu\pi(a) \otimes s \otimes \pi(c) + \phi_1(r', r_1(x)).$$

We know that $r_1(x) \prec x$, and we may write it as a finite sum $r_1(x) = \sum_i \mu_i q_i$. Using the inductive hypothesis, we deduce that $\phi_1(r, x) \in \langle \overline{\mathcal{L}}_1^{\prec}(p)\rangle_{\mathbb{Z}}$. □

Lemma 6.3. *For all $x \in A \otimes_E k\mathcal{A}_1 \otimes_E A$, x belongs to the kernel of $\delta_0 \circ d_1(x)$.*

Proof. Let x be an element of $A \otimes_E k\mathcal{A}_1 \otimes_E A$. Since these maps are morphisms of A-bimodules, we may suppose $x = 1 \otimes s \otimes 1$, with $s \in \mathcal{A}_1$. A direct computation gives

$$\begin{aligned}
\delta_0(d_1(1 \otimes s \otimes 1)) &= \delta_0(\phi_0(s) - \phi_0(\beta(s))) \\
&= \pi(s) \otimes 1 - 1 \otimes \pi(s) - \pi(\beta(s)) \otimes 1 + 1 \otimes \pi(\beta(s)) \\
&= 0. \qquad \square
\end{aligned}$$

Lemma 6.4. *Given $a, c \in Q_{\geq 0}$ and $p = \sum_{i=1}^{n} \lambda_i p_i \in kQ$, with $p_i \in Q_{\geq 0}$ for all i, we obtain the equality*

$$\phi_0(apc) = \phi_0(a)\pi(pc) + \pi(a)\phi_0(p)\pi(c) + \pi(ap)\phi_0(c).$$

The proof is immediate using the definition of ϕ_0 and k-linearity of ϕ_0 and π.

Next we prove the last of the preparatory lemmas.

Lemma 6.5. *Given $p \in Q_{\geq 0}$ and a reduction $r = (r_n, \ldots, r_1)$, with r_i a basic reduction for all i such that $1 \leq i \leq n$, there is an equality*

$$d_1(\phi_1(r_1, p)) = \phi_0(p) - \phi_0(r(p)).$$

Proof. We will prove the result by induction on n. We will denote $r_i = r_{a_i,s_i,c_i}$.

For $n = 1$, there are two cases. The first one is when $p \neq a_1 s_1 c_1$. In this situation, $r(p) = r_1(p) = p$, $\phi_1(r_1, p) = 0$ and so the equality is trivially true. In the second case, $p = a_1 s_1 c_1$, $\phi_1(r_1, p) = \pi(a_1) \otimes s_1 \otimes \pi(c_1)$ and $r(p) = r_1(p) = a_1 \beta(s_1) c_1$. Moreover,

$$\begin{aligned} d_1(\phi_1(r_1, p)) + \phi_0(r_1(p)) &= d_1(\pi(a_1) \otimes s_1 \otimes \pi(c_1)) + \phi_0(a_1 \beta(s_1) c_1) \\ &= \pi(a_1)\phi_0(s_1)\pi(c_1) - \pi(a_1)\phi_0(\beta(s_1))\pi(c_1) + \phi_0(a_1\beta(s_1)c_1). \end{aligned}$$

Using Lemma 6.4, the last term equals

$$\phi_0(a_1)\pi(\beta(s_1)c_1) + \pi(a_1)\phi_0(\beta(s_1))\pi(c_1) + \pi(a_1\beta(s_1))\phi_0(c_1),$$

so the whole expression is

$$\begin{aligned} &\pi(a_1)\phi_0(s_1)\pi(c_1) + \phi_0(a_1)\pi(\beta(s_1)c_1) + \pi(a_1\beta(s_1))\phi_0(c_1) \\ &= \pi(a_1)\phi_0(s_1)\pi(c_1) + \phi_0(a_1)\pi(s_1c_1) + \pi(a_1s_1)\phi_0(c_1), \end{aligned}$$

and using again Lemma 6.4, this equals $\phi_0(p)$.

Suppose the result holds for $n - 1$. As usual, we denote $r' = (r_n, \ldots, r_2)$.

Since $r(p) = r'(r_1(p))$,

$$\begin{aligned} d_1(\phi_1(r, p)) + \phi_0(r(p)) &= d_1(\phi_1(r_1, p)) + d_1(\phi_1(r', r_1(p))) + \phi_0(r'(r_1(p))) \\ &= d_1(\phi_1(r_1, p)) + \phi_0(r_1(p)) \\ &= \phi_0(p). \qquad \square \end{aligned}$$

Consider now an element $p \in \mathcal{A}_2$. By definition we write $p = u_0u_1u_2 = v_2v_1v_0$ where u_0u_1 and v_1v_0 are paths in $\mathcal{A}_1$ dividing p. Suppose $r = r_{a,s,c}$ is a basic reduction such that $r(p) \neq p$. We deduce that either $s = u_0u_1$ or $s = v_1v_0$. For an arbitrary reduction $r = (r_n, \ldots, r_1)$, we will say that r *starts on the left of* p if $r_1 = r_{a,s,c}$, $s = u_0u_1$ and

$asc = p$, and we will say that r *starts on the right of* p if $r_1 = r_{a,s,c}$, $s = v_1v_0$ and $asc = p$.

Proposition 6.6. *Let $\{r^p\}_{p\in\mathcal{A}_2}$ and $\{t^p\}_{p\in\mathcal{A}_2}$ be two sets of reductions such that $r^p(p)$ and $t^p(p)$ belong to $k\mathcal{B}$, r^p starts on the left of p and t^p starts on the right of p. Consider $d_2 : A\otimes_E k\mathcal{A}_2\otimes_E A \to A\otimes_E k\mathcal{A}_1\otimes_E A$ the map of A-bimodules defined by $d_2(1\otimes p\otimes 1) = \phi_1(t^p, p) - \phi_1(r^p, p)$.*

The sequence

$$A\otimes_E k\mathcal{A}_2\otimes_E A \xrightarrow{d_2} A\otimes_E k\mathcal{A}_1\otimes_E A \xrightarrow{d_1} A\otimes_E k\mathcal{A}_0\otimes_E A \xrightarrow{\delta_0} A\otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

is exact.

Proof. To check that d_2 is well defined, consider the map $\tilde{d}_2 : A\times k\mathcal{A}_2\times A \to A\otimes_E k\mathcal{A}_1\otimes_E A$ defined by $\tilde{d}_2(x, p, y) = x\phi_1(t^p, p)y - x\phi_1(r^p, p)y$, for all $x, y \in A$, which is clearly multilinear; taking into account the definition of ϕ_1, it is such that $\tilde{d}_2(xe, p, y) = \tilde{d}_2(x, ep, y)$ and $\tilde{d}_2(x, pe, y) = \tilde{d}_2(x, p, ey)$ for all $e \in E$, so it induces d_2 on $A\otimes_E k\mathcal{A}_2\otimes_E A$.

The sequence is a complex:

- $\delta_{-1}\circ\delta_0 = 0$ and $\delta_0\circ d_1 = 0$ follow from Lemma 6.3.
- Given $p \in \mathcal{A}_2$, $d_1(d_2(1\otimes p\otimes 1)) = d_1(\phi_1(t^p, p) - \phi_1(r^p, p))$. Using Lemma 6.5, this last expression equals $\phi_0(p) - \phi_0(t^p(p)) - \phi_0(p) + \phi_0(r^p(p))$, which is, by Remark 2.5.1, equal to $-\phi_0(\beta(p)) + \phi_0(\beta(p))$, so $d_1\circ d_2 = 0$.

It is exact:

- We already know that this is true at A and at $A\otimes_E A$.
- Given $s \in \mathcal{A}_1$, $d_1(1\otimes s\otimes 1) - \delta_1(1\otimes s\otimes 1)$ belongs to $\langle\overline{\mathcal{L}}_0^{\prec}(s)\rangle_k$: indeed, notice that $\delta_1(1\otimes s\otimes 1) = \phi_0(s)$, and $\phi_0(\beta(s))$ belongs to $\langle\overline{\mathcal{L}}_0^{\prec}(s)\rangle_k$ since $\beta(s) \prec s$. It follows that
$$d_1(1\otimes s\otimes 1) - \delta_1(1\otimes s\otimes 1) = -\phi_0(\beta(s)) \in \langle\overline{\mathcal{L}}_0^{\prec}(s)\rangle_k.$$
- Given $p \in \mathcal{A}_2$, we will now prove that $(d_2 - \delta_2)(1\otimes p\otimes 1)$ belongs to $\langle\overline{\mathcal{L}}_1^{\prec}(p)\rangle_k$. We may write $p = u_0u_1u_2 = v_2v_1v_0$, as we did just before this proposition and thus $\delta_2(1\otimes p\otimes 1) = \pi(v_2)\otimes v_1v_0\otimes 1 - 1\otimes u_0u_1\otimes\pi(u_2)$. Besides, if $r^p = (r_n, \dots, r_1)$ and $t^p = (t_m, \dots, t_1)$ with t_i and r_j basic reductions, the fact that r^p starts on the left and t^p starts on the right of p gives
$$(d_2 - \delta_2)(1\otimes p\otimes 1) = \phi_1(t'^p, t_1(p)) - \phi_1(r'^p, r_1(p)),$$
where $t'^p = (t_m, \dots, t_2)$ and $r'^p = (r_n, \dots, r_2)$. Since $t_1(p) \prec p$ and $r_1(p) \prec p$, Lemma 6.2 allows us to deduce the result.

Finally, Theorem 4.1 implies that the sequence considered is exact. □

Remark 6.6.1. Given $a \in \mathcal{A}_0 = Q_1$, we have that $\overline{\mathcal{L}}^{\prec}_{-1}(a) = \emptyset$, so for any morphism of A-bimodules $d : A \otimes_E k\mathcal{A}_0 \otimes_E A \to A \otimes_E k\mathcal{A}_{-1} \otimes_E A$ such that $(d - \delta_0)(1 \otimes a \otimes 1)$ belongs to $\langle \overline{\mathcal{L}}^{\prec}_{-1}(a)\rangle_k$, it must be $d = \delta_0$.

On the other hand, given $s \in \mathcal{A}_1$, write $\beta(s) = \sum_{i=1}^m \lambda_i b_i$. Let $r = r_{a,s',c}$ be a basic reduction such that $r(s) \neq s$. We must have $s' = s$ and $a, c \in Q_0$ must coincide with the source and target of s, respectively. In other words, the only basic reduction such that $r(s) \neq s$ is $r_{a,s,c}$ with a and c as we just said, and in this case $r(s) = \beta(s) \in k\mathcal{B}$.

In this situation

$$\{\lambda q \in k^\times Q_{\geq 0} : \lambda q \prec s\} = \{\lambda_1 b_1, \dots, \lambda_m b_m\},$$

and writing $b_i = b_i^{n_i} \cdots b_i^1$ with $b_i^j \in Q_1$,

$$\overline{\mathcal{L}}^{\prec}_0(s) = \bigcup_{i=1}^{N} \{\lambda_i \pi(b_i^{n_i} \cdots b_i^2) \otimes b_i^1 \otimes 1, \dots, \lambda_i \otimes b_i^{n_i} \otimes \pi(b_i^{n_i-1} \cdots b_i^1)\}.$$

If $d : A \otimes_E k\mathcal{A}_1 \otimes_E A \to A \otimes_E k\mathcal{A}_0 \otimes_E A$ verifies $(d - \delta_1)(1 \otimes s \otimes 1) \in \overline{\mathcal{L}}^{\prec}_0(s)$ and $\delta_0 \circ d(s) = 0$ for all $s \in \mathcal{A}_1$, then there exists $\gamma_i^j \in k$ such that

$$d(1 \otimes s \otimes 1) = \phi_0(s) - \sum_{i=1}^m \sum_{j=1}^{n_i} \gamma_i^j \lambda_i \pi(b_i^{n_i} \cdots b_i^{j+1}) \otimes b_i^j \otimes \pi(b_i^{j-1} \cdots b_i^1).$$

From this, applying δ_0 and reordering terms we can deduce that $\gamma_i^j = 1$ for all i, j. We conclude that the unique morphism with the desired properties is d_1.

7. Examples

In this section we construct explicitly projective bimodule resolutions of some algebras using the methods we developed in previous sections.

Given an algebra $A = kQ/I$, we proved in Lemmas 2.9 and 2.10 that it is always possible to construct a reduction system $\mathcal{R}$ such that every path is reduction-unique. However, it is not always easy to follow the prescriptions given by these lemmas for a concrete algebra. Moreover, the reduction system obtained from a *deglex* order $\leq_\omega$ may be sometimes less convenient than other ones. In fact, describing the set $\mathsf{tip}(I)$ is not in general an easy task.

Bergman's Diamond Lemma is the tool we use to effectively compute a reduction system in most cases. Next we sketch this procedure, which is also described in [3, Section 5].

The two sided ideal I is usually presented by giving a set $\{x_i\}_{i\in\Gamma} \subseteq kQ$ of generating relations. If we fix a well-order on $Q_0 \cup Q_1$, a function $\omega : Q_1 \to \mathbb{N}$ and consider the total order $\leq_\omega$ on $Q_{\geq 0}$, we can easily write $x_i = s_i - f_i$, and we can eventually rescale x_i so that s_i is monic, with $s_i >_\omega f_i$ for all i and define the reduction system $\mathcal{R} = \{(s_i, f_i)\}_{i\in\Gamma}$.

Every path p will be reduction-finite with respect to $\mathcal{R}$. Bergman's Diamond Lemma says that every path is reduction-unique if and only if for every path $p \in \mathcal{A}_2$ there are reductions r, t with r starting on the left and t starting on the right of p such that $r(p) = t(p)$. This last situation is described by saying that p is *resolvable*. The set $\mathcal{A}_2$ is usually finite and so there is a finite number of conditions to check.

In case there exists a non-resolvable ambiguity $p \in \mathcal{A}_2$, choose any two reductions r, t starting on the left and on the right respectively with $r(p)$ and $t(p)$ both irreducible. The element $r(p) - t(p)$ belongs to $I \setminus \{0\}$. We can write $r(p) - t(p) = s - f$ with $f <_\omega s$ and add the element (s, f) to our reduction system, and so p is now resolvable. New ambiguities may now appear, so it is necessary to iterate this process, which may have infinitely many steps, but we will arrive to a reduction system $\mathcal{R}$ satisfying condition $(\Diamond)$.

Next we give an example to illustrate this procedure, which will be also useful to exhibit a case where another reduction system found in an alternative way is better than the prescribed one.

Example 7.0.1. Consider the algebra of Example 2.10.1. Let $x < y < z$ and $\omega(x) = \omega(y) = \omega(z) = 1$. The ideal I is presented as the two sided ideal generated by the element $x^3 + y^3 + z^3 - xyz$. We see that $z^3 = \mathsf{tip}(z^3 - (xyz - x^3 - y^3))$, so we start considering the reduction system $\mathcal{R} = \{(z^3, xyz - x^3 - y^3)\}$. Notice that $\mathcal{A}_2 = \{z^4\}$. If we apply the reduction $r_{z,z^3,1}$ to z^4 we obtain $zxyz - zx^3 - zy^3$ which is irreducible. On the other hand, if we apply the reduction $r_{1,z^3,z}$ to z^4 we obtain $xyz^2 - x^3z - y^3z$ which is also irreducible and different from the first one. The difference between them is $xyz^2 - x^3z - y^3z - zxyz + zx^3 + zy^3$, so we add $(xyz^2, x^3z + y^3z + zxyz - zx^3 - zy^3)$ to the reduction system $\mathcal{R}$. Notice that now the set $\mathcal{A}_2$ is $\{z^4, xyz^3\}$. Applying reductions on the left and on the right to the element xyz^3 we obtain again two different irreducible elements and, proceeding as before, we see that we have to add the element $(y^3z^2, -x^3z^2 - z^2xyz + z^2x^3 + z^2y^3 + xyxyz - xyx^3 - xy^4)$ to our reduction system $\mathcal{R}$. We obtain the new ambiguity y^3z^3 which is not difficult to see that it is resolvable. Thus, the reduction system

$$\mathcal{R}_1 = \{(z^3, xyz - x^3 - y^3), (xyz^2, x^3z + y^3z + zxyz - zx^3 - zy^3),$$
$$(y^3z^2, -x^3z^2 - z^2xyz + z^2x^3 + z^2y^3 + xyxyz - xyx^3 - xy^4)\},$$

satisfies condition $(\Diamond)$.

There is another reduction system for this algebra, namely $\mathcal{R}_2 = \{(xyz, x^3 + y^3 + z^3)\}$. Let us denote $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ the respective set of n-ambiguities. Notice that $z^{\frac{3}{2}(n+1)} \in \mathcal{A}_n^1$ for n odd and $z^{\frac{3}{2}n+1} \in \mathcal{A}_n^1$ for n even, so $\mathcal{A}_n^1$ is not empty for all $n \in \mathbb{N}$. On the other hand, $\mathcal{A}_n^2$ is empty for all $n \geq 2$. We conclude that using $\mathcal{R}_2$ we will obtain a resolution of length 2, with differentials given explicitly by Proposition 6.6, and using $\mathcal{R}_1$ the resolution obtained will have infinite length. This shows how different can the resolutions from different reduction systems be.

Notice that $\mathcal{R}_2$ cannot be obtained by the procedure described above by any choice of order on $Q_0 \cup Q_1$ and weight ω. The algebra $A = k < x, y, z > /(xyz - x^3 - y^3 - z^3)$ is in fact a 3-Koszul algebra. Indeed, denoting by V the k-vector space spanned by x, y, z and by R the one dimensional k-vector space spanned by the relation $xyz - x^3 - y^3 - z^3$, it is straightforward that

$$R \otimes V \otimes V \cap V \otimes V \otimes R = \{0\},$$

and so the intersection is a subset of $V \otimes R \otimes V$. Theorem 2.5 of [5] guarantees that A is 3-Koszul.

The resolution we obtain from the reduction system $\mathcal{R}_2$ is the isomorphic to the Koszul resolution, since it is minimal, see Theorem 8.1. As we shall see, this is a particular case of a general situation.

7.1. *The algebra counterexample to Happel's question*

Let ξ be an element of the field k and let A be the k-algebra with generators x and y, subject to the relations $x^2 = 0 = y^2$, $yx = \xi xy$. Choose the order $x < y$ with weights $\omega(x) = \omega(y) = 1$ and fix the reduction system $\mathcal{R} = \{(x^2, 0), (y^2, 0), (yx, \xi xy)\}$. The set $\mathcal{B}$ of irreducible paths is thus $\{1, x, y, xy\}$. It is easy to verify that $\mathcal{A}_2 = \{x^3, yx^2, y^2x, y^3\}$ and that all paths in $\mathcal{A}_2$ are reduction-unique. Bergman's Diamond Lemma guarantees that $\mathcal{R}$ satisfies $(\Diamond)$.

The only path of length 2 not in S is xy; Proposition 3.4 implies that for each n, $\mathcal{A}_n$ is the set of paths of length $n + 1$ not divisible by xy,

$$\mathcal{A}_n = \{y^s x^t : s + t = n + 1\}.$$

Lemma 7.1. *The following complex provides the beginning of an A-bimodule projective resolution of the algebra A*

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \to 0$$

where d_1 is the A-bimodule map such that

$$d_1(1 \otimes x^2 \otimes 1) = x \otimes x \otimes 1 + 1 \otimes x \otimes x,$$
$$d_1(1 \otimes y^2 \otimes 1) = y \otimes y \otimes 1 + 1 \otimes y \otimes y,$$
$$d_1(1 \otimes yx \otimes 1) = y \otimes x \otimes 1 + 1 \otimes y \otimes x - \xi x \otimes y \otimes 1 - \xi \otimes x \otimes y$$

and d_2 is the A-bimodule morphism such that

$$d_2(1 \otimes y^3 \otimes 1) = y \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes y,$$
$$d_2(1 \otimes y^2x \otimes 1) = y \otimes yx \otimes 1 + \xi \otimes yx \otimes y + \xi^2 x \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes x,$$

$$d_2(1 \otimes yx^2 \otimes 1) = y \otimes x^2 \otimes 1 - 1 \otimes yx \otimes x - \xi x \otimes yx \otimes 1 - \xi^2 \otimes x^2 \otimes y$$
$$d_2(1 \otimes x^3 \otimes 1) = x \otimes x^2 \otimes 1 - 1 \otimes x^2 \otimes x.$$

Proof. We apply Proposition 6.6 to the following sets $\{r^p\}_{p\in\mathcal{A}_2}$ of left reductions and $\{t^p\}_{p\in\mathcal{A}_2}$ of right reductions, where

$$\begin{aligned} r^{y^3} &= r_{1,y^2,y}, & r^{y^2x} &= r_{1,y^2,x}, \\ r^{yx^2} &= (r_{1,x^2,y}, r_{x,yx,1}, r_{1,yx,x}), & r^{x^3} &= r_{1,x^2,x}, \\ t^{y^3} &= r_{y,y^2,1}, & t^{y^2x} &= (r_{x,y^2,1}, r_{1,yx,y}, r_{y,yx,1}), \\ t^{yx^2} &= r_{y,x^2,1}, & t^{x^3} &= r_{x,x^2,1}. \quad \square \end{aligned}$$

One can find an A-bimodule resolution of A in [9] and in [7]; the authors also compute the Hochschild cohomology of A therein. We recover this resolution with our method.

Given $q \in \mathcal{A}_n$, there are $s, t \in \mathbb{N}$ such that $s+t = n+1$ and $q = y^s x^t$. Suppose $q = apc$ with $p = y^{s'} x^{t'} \in \mathcal{A}_{n-1}$ and $a, c \in Q_{\geq 0}$. Since $s + t = n + 1$ and $s' + t' = n$, either a belongs to Q_0 and $c = x$ or $a = y$ and $c \in Q_0$. As a consequence of this fact, the maps

$\delta_n : kQ \otimes_E k\mathcal{A}_n \otimes_E kQ \to kQ \otimes_E k\mathcal{A}_{n-1} \otimes_E A$ are

$$\delta_n(1 \otimes y^s x^t \otimes 1) = \begin{cases} y \otimes y^{s-1}x^t \otimes 1 + (-1)^{n+1} \otimes y^s x^{t-1} \otimes x, & \text{if } s \neq 0 \text{ and } t \neq 0, \\ y \otimes y^n \otimes 1 + (-1)^{n+1} \otimes y^n \otimes y, & \text{if } t = 0, \\ x \otimes x^n \otimes 1 + (-1)^{n+1} \otimes x^n \otimes x, & \text{if } s = 0, \end{cases} \tag{7.1}$$

Moreover, given a basic reduction $r = r_{a,s,c}$, the fact that s belongs to $S = \{x^2, y^2, yx\}$ implies that $r(y^s x^t)$ is either 0 or $\xi y^{s-1} xyx^{t-1}$. Considering the reduction system $\mathcal{R}$, if $s \neq 0$ and $t \neq 0$, then

$$\overline{\mathcal{L}}^{\prec}_{n-1}(y^s x^t) = \{\xi^s x \otimes y^s x^{t-1} \otimes 1, \xi^t \otimes y^{s-1} x^t \otimes y\}.$$

In case $s = 0$ or $t = 0$, the set $\overline{\mathcal{L}}^{\prec}_{n-1}(y^s x^t)$ is empty.

The computation of $d_2 - \delta_2$ suggests the definition of the maps

$$d_n : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$$

as follows

$$d_n(1 \otimes y^s x^t \otimes 1) = \delta_n(1 \otimes y^s x^t \otimes 1) + \epsilon(\xi^s x \otimes y^s x^{t-1} \otimes 1 + \xi^t \otimes y^{s-1} x^t \otimes y)$$

where ϵ denotes a sign depending on s, t, n. The equality $d_{n-1} \circ d_n = 0$ shows that making the choice $\epsilon = (-1)^s$ does the job.

Finally, Theorem 4.1 shows that the complex

$$\cdots \longrightarrow A \otimes_E k\mathcal{A}_n \otimes_E A \xrightarrow{d_n} \cdots \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

with

$$d_n(1 \otimes y^s x^t \otimes 1) = y \otimes y^{s-1}x^t \otimes 1 + (-1)^{n+1} 1 \otimes y^s x^{t-1} \otimes x \\ + (-1)^s \xi^s x \otimes y^s x^{t-1} \otimes 1 + (-1)^s \xi^t \otimes y^{s-1} x^t \otimes y,$$

for $s > 0$ and $t > 0$, and

$$d_n(1 \otimes y^{n+1} \otimes 1) = y \otimes y^n \otimes 1 + (-1)^{n+1} 1 \otimes y^n \otimes y,$$
$$d_n(1 \otimes x^{n+1} \otimes 1) = x \otimes x^n \otimes 1 + (-1)^{n+1} 1 \otimes x^n \otimes x,$$

is a projective bimodule resolution of A.

Again, the algebra A is Koszul, see for example [6] and the resolution obtained using our procedure is isomorphic to the Koszul resolution, which is the minimal one, see Theorem 8.1.

7.2. *Quantum complete intersections*

These algebras generalize the previous case. Instead of the relations $x^2 = 0 = y^2$, $yx = \xi xy$, we have $x^n = 0 = y^m$, $yx = \xi xy$, where n and m are fixed positive integers, $n, m > 1$.

We still denote the algebra by A. Consider the order $x < y$ with weights $\omega(x) = \omega(y) = 1$. The set of 2-ambiguities associated to the reduction system $\mathcal{R} = \{(x^n, 0), (y^m, 0), (yx, \xi xy)\}$ is $\mathcal{A}_2 = \{y^{m+1}, y^m x, yx^n, x^{n+1}\}$, and the set of irreducible paths is $\mathcal{B} = \{x^i y^j \in k\langle x, y\rangle : 0 \le i \le n-1, 0 \le j \le m-1\}$. We easily check that every path in $\mathcal{A}_2$ is reduction-unique and using Bergman's Diamond Lemma, we conclude that $\mathcal{R}$ satisfies $(\lozenge)$, Also, $\mathcal{A}_1 = S = \{y^m, yx, x^n\}$ and $\mathcal{A}_3 = \{y^{2m}, y^{m+1}x, y^m x^n, yx^{n+1}, x^{2n}\}$.

Denote by $\varphi : \mathbb{N}_0^2 \to \mathbb{N}_0$ the map

$$\varphi(s, n) = \begin{cases} \frac{s}{2} n & \text{if } s \text{ is even,} \\ \frac{s-1}{2} n + 1 & \text{if } s \text{ is odd.} \end{cases} \tag{7.2}$$

Given $N \in \mathbb{N}$, the set of N-ambiguities is $\mathcal{A}_N = \{y^{\varphi(s,m)} x^{\varphi(t,n)} : s + t = N + 1\}$. We will sometimes write (s, t) instead of $y^{\varphi(s,m)} x^{\varphi(t,n)} \in \mathcal{A}_N$.

We first compute the beginning of the resolution.

Lemma 7.2. *The following complex provides the beginning of a projective resolution of A as A-bimodule:*

$$A \otimes_E k\mathcal{A}_2 \otimes_E A \xrightarrow{d_2} A \otimes_E k\mathcal{A}_1 \otimes_E A \xrightarrow{d_1} A \otimes_E k\mathcal{A}_0 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where d_1 and d_2 are morphisms of A-bimodules given by the formulas

$$d_1(1 \otimes x^n \otimes 1) = \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-1-i},$$

$$d_1(1 \otimes y^m \otimes 1) = \sum_{i=0}^{m-1} y^i \otimes y \otimes y^{m-1-i},$$

$$d_1(1 \otimes yx \otimes 1) = 1 \otimes y \otimes x + y \otimes x \otimes 1 - \xi \otimes x \otimes y - \xi x \otimes y \otimes 1$$

$$d_2(1 \otimes y^{m+1} \otimes 1) = y \otimes y^m \otimes 1 - 1 \otimes y^m \otimes y,$$

$$d_2(1 \otimes y^m x \otimes 1) = \sum_{i=0}^{m-1} \xi^i y^{m-1-i} \otimes yx \otimes y^i + \xi^m x \otimes y^m \otimes 1 - 1 \otimes y^m \otimes x$$

$$d_2(1 \otimes yx^n \otimes 1) = y \otimes x^n \otimes 1 - \sum_{i=0}^{n-1} \xi^i x^i \otimes yx \otimes x^{n-1-i} - \xi^n \otimes x^n \otimes y,$$

$$d_2(1 \otimes x^{n+1} \otimes 1) = x \otimes x^n \otimes 1 - 1 \otimes x^n \otimes x.$$

Proof. It is straightforward, using Proposition 6.6 applied to the set $\{r^p\}_{p\in\mathcal{A}_2}$ of left reductions, where

$$r^{y^{m+1}} = r_{1,y^m,y}, \qquad r^{y^m x} = r_{1,y^m,x},$$
$$r^{yx^n} = (r_{1,x^n,y}, \dots, r_{x,yx,x^{n-2}}, r_{1,yx,x^{n-1}}) \qquad r^{x^{n+1}} = r_{1,x^n,x},$$

and the set $\{t^p\}_{p\in\mathcal{A}_2}$ of right reductions, where

$$t^{y^{m+1}} = r_{y,y^m,1}, \qquad t^{y^m x} = (r_{x,y^m,1}, \dots, r_{y^{m-2},yx,y}, r_{y^{m-1},yx,1}),$$
$$y^{yx^n} = r_{y,x^n,1}, \qquad t^{x^{n+1}} = r_{x,x^n,1}. \qquad \square$$

Of course we want to construct the rest of the resolution. Denote $(s,t) = y^{\varphi(s,m)}x^{\varphi(t,n)} \in \mathcal{A}_N$. We will first describe the set $\overline{\mathcal{L}}^{\prec}_{N-1}(s,t)$. There are four cases, depending on the parity of s, t and N. With this in view, it is useful to make some previous computations that we list below.

(1) For s even, for all j, $0 \le j \le m-1$, $y^{\varphi(s,m)} = y^{m-1-j}y^{\varphi(s-1,m)}y^j$.
(2) For s odd, $y^{\varphi(s,m)} = yy^{\varphi(s-1,m)} = y^{\varphi(s-1,m)}y$.

(3) For t even, for all i, $0 \leq i \leq n-1$, $x^{\varphi(t,n)} = x^i x^{\varphi(t-1,n)} x^{n-i-1}$.
(4) For t odd, $x^{\varphi(t,n)} = x x^{\varphi(t-1,n)} = x^{\varphi(t-1,n)} x$.

First case: N even, s even, t odd,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j\}_{j=1}^{m-1} \cup \{\xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1\}.$$

Second case: N even, s odd, t even,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y\} \cup \{\xi^{\varphi(s,m)i} x^i \otimes (x,t-1) \otimes x^{n-1-i}\}_{i=1}^{n-1}.$$

Third case: N odd, s even, t even,

$$\begin{aligned}\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{&\xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j\}_{j=1}^{m-1} \\ &\cup \{\xi^{\varphi(s,m)i} x^i \otimes (s,t-1) \otimes x^{n-1-i}\}_{i=1}^{n-1}.\end{aligned}$$

Fourth case: N, s and t odd,

$$\overline{\mathcal{L}}^{\prec}_{N-1}(s,t) = \{\xi^{\varphi(t,n)} 1 \otimes (s-1,t) \otimes y, \xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1\}.$$

Remark 7.2.1. We observe that, analogously to the case $n = m = 2$,

$$(d_1 - \delta_1)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_0(s,t)} u,$$

$$(d_2 - \delta_2)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_1(s,t)} u.$$

Proposition 5.8 for $R = \mathbb{Z}$ guarantees that there exist A-bimodule maps $d_N : A \otimes_E k\mathcal{A}_N \otimes_E A \to A \otimes_E k\mathcal{A}_{N-1} \otimes_E A$ such that $(d_N - \delta_N)(1 \otimes (s,t) \otimes 1) \in \langle \overline{\mathcal{L}}^{\prec}_{N-1}(s,t) \rangle_{\mathbb{Z}}$ and, most important, the complex $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ is a projective resolution of A as A-bimodule.

We are not yet able at this point to give the explicit formulas of the differentials.

In order to illustrate the situation, let us describe what happens for $N = 3$. We know after the mentioned proposition that there exist $t_1, t_2 \in \mathbb{Z}$ such that

$$\begin{aligned} d_3(1 \otimes y^{m+1} x \otimes 1) &= d_3(1 \otimes (3,1) \otimes 1) \\ &= \delta_3(1 \otimes (3,1) \otimes 1) + t_1 \xi \otimes (2,1) \otimes y + t_2 \xi^3 x \otimes (3,0) \otimes 1 \\ &= y \otimes y^m x \otimes 1 + 1 \otimes y^{m+1} \otimes x + t_1 \xi \otimes y^m x \otimes y + t_2 \xi^3 x \otimes y^{m+1} \otimes 1. \end{aligned}$$

Of course, $d_2 \circ d_3 = 0$. It follows from this equality that $t_1 = t_2 = -1$. This example motivates the following lemma, stated in terms of the preceding notations.

Lemma 7.3. *The A-bimodule morphisms* $d_N : A \otimes_E k\mathcal{A}_N \otimes_E A \to A \otimes_E k\mathcal{A}_{N-1} \otimes_E A$ *defined by the formula*

$$d_N(1 \otimes (s,t) \otimes 1) = \delta_N(1 \otimes (s,t) \otimes 1) + (-1)^s \sum_{u \in \overline{\mathcal{L}}^{\prec}_{N-1}(s,t)} u$$

satisfy the hypotheses of Theorem 4.1.

Proof. It is straightforward. □

We gather all the information we have obtained about the projective bimodule resolution of A in the following proposition.

Proposition 7.4. *The complex of A-bimodules* $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$, *with*

$$\mathcal{A}_N = \{y^{\varphi(s,m)} x^{\varphi(t,n)} : s + t = N + 1\}$$

and differentials defined as follows is exact.

(1) For N even, s even and t odd,

$$d_N(1 \otimes (s,t) \otimes 1) = y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^s \xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j + (-1)^{N+1} 1 \otimes (s,t-1) \otimes x + (-1)^s \xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1.$$

(2) For N even, s odd and t even,

$$d_N(1 \otimes (s,t) \otimes 1) = y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y + (-1)^{N+1} 1 \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{\varphi(s,m)i} x^i \otimes (s,t-1) \otimes x^{n-1-i}$$

(3) For N odd, s and t even,

$$d_N(1 \otimes (s,t) \otimes 1) = y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^s \xi^{\varphi(t,n)j} y^{m-1-j} \otimes (s-1,t) \otimes y^j + (-1)^{N+1} 1 \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{\varphi(s,m)i} x^i \otimes (s,t-1) \otimes x^{n-1-i}$$

(4) For N, s and t odd,

$$d_N(1 \otimes (s,t) \otimes 1) = y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{\varphi(t,n)} \otimes (s-1,t) \otimes y \\ + (-1)^{N+1} 1 \otimes (s,t-1) \otimes x + (-1)^s \xi^{\varphi(s,m)} x \otimes (s,t-1) \otimes 1.$$

Again, we obtain the minimal resolution of A, even for $n \neq 2$ or $m \neq 2$, when the algebra is not homogeneous.

7.3. Down–up algebras

Given $\alpha, \beta, \gamma \in k$, we will denote $A(\alpha,\beta,\gamma)$ the quotient of $k\langle d,u\rangle$ by the two sided ideal I generated by relations

$$d^2u - \alpha dud - \beta ud^2 - \gamma d = 0,$$
$$du^2 - \alpha udu - \beta u^2 d - \gamma u = 0.$$

Down–up algebras have been deeply studied since they were defined in [11]. We can mention the articles [15,13,8,16,14,21,22,24–28], in which the authors prove diverse properties of down–up algebras. It is well known that they are noetherian if and only if $\beta \neq 0$ [22]. They are graded with $\mathsf{dg}(d) = 1$, $\mathsf{dg}(u) = -1$, and they are filtered if we consider d and u of weight 1. If $\gamma = 0$ they are also graded by this weight.

Down–up algebras are 3-Koszul if $\gamma = 0$, and if $\gamma \neq 0$, they are PBW deformations of 3-Koszul algebras [8].

Little is known about their Hochschild homology and cohomology, except for the center, described in [31] and [24]. We apply our methods to construct a projective resolution of A as A-bimodule, and then use this resolution to compute $H^\bullet(A, A^e)$ and prove that in the noetherian case, $A(\alpha,\beta,\gamma)$ is 3-Calabi–Yau if and only if $\beta = -1$. Moreover, in this situation we exhibit a potential $\Phi(d,u)$ such that the relations are in fact the cyclic derivatives $\partial_u\Phi$ and $\partial_d\Phi$, respectively.

We briefly recall that a d-Calabi–Yau algebra is an associative algebra such that there is an isomorphism f of A-bimodules

$$Ext^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A & \text{if } i = d. \end{cases} \tag{7.3}$$

The A-bimodule outer structure of A^e is used when computing $Ext^i_{A^e}(A, A^e)$, while the isomorphism f takes account of the inner bimodule structure of A^e. Bocklandt proved in [10] that graded Calabi–Yau algebras come from a potential and Van den Bergh [30] generalized this result to complete algebras with respect to the I-adic topology.

We fix a lexicographical order such that $d < u$, with weights $\omega(d) = 1 = \omega(u)$. The reduction system $\mathcal{R} = \{(d^2u, \alpha dud + \beta ud^2 + \gamma d), (du^2, \alpha udu + \beta u^2 d + \gamma u)\}$ has $\mathcal{B} = \{u^i(du)^k d^j : i,k,j \in \mathbb{N}_0\}$ as set of irreducible paths and $\mathcal{A}_2 = \{d^2u^2\}$; using

Bergman's Diamond Lemma we see that $\mathcal{R}$ satisfies condition ($\diamondsuit$). Also, $\mathcal{A}_0 = \{d, u\}$ and $\mathcal{A}_n = \emptyset$ for all $n \geq 3$. The set $\mathcal{B}$ is the k-basis already considered in [11].

The reductions $r^{d^2u^2} = (r_{u,d^2u,1}, r_{1,d^2u,u})$ and $t^{d^2u^2} = (t_{1,du^2,d}, t_{d,du^2,1})$ are respectively left and right reductions of d^2u^2.

In view of Proposition 6.6 and observing that δ_{-1} is in fact an epimorphism and that $\mathcal{A}_3 = \emptyset$, the following complex gives a free resolution of A as A-bimodule:

$$0 \longrightarrow A \otimes_E kd^2u^2 \otimes_E A \xrightarrow{d_2} A \otimes_E (kd^2u \oplus kdu^2) \otimes_E A \xrightarrow{d_1} A \otimes_E (kd \oplus ku) \otimes_E A$$
$$\xrightarrow{\delta_0} A \otimes_E A \xrightarrow{\delta_{-1}} A \longrightarrow 0$$

where

$$\begin{aligned} d_1(1 \otimes d^2u \otimes 1) &= 1 \otimes d \otimes du + d \otimes d \otimes u + d^2 \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes d^2 + u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1, \\ d_1(1 \otimes du^2 \otimes 1) &= 1 \otimes d \otimes u^2 + d \otimes u \otimes u + du \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes ud + u \otimes u \otimes d + u^2 \otimes d \otimes 1) - \gamma \otimes u \otimes 1, \end{aligned}$$

and

$$d_2(1 \otimes d^2u^2 \otimes 1) = d \otimes du^2 \otimes 1 + \beta \otimes du^2 \otimes d - 1 \otimes d^2u \otimes u - \beta u \otimes d^2u \otimes 1.$$

As we have proved in general, the map d_2 takes into account the reductions applied to the ambiguity.

Proposition 7.5. *Suppose that $\beta \neq 0$. The algebra $A(\alpha, \beta, \gamma)$ is 3-Calabi–Yau if and only if $\beta = -1$.*

Proof. We need to compute $Ext^\bullet_{A^e}(A, A^e)$. We apply the functor $Hom_{A^e}(-, A^e)$ to the previous resolution, and we use that for any finite dimensional vector space V which is also an E-bimodule, the space $Hom_{A^e}(A \otimes_E V \otimes_E A, A^e)$ is isomorphic to $Hom_{E^e}(V, A^e)$, and this last one is, in turn, isomorphic to $A \otimes_E V^* \otimes_E A$. All the isomorphisms are natural. The explicit expression of the last isomorphism is, fixing a k-basis $\{v_1, \dots, v_n\}$ of V and its dual basis $\{\varphi_1, \dots, \varphi_n\}$ of V^*,

$$\begin{aligned} A \otimes_E V^* \otimes_E A &\to Hom_{E^e}(V, A^e) \\ a \otimes \varphi \otimes b &\mapsto [v \mapsto \varphi(v) b \otimes a] \end{aligned}$$

with inverse $f \mapsto \sum_{i,j} b^i_j \otimes \varphi_i \otimes a^i_j$, where $f(v_i) = \sum_j a^i_j \otimes b^i_j$.

After these identifications, we obtain the following complex of k-vector spaces whose homology is $Ext^{\bullet}_{A^e}(A, A^e)$

$$0 \longrightarrow A \otimes_E A \xrightarrow{\delta_0^*} A \otimes_E (kD \oplus kU) \otimes_E A \xrightarrow{d_1^*} A \otimes_E (kD^2U \oplus kDU^2) \otimes_E A$$
$$\xrightarrow{d_2^*} A \otimes_E kD^2U^2 \otimes_E A \longrightarrow 0,$$

where $\{D, U\}$ denotes the dual basis of $\{d, u\}$ and, accordingly, we denote with capital letters the dual bases of the other spaces.

The maps in the complex are, explicitly:

$$\begin{aligned}
\delta_0^*(1 \otimes 1) &= 1 \otimes D \otimes d - d \otimes D \otimes 1 + 1 \otimes U \otimes u - u \otimes U \otimes 1 \\
d_1^*(1 \otimes U \otimes 1) &= 1 \otimes D^2U \otimes d^2 - \alpha d \otimes D^2U \otimes d - \beta d^2 \otimes D^2U \otimes 1 + u \otimes DU^2 \otimes d \\
&\quad + 1 \otimes DU^2 \otimes du - \alpha du \otimes DU^2 \otimes 1 - \alpha \otimes DU^2 \otimes ud - \beta ud \otimes DU^2 \otimes 1 \\
&\quad - \beta d \otimes DU^2 \otimes u - \gamma \otimes DU^2 \otimes 1, \\
d_1^*(1 \otimes D \otimes 1) &= du \otimes D^2U \otimes 1 + u \otimes D^2U \otimes d - \alpha ud \otimes D^2U \otimes 1 - \alpha \otimes D^2U \otimes du \\
&\quad - \beta d \otimes D^2U \otimes u - \beta \otimes D^2U \otimes ud - \gamma \otimes D^2U \otimes 1 + u^2 \otimes DU^2 \otimes 1 \\
&\quad - \alpha u \otimes DU^2 \otimes u - \beta \otimes DU^2 \otimes u^2, \\
d_2^*(1 \otimes DU^2 \otimes 1) &= 1 \otimes D^2U^2 \otimes d + \beta d \otimes D^2U^2 \otimes 1, \\
d_2^*(1 \otimes D^2U \otimes 1) &= -u \otimes D^2U^2 \otimes 1 - \beta \otimes D^2U^2 \otimes u.
\end{aligned}$$

Consider the following isomorphisms of A-bimodules

$$\begin{aligned}
&\psi_0 : A \otimes_E A \to A \otimes_E kd^2u^2 \otimes_E A, \\
&\qquad \psi_0(1 \otimes 1) = 1 \otimes d^2u^2 \otimes 1, \\
&\psi_1 : A \otimes_E (kD \oplus kU) \otimes_E A \to A \otimes_E (kd^2u \oplus kdu^2) \otimes_E A \\
&\qquad \psi_1(1 \otimes D \otimes 1) = 1 \otimes du^2 \otimes 1, \text{ and } \psi_1(1 \otimes U \otimes 1) = 1 \otimes d^2u \otimes 1 \\
&\psi_2 : A \otimes_E (kD^2U \oplus kDU^2) \otimes_E A \to A \otimes_E (kd \oplus ku) \otimes_E A, \\
&\qquad \psi_2(1 \otimes D^2U \otimes 1) = 1 \otimes u \otimes 1, \text{ and } \psi_2(1 \otimes DU^2 \otimes 1) = 1 \otimes d \otimes 1 \\
&\psi_3 : A \otimes_E kD^2U^2 \otimes_E \to A \otimes_E A \\
&\qquad \psi_3(1 \otimes D^2U^2 \otimes 1) = 1 \otimes 1.
\end{aligned}$$

It is straightforward to verify that the following diagram commutes, thus inducing isomorphisms between the homology spaces of both horizontal sequences:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A|A & \xrightarrow{\delta_0^*} & A|(k\mathcal{A}_0)^*|A & \xrightarrow{d_1^*} & A|(k\mathcal{A}_1)^*|A & \xrightarrow{d_2^*} & A|(k\mathcal{A}_2)^*|A & \longrightarrow & 0 \\
 & & \downarrow \psi_0 & & \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 & & \\
0 & \longrightarrow & A|k\mathcal{A}_2|A & \xrightarrow{\bar{d}_0} & A|k\mathcal{A}_1|A & \xrightarrow{\bar{d}_1} & A|k\mathcal{A}_0|A & \xrightarrow{\bar{d}_2} & A|A & \longrightarrow & 0
\end{array}$$

where $|$ denotes $\otimes_E$ and $\bar{d}_0$ is given by

$$\bar{d}_0(1 \otimes d^2u^2 \otimes 1) = 1 \otimes du^2 \otimes d - d \otimes du^2 \otimes 1 - u \otimes d^2u \otimes 1 + 1 \otimes d^2u \otimes u.$$

$\bar{d}_1$ is

$$\begin{aligned}
\bar{d}_1(1 \otimes d^2u \otimes 1) = {} & 1 \otimes d \otimes du - \beta d \otimes d \otimes u - \beta d^2 \otimes u \otimes 1 \\
& - \alpha(1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1) \\
& - \beta(-\beta^{-1} \otimes u \otimes d^2 - \beta^{-1}u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1 \\
\bar{d}_1(1 \otimes du^2 \otimes 1) = {} & -\beta \otimes d \otimes u^2 - \beta d \otimes u \otimes u + du \otimes u \otimes 1 \\
& - \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1) \\
& - \beta(1 \otimes u \otimes ud - \beta^{-1}u \otimes u \otimes d - \beta^{-1}u^2 \otimes d \otimes 1) - \gamma \otimes u \otimes 1
\end{aligned}$$

and $\bar{d}_2$ is

$$\bar{d}_2(1 \otimes u \otimes 1) = -\beta \otimes u - u \otimes 1, \qquad \bar{d}_2(1 \otimes d \otimes 1) = 1 \otimes d + \beta d \otimes 1.$$

From this we deduce that $HH^3(A, A^e) \cong A \otimes_E A/(Im\bar{d}_2)$. Let σ be the algebra automorphism of A defined by $\sigma(d) = -\beta d$, $\sigma(u) = -\beta^{-1}u$. Recall that A_σ is the A-bimodule with A as underlying vector space and action of $A \otimes_k A^{op}$ given by: $(a \otimes b) \cdot x = ax\sigma(b)$, that is, it is twisted on the right by the automorphism σ.

It is easy to see that if $\beta \neq 0$ then $A_\sigma \cong A \otimes_E A/(Im\bar{d}_2) \cong HH^3(A, A^e)$ as A-bimodules. If $\beta = 0$ then the action on the left by u on $HH^3(A, A^e)$ is zero and then $A \ncong HH^3(A, A^e)$ since the action on the left by u on A is injective. We conclude after a short computation that $HH^3(A, A^e) \cong A$ if and only if $\beta = -1$. Notice that for $\beta = -1$ the complex in the second line of the diagram above is the resolution of A. As a consequence, A is 3-Calabi–Yau if and only if $\beta = -1$. In this case the potential Φ equals $d^2u^2 + \frac{\alpha}{2}dudu + \gamma du$. For $\beta \neq 0, -1$, we shall see in a forthcoming article that A is twisted 3-Calabi–Yau algebra [12], coming from a twisted potential. □

8. Final remarks

We have studied some examples of algebras, in particular of N-Koszul algebras for which we managed to obtain the minimal resolution using our methods. This fact can be stated in general as follows.

Theorem 8.1. *Given an algebra $A = kQ/I$ such that*

(1) there is a reduction system $\mathcal{R} = \{(s_i, f_i)\}_i$ for I satisfying $(\Diamond)$ with s_i and f_i homogeneous of length $N \geq 2$ for all i,
(2) for all $n \in \mathbb{N}$, the length of the elements of $\mathcal{A}_n$ is strictly smaller than the length of the elements of $\mathcal{A}_{n+1}$.

The resolutions of A as A-bimodule obtained using Theorem 4.1 and Theorem 4.2 are minimal.

Proof. Let $(A \otimes_E k\mathcal{A}_\bullet \otimes_E A, d_\bullet)$ be a resolution of A as A-bimodule obtained using Theorem 4.1 or Theorem 4.2. Denote by $|c|$ the length of a path $c \in Q_{\geq 0}$. Condition (1) guarantees that for all paths p, q such that $\lambda p \preceq q$ for some $\lambda \in k^\times$, we have $|p| = |q|$. Let $n \geq 0$, $q \in \mathcal{A}_n$ and $\lambda\pi(b) \otimes p \otimes \pi(b') \in \overline{\mathcal{L}}^{\prec}_{n-1}(q)$. Since $p \in \mathcal{A}_{n-1}$, condition (2) says that $|p| < |q|$. On the other hand, $\lambda bpb' \prec q$ and then $|bpb'| = |q|$. We deduce that $b \in Q_{\geq 1}$ or $b' \in Q_{\geq 1}$. As a consequence, $Im(d_n)$ is contained in $J \otimes_E k\mathcal{A}_{n-1} \otimes_E A \cup A \otimes_E k\mathcal{A}_{n-1} \otimes_E J$, where J is the ideal generated by the arrows and therefore the resolution of A is minimal. □

Remark 8.1.1. The conclusion holds in a more general situation, which includes example in Subsection 7.2. It is sufficient to have a reduction system satisfying (1) and such that the ambiguities p that appear when reducing a given $n+1$-ambiguity q are of length strictly smaller than the length of q.

Remark 8.1.2. In Example 7.0.1, the reduction system $\mathcal{R}_2$ satisfies the conditions of Theorem 8.1, while $\mathcal{R}_1$ does not satisfy (2).

Notice that if $\mathcal{R}$ is a reduction system for an algebra for which there is a non-resolvable ambiguity, then, even if we complete it like we did in Example 7.0.1, the resolutions obtained using Theorem 4.1 and Theorem 4.2 will not be minimal.

We end this article proving a generalization of Prop. 8 of [18] and a corollary.

Proposition 8.2. *Let $A = kQ/I$, where Q is a finite quiver, kQ is the path algebra graded by the length of paths and I a homogeneous ideal with respect to this grading, contained in $Q_{\geq 2}$. Let $\mathcal{R}$ be a reduction system satisfying conditions (1) and (2) of Theorem 8.1 and let A_S be the associated monomial algebra. The algebra A_S is N-Koszul if and only if A is an N-Koszul algebra.*

Proof. The projective bimodules appearing in the minimal resolution of A_S are in one-to-one correspondence with those appearing in the resolution of A, so either both of them are generated in the correct degrees or none is. □

This proposition, together with Thm. 3 of [17] give the following result.

Corollary 8.3. *If A has a reduction system $\mathcal{R}$ satisfying condition (1) of Theorem 8.1 and such that $S \subseteq Q_2$, then A is Koszul.*

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