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Bruno Da Ré, Mariela Rubin
Paula Teijeiro

Metainferential Paraconsistency

Abstract. In this article, our aim is to take a step towards a full understanding of the notion of paraconsistency in the context of metainferential logics. Following the work initiated by Barrio et al. [2018], we will consider a metainferential logic to be paraconsistent whenever the metainferential version of Explosion (or meta-Explosion) is invalid. However, our contribution consists in modifying the definition of meta-Explosion by extending the standard framework and introducing a negation for inferences and metainferences. From this new perspective, Tarskian paraconsistent logics such as **LP** will not turn out to be metainferentially paraconsistent, in contrast to, for instance, non-transitive logics like **ST**. Finally, we will end up by defining a logic which is metainferentially paraconsistent at every level, and discussing whether this logic is uniform through translations.

Keywords: paraconsistency; metainferential logics; uniformity

1. Introduction

The aim of this article is to provide a new answer to the question of what a paraconsistent logic is. To do so, we need to consider two topics: what a logic is and what paraconsistency is.

Regarding the former, a logic is usually defined as a set of (valid) inferences (of a given language). Two logics are considered to be equivalent if they share the same set of valid inferences. However, in recent papers [see, e.g., Cobreros et al., 2012, 2014; Ripley, 2012], among others, have argued that dropping transitivity can solve the semantic paradoxes. For that purpose, they develop the logic **ST**, used to later build a non-transitive theory of truth. One of the most fundamental

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features of **ST** is that although it has the same set of valid inferences as classical logic, it differs at the level of metainferences. Intuitively, just as an inference is a relation between sentences, where premises are meant to justify the conclusion, a metainference is a relation between inferences; that is, inferences themselves play the part of premises and conclusions. The rule of Cut —usually taken to codify the transitivity of logical consequence— is present in many sequent calculi for classical logic and is the metainference abandoned in **ST**:

$$\text{Cut}\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

Barrio et al. [2015] and Dicher and Paoli [2019] have emphasized that **ST** is not classical logic[1], precisely because the absence of Cut makes classical logic and **ST** different regarding the metainferences they determine as valid. So, Barrio et al. [2020] (from now on BPS) and Pailos [2019a,b] introduced the logic **TS**/**ST**, whose consequence relation is defined for metainferences, and so is a *metainferential logic*. **TS**/**ST** is characterized through the non-transitive logic **ST** and the non-reflexive logic **TS** [introduced in Cobreros et al., 2012],[2] While **ST** coincides with classical logic in each valid inference (but not in every classically valid metainference), **TS**/**ST** recovers every classically valid metainference. However, it fails to validate some classically valid meta-metainferences (metainferences of level 2). Thus, a new logic for meta-metainferences based on **TS**/**ST** can be defined to recover them. But then again, it fails to validate classically valid metainferences at higher levels (metainferences of level $n > 2$).

These are only the first steps of a hierarchy of metainferential logics such that each step recaptures more classically valid metainferences than the lower metainferential logics in the hierarchy, although none of them is entirely classical. And, eventually, it is possible to define a logic as the union of all of the metainferential logics of the hierarchy that recaptures every classically valid metainference of any level, as Pailos [2019b] and Scambler [2020] do.

The moral of these new developments is that metainferences matter when we define what a logic is. So, in this article, we will consider the logics defined not only by their inferences but also by their metainferences.

[1] Actually, these theorists have related **ST** with Priest's paraconsistent logic **LP**. We will go back to this point in later sections.

[2] We will formally define these logics in the next section.

Regarding paraconsistency, the philosophical idea behind this technical property is the failure of the principle known as (ECQ) *ex contradictione quodlibet* (from a contradiction, anything follows), which is also known as 'Explosion'. This principle, usually formalized as a rule in inferential logics, is of course classically valid. In other words, in classical logic a contradiction entails any formula, and that is the reason why classical theories cannot deal with contradictions. However, there are many logics that were designed to deal with inconsistent information, or to model inconsistent situations, e.g., the logic **LP** [see Asenjo, 1966; Priest, 1979]. These logics reject ECQ and the rules that formalize it.

Once we incorporate metainferences in the characterization of logics, we need to rethink how to formalize the principle ECQ. In [Barrio et al., 2018], BPS have discussed some ways of doing it and have proposed what we will call the notion of *BPS-paraconsistency*. To do so, the authors formalise the principle ECQ using a metainference (BPS-Meta Explosion). However, we think that this formalization of ECQ for metainferences has several problems. The main issue is related to the fact that the authors express the contradiction between inferences by appealing to the contradiction between formulas. We believe that once we take seriously the fact that metainferential logics are logics about inferences or metainferences, the contradiction between inferences and metainferences should be independent of the formulas appearing in them.

Luckily, we can reformulate the notion of metainferential paraconsistency in the light of the most recent developments made by Da Ré et al. [2020]. The idea is to extend the framework with a kind of negation for inferences and metainferences and to express the contradictions without appealing to the negation of formulas. It is important to mention that these resources were not at the disposal of BPS when they introduced this definition, so we can take the following work as an extension of BPS's ideas, rather than as a rival proposal.

Once we introduce our notion of paraconsistency, we will apply it to some metainferential logics and we will design a logic which is paraconsistent at every level of the hierarchy. Also, we will evaluate whether this coherent policy regarding paraconsistency can be extended to other properties and whether we can say that all of the logics in this hierarchy are actually the very same logic.

So, to lay out all of these ideas in order, the structure of this article is as follows. In section 2, we will present some technical resources that will be of use in the rest of the article. In section 3, we will explore what it means for a logic to be paraconsistent. In order to do it, we will introduce its traditional definition, following BPS [Barrio et al., 2018], and then we will expand it by means of an inferential and a metainferential negation. In section 4, we will examine how to get a logic that is paraconsistent at every metainferential level as well as at the inferential level, that is, a purely paraconsistent logic. Then, we will explore whether this research into pure paraconsistency can be extended to other properties of a logic, such as uniformity. Finally, in Section 5, we will end up with some conclusions and proposals for future work.

2. Technicalities

In this section, we will present some technical resources that we will use throughout this article. In particular, we will explain how to understand validity in each logic.

Let $\mathcal{L}$ be a propositional language and let $For(\mathcal{L})$ denote the set of all well-formed formulas of $\mathcal{L}$. We will let Greek capital letters be variables for sets of formulas and their lowercase counterparts schematically stand for individual formulas.

An inference of $\mathcal{L}$ is an ordered pair $\langle \Gamma, \Delta \rangle$ where $\Gamma, \Delta \subseteq For(\mathcal{L})$ (written $\Gamma \Rightarrow_0 \Delta$)[3]. $SEQ^0(\mathcal{L})$ will denote the set of all inferences of $\mathcal{L}$.

All *inferential logics* are sets of inferences. In this article we will work with four inferential logics: $\mathbf{K_3}$ [Kleene, 1952], **LP**, **ST** and **TS**. All four of them can be characterised with three valued valuations respecting the usual Strong Kleene truth tables as depicted below:

	$\neg$
t	**f**
i	**i**
f	**t**

$\wedge$	**t**	**i**	**f**
t	**t**	**i**	**f**
i	**i**	**i**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**i**	**f**
t	**t**	**t**	**t**
i	**t**	**i**	**i**
f	**t**	**i**	**f**

In general, we say an inference $\Gamma \Rightarrow \Delta$ is valid in a logic $\boldsymbol{L}$ if and only if all valuations satisfy it, which means that for every v, it either satisfies some $\delta \in \Delta$ or it does not satisfy some $\gamma \in \Gamma$. What it means for a valuation to satisfy a formula as a premise or as a conclusion of an inference varies from logic to logic. As is shown in the definitions below, in some logics, formulas are satisfied when they get a designated value,

[3] We will drop the 0 sub-index when the context allows for no ambiguity.

whereas in others, standards of satisfaction change between premises and conclusions (the standard consisting of the set $\{\mathbf{t}, \mathbf{i}\}$ is usually called "tolerant", whereas the one consisting only of the singleton $\{\mathbf{t}\}$ is called "strict" [see, e.g., Cobreros et al., 2012]). So for any valuation v and any inference $\Gamma \Rightarrow \Delta$ we say that:

- v *lp-satisfies* $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathbf{LP}} \Gamma \Rightarrow \Delta$) if and only if either $v(\gamma) \in \{\mathbf{f}\}$ for some $\gamma \in \Gamma$ or $v(\delta) \in \{\mathbf{t}, \mathbf{i}\}$ for some $\delta \in \Delta$;
- v *k3-satisfies* $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathbf{K_3}} \Gamma \Rightarrow \Delta$) if and only if either $v(\gamma) \in \{\mathbf{f}, \mathbf{i}\}$ for some $\gamma \in \Gamma$ or $v(\delta) \in \{\mathbf{t}\}$ for some $\delta \in \Delta$;[4]
- v *st-satisfies* $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta$) if and only either if $v(\gamma) \in \{\mathbf{f}, \mathbf{i}\}$ for some $\gamma \in \Gamma$ or $v(\delta) \in \{\mathbf{t}, \mathbf{i}\}$ for some $\delta \in \Delta$;
- v *ts-satisfies* $\Gamma \Rightarrow \Delta$ (written $v \vDash_{\mathbf{TS}} \Gamma \Rightarrow \Delta$) if and only if either $v(\gamma) \in \{\mathbf{f}\}$ for some $\gamma \in \Gamma$ or $v(\delta) \in \{\mathbf{t}\}$ for some $\delta \in \Delta$.

Taking these definitions into account, we note that **LP** is a logic in which the inference $\varphi \wedge \neg\varphi \Rightarrow \psi$ fails, since the valuation $v(\varphi) = v(\neg\varphi) = \mathbf{i}$ and $v(\psi) = \mathbf{f}$ is such that lp-satisfies its premises but it does not lp-satisfy its conclusion. By contrast, $\mathbf{K_3}$ is a logic in which $\psi \Rightarrow \varphi \vee \neg\varphi$ fails, for the valuation $v(\varphi) = v(\neg\varphi) = \mathbf{i}$ and $v(\psi) = \mathbf{t}$ is such that it $k3$-satisfies its premise and it does not $k3$-satisfy its conclusion.

At the same time, it is worth noticing that, as shown in [Cobreros et al., 2012], **ST** has the same inferences as **CL**, while **TS** has no valid inferences. The interesting thing about them is that even if inferentially there is no difference between **ST** and **CL**, as we will see, the metainferences they determine are quite different. In order to state it in more formal terms, we introduce the following definition that we take from [Da Ré et al., 2020]:

A *metainference of level* n, for $n \geqslant 1$, is an ordered pair $\langle \Gamma, \Delta \rangle$ where $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$ and $\Delta \subseteq SEQ^{n-1}(\mathcal{L})$ (written $\Gamma \Rightarrow_n \Delta$). $SEQ^n(\mathcal{L})$ is the set of all metainferences of level n on $\mathcal{L}$.[5]

In [Pailos, 2019a], the author introduced sixteen mixed and impure meta inferential consequence relations of level 1 based on **LP**, $\mathbf{K_3}$, **ST** and **TS**. Each of these consequence relations has the structure $\boldsymbol{L}_1/\boldsymbol{L}_2$, where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are *possibly different* inferential consequences relations.

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[4] We present $\mathbf{K_3}$ here because even if it is not a paraconsistent logic, it will come in handy later on when we analyze some metainferential logics.

[5] Notice that this definition allows not only premises but also conclusions to be multiple, generalizing the usual framework, as the authors do in [Da Ré et al., 2020].

The main idea is that $\boldsymbol{L}_1$ represents the standard for the premises of the metainference and $\boldsymbol{L}_2$ stands for the canon for the conclusion. This can be generalized to metainferential logics of any level (we adapt this definition from [Da Ré et al., 2020]):

A metainference (of any level $n \geqslant 1$) $\Gamma \Rightarrow_n \Delta$ is l_1/l_2-*satisfied* by a valuation v if and only if v either l_2-satisfies some member of Δ or it does not l_1-satisfy some member of Γ. A metainference is *locally valid* in $\boldsymbol{L} = \boldsymbol{L}_1/\boldsymbol{L}_2$ if and only if every valuation l_1/l_2-satisfies it.

All metainferential logics of level n are sets of metainferences of level n. A logic $\boldsymbol{L}$ is a set $\bigcup_{n\in\omega} L_n$, where $\boldsymbol{L}_n$ is an inferential logic (when $n = 0$) and a metainferential logic of level n (when $1 \leqslant n < \omega$). In other words, a logic is a union of an inferential logic and metainferential logics of every level.

As we said, even though **ST** coincides with **CL** at the level of inferences, the metainferential logic **ST**/**ST** is much weaker. Besides Cut, there are many other metainferences which are rendered invalid, such as the following (here represented in a vertical fashion, as we did in the introduction and is customary in sequent calculi):

$$\frac{\Gamma \Rightarrow \Delta, \varphi \to \psi \qquad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi, \Delta}$$

At the same time, despite **TS** being empty, **TS**/**TS** is far from vacuous or trivial, since it validates many metainferences, such as the one above, and invalidates many others, such as the following (which has empty premises and two conclusions):

$$\frac{}{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \neg\varphi}$$

As has been proven many times in the literature [see, e.g., Barrio et al., 2015; Dicher and Paoli, 2019; Pynko, 2010] the metainferential logic **ST**/**ST** coincides via a suitable translation with **LP**,[6] while it was also suggested that the metainferential logic **TS**/**TS** coincides via a suitable translation with $\mathbf{K_3}$.[7] At the same time, as the author points out in [Pailos, 2019a], **TS**/**ST** validates every classical metainference of level 1, while **ST**/**TS** has no valid metainferences of level 1.

Before moving on to the next section, there are two clarifications left to make. First, in order to refer to a logic *simpliciter* (i.e., the union of all

[6] We are not using the same terminology as in the literature but an equivalent one.

[7] As far as we know this was not actually proved, although it can be easily done.

natural levels), we will take some level n as its denomination, and assume the lower levels can be inferred as follows: if the label is $\boldsymbol{L}_1/\boldsymbol{L}_2$, the metainferential logic below is $\boldsymbol{L}_2$. This assumption is usually implicitly adopted in the literature [see, e.g., Barrio et al., 2018, 2020; Pailos, 2019a], although never explicitly stated. Moreover, it is also assumed that an inferential logic determines all the metainferential logics above it. For instance, **LP** is taken to be the union of **LP**, **LP**/**LP** and so on. We will not follow this second convention, and will instead indicate the logic of each level whenever necessary. In the case of **LP**, we will use the name $\mathbf{LP}_\omega$ for the whole set of metainferential logics.

In the next section, we will explore how paraconsistency is treated in the literature and we will expand the usual notion of a paraconsistent logic in order to deal with metainferential logics.

3. What is a paraconsistent logic?

In this section, we will present a new technical definition of what paraconsistency is, taking into account not only inferences but also metainferences. To do so, however, first, we will present some philosophical motivations concerning the concept of paraconsistency. Then, we will introduce the traditional definitions of what it means for an inferential logic to be paraconsistent. Next, we will introduce an appropriate notion for the metainferential case. Before doing so, we will review the BPS proposal and we will argue against it. Finally, to deal with the problems the BPS proposal has, we will introduce some changes in the framework and, following [Da Ré et al., 2020], we will introduce a metainferential negation, which will allow us to define metainferential paraconsistency.

3.1. Paraconsistency: the philosophical interpretations

In this section, we will briefly describe some of the more well-known philosophical insights behind the notion of paraconsistency, and we will present the main motivations that philosophers and paraconsistent logicians have followed to develop paraconsistent logics.

The key idea of paraconsistency is the failure of the principle known as (ECQ) *ex contradictione quodlibet* (from a contradiction, anything follows), also known as Explosion, which of course is classically valid.

Although all the paraconsistent theorists reject ECQ, there have been many different particular paraconsistent formal logics that have been proposed, under many different motivations. To schematically classify these positions, we can revisit the taxonomy made by Urbas [1990]. In his article, the author mentions at least three different main motivations related to the development of paraconsistent logics.[8]

First of all, the dialetheic position. This is a metaphysical stance that considers that there are true contradictions. This position is also called *Dialetheism*, and those who embrace it are usually called *dialetheists*. Of course, from accepting that there are some true contradictions it does not follow that every contradiction is true (trivialism). In this vein, most philosophers and logicians who have developed paraconsistent logics in order to face paradoxical phenomena are dialetheists. We may mention [Asenjo, 1966; Beall, 2009; Priest, 1979, 2006; Priest et al., 2018; Routley and Meyer, 1976] among many others [see Priest et al., 2018 for a comprehensive survey of dialetheism].

Secondly, it is important to mention the pragmatic position. The theorists classified under this position develop paraconsistent logics to be able to deal with inconsistent information, e.g., databases, collections of statements, beliefs, scientific theories. However, these logicians and philosophers are not necessarily committed to a metaphysical position, as the dialetheists are. In this line, we should mention the development of the *Logics of Formal Inconsistency*. Among many others, we should mention the Brazilian tradition initiated by Da Costa [1974], and continued and developed among others by Carnielli et al. [2007] [see Carnielli and Coniglio, 2016, for a comprehensive study of this tradition].

Lastly, there are other motivations for embracing a paraconsistent logic that are neither related to dealing with inconsistent information nor with the metaphysical claim that there are true contradictions. In the relevance tradition, for instance, the failure of ECQ is due to the fact that the conclusion is irrelevant with respect to the premises [see Mares, 2020, for an introduction to relevance logics].

[8] Also, recently [Carnielli and Rodrigues, 2019] have proposed another classification of the paraconsistent positions, focusing on their metaphysical commitments. We won't engage here with metaphysical concerns, preferring instead to remain neutral and hope that everything we say here will be sufficiently general to avoid having to take any metaphysical stance.

It is not our aim to be exhaustive about the philosophical motivation for being a paraconsistent theorist, but just to mention some of the most representative positions. For a comprehensive presentation of paraconsistency and paraconsistent logics, [see, e.g., Carnielli and Coniglio, 2016; Priest et al., 2018b].

One thing all these approaches have in common is their rejection of ECQ. Nonetheless, this rejection has to be interpreted substantially for it to be a significant thesis[9]. In other words, it is not sufficient that there is *some* monadic operator which does not satisfy it, given that there are plenty operators of that kind, even definable in classical settings (a constant operator mapping all values to **t**, for instance). Instead, the expression which does not comply with the principle has to be rightfully considered a negation.

There is probably no fixed set of positive principles that sets the issue of what qualifies as negation without, for example, begging the question and excluding, by fiat, the possibility of paraconsistency. Many agree that being a contradictory-forming operator is enough, although there is again dissent on how to characterise contradictions. For instance, according to [Priest et al., 1989a], it means to validate the principle of non-contradiction, which speaks against at least some logics of formal inconsistency being genuine paraconsistent logics. In a similar vein, De and Omori [2018] claim that to understand $\neg$ as a negation, one must have the following:

$$\neg\varphi \text{ is true iff } \varphi \text{ is false and } \neg\varphi \text{ is false iff } \varphi \text{ is true.}$$

This qualifies **LP**'s interpretation of $\neg$ as a *bona fide* negation, and excludes some others, like the intuitionistic case. However, Slater [1995] argues that **LP**'s negation does not form contradictories, because the possibility of both φ and $\neg\varphi$ being true classifies them as subcontraries. This argument can be extended to any logic that can make both φ and $\neg\varphi$ true for some sentence φ. Summing up, to accept the characterization of negation as a contradictory-forming operator seems to create a faultless disagreement situation, where parties talk past each other because they do not share the concept of contradictories. Another option for the paraconsistent theorist is to defend the weaker claim that negation is a subcontraries-forming operator instead, in the sense that at least one of φ and $\neg\varphi$ has to be true.[10] This claim, it is only fair to notice, has been explicitly rejected by defenders of paraconsistent logic themsleves. In the case of LFI's, they argue that if paraconsistent negation were merely a subcontraries-operator, it would not be possible to define classical negation from it [see Carnielli and Coniglio, 2016]. In the case

[9] We want to thank our anonymous referees for pressing us on this point.

[10] Dually, a contraries-forming operator, in the case of a paracomplete theorist.

of [Priest et al., 1989a], they dismiss this position on the grounds that formulas do not have unique subcontraries (up to equivalence), in the way that they have unique contradictories. Be that as it may, accepting the negation-as-subcontrary stance has the advantage of characterizing the whole family of logics we have considered so far in this article, and does not pit the different paraconsistent logics against each other, which for our purposes is enough. We will only briefly come back to this point in the next section, when discussing metainferential negation.

Leaving then the issue of negation aside for now, we find that the other thing all paraconsistent logics share is an assumption regarding what sort of objects can form a contradiction; that is, formulas. However, the emergence of metainferential logics [Barrio et al., 2018, 2020; Pailos, 2019b; Scambler, 2020] has challenged this assumption: metainferences are as important as inferences, and thus any formalization of ECQ has to take metainferences into account. In this vein, Barrio et al. [2018] develop a metainferential principle of Explosion and define a new notion of paraconsistency. In the next section, we will introduce in detail such definition. Afterward, we will extend the work made by Barrio et al. [2018], and using some machinery recently developed in [Da Ré et al., 2020] we will present our own definition.

3.2. Paraconsistency: the technical definition

Let us start first with the usual notion of inferential paraconsistency. As we mentioned in the previous section, the main idea of paraconsistency is the failure of ECQ: *from a contradiction anything follows*. However, how to formalize this failure depends on what a contradiction is, which in turn depends on the objects the logic talks about. In particular, inferential logics describe inferential patterns between (sets of) formulas, and thus, a contradiction can only be formed between formulas. So, e.g., Ripley [2015] (and similarly Carnielli and Coniglio [2016]) define inferential paraconsistency as follows[11]:

A logic $\boldsymbol{L}$ is *inferentially paraconsistent* if and only if $\nvDash_{\boldsymbol{L}} \varphi \wedge \neg\varphi \Rightarrow \psi$, for some $\varphi, \psi \in For(\mathcal{L})$ or $\nvDash_{\boldsymbol{L}} \varphi, \neg\varphi \Rightarrow \psi$, for some $\varphi, \psi \in For(\mathcal{L})$.

Notice that these two inferences are just two possible ways of formalizing ECQ. In this context, all of the logics will either validate or invalidate both of these inferences, so we omit this distinction and de-

[11] Here we are adapting the terminology to this article.

fine inferential paraconsistency regardless of the way the formulas that form the contradiction are connected.[12]

Now we can look back to the logics we have presented in the previous section and check which of them are paraconsistent. First, any logic containing **LP** or **TS** as its inferential part is inferentially paraconsistent, since any valuation v such that $v(\varphi) = \mathbf{i}$ and $v(\psi) = \mathbf{f}$ is a counterexample to the inference $\varphi, \neg\varphi \Rightarrow \psi$ in both logics. On the other hand, as it is widely known, it is straightforward to check that a logic containing $\mathbf{K_3}$ and **ST** as its inferential part is not inferentially paraconsistent.

So far, we have just defined what it means for a logic to be inferentially paraconsistent. However, as the title of this article suggests, we are interested in investigating how to extend this definition to metainferential logics. In other words, we would like to answer the question: what makes a logic metainferentially paraconsistent? It is worth mentioning that we are not the first to try to address this issue. Barrio et al. [2018], when analyzing how to formalize the idea of ECQ, claimed the following:

> An inference with an inconsistent premise set implies any conclusion. As is well-known, inconsistent premise sets for inferences are sets that include (some instance of) the (schematic) formulae A and $\neg A$ [...] The question is now, how to adapt this idea to the case of meta-inferences. For us, the most reasonable take is to say that a meta-inference with an inconsistent premise set implies any conclusion.
>
> [Barrio et al., 2018, p. 94]

So, to formalize these ideas, Barrio et al. [2018, p. 95] provide the following definition:[13] a logic is *BPS-paraconsistent* if and only if either Explosion is invalid or the following metainference is locally invalid:

$$\text{BPS-Meta Explosion} \quad \frac{\Rightarrow \varphi \qquad \Rightarrow \neg\varphi}{\Rightarrow \psi}$$

[12] Notice that although these two rules are very closely related they are not equivalent. For example, in contexts where structural contraction or weakening fail, some formulation could be stronger than the other. Also, subvaluationist and supervaluationist logics are sensitive to this distinction. In this article we will not expand on these systems but we hope to do it in future works, since our definitions intend to be quite general.

[13] Here we are adapting the notation to the terminology adopted in this article. Also, we refer in the definition to *BPS-paraconsistency*, instead of *paraconsistency*, since we will keep the latter for our proposal.

Also, we can say that a logic is BPS-metainferentially paraconsistent if BPS-meta Explosion is locally invalid in it.[14] Now, let us apply this definition to some of the logics we have presented in the previous section. First, any logic containing **LP**/**LP** is BPS-metainferentially paraconsistent (the same valuation defined to invalidate Explosion is a counterexample to the instance of BPS-Meta Explosion formed by replacing φ by p and ψ by q). On the other hand, although **ST** is not inferentially paraconsistent, **ST**/**ST** is indeed BPS-metainferentially paraconsistent (again the same valuation is a counterexample to BPS-meta Explosion in **ST**). In the case of **TS**/**TS** it is easy to check that it is not BPS-metainferentialy paraconsistent since although **TS** is an empty inferential logic, **TS**/**TS** locally validates many metainferences, including BPS-Meta Explosion. Finally, regarding $\mathbf{K_3}/\mathbf{K_3}$, it is not BPS-metainferentially paraconsistent.[15]

So, the authors consider that in order to formalize the contradiction between two inferences we need to use the negation of the object language. Although we agree that we need to consider sets of inconsistent inferences, we think that the definition given by BPS has some problems.

First of all, the BPS definition is based on formulas and not on inferences. In other words, although the authors claim that the inconsistency in a metainference has to be formed between inferences, when they define BPS-meta Explosion the contradiction between the inferences is derived from the contradiction between the formulas which appear in the inferences and not between the inferences themselves. However, since metainferences talk about inferences, the contradiction should be formed between inferences independently of the linguistic resources of the object language. In other words, since metainferential logics describe inferential patterns between inferences, the definition of metainferential properties (in this case, metainferential paraconsistency) should be independent of the presence of any operator for the formulas.

Related to this, it is worth mentioning that many logics[16] are BPS-metainferentially paraconsistent just because they are inferentially para-

[14] Notice that BPS-Meta Explosion is a metainference of level 1. However, Barrio et al. [2018] seem to implicitly assume a generalization of this definition to metainferences of arbitrary level n.

[15] Note that no valuation can ts-satisfy nor k3-satisfy the premises of BPS- Meta Explosion.

[16] Although we will not expand on this point, at least it will hold in any logic where the internal and the external notion of consequence coincide. See, e.g., [Avron, 1991] for a definition of the internal and the external consequence relations.

consistent. This is the case, for instance of **LP**, whose associated metainferential consequence relation (**LP**/**LP**) is BPS-metainferentially paraconsistent, since the way in which BPS-Meta Explosion is defined only focuses on the contradiction between formulas. However, if the concept of metainferential paraconsistency is taken to merely be a subproduct of its more fundamental inferential counterpart, its theoretical interest becomes rather dull.

Finally, we think that the BPS formalization does not capture the structure of the main idea of ECQ (*from a contradiction anything follows*), since it does not represent the idea of *contradiction* and it does not represent the idea of *anything*. Regarding the contradiction, schematically we think of a contradiction as formed by a pair of objects, A and *not*A where *not* is some kind of contradiction-forming operator. This is the way in which Explosion is formulated in the inferential case, where the operator *not* is none other than the negation connective $\neg$. However, BPS-meta Explosion does not have this structure (partly, because of the lack of resources of the language). Regarding the *anything* part of ECQ, notice that the conclusion of BPS-meta Explosion does not have the form of any inference, but the form of one particular kind of inference ($\Rightarrow \psi$), i.e., one without premises and with just one conclusion.

In a nutshell, BPS define meta Explosion using the internal operator $\neg$, which seems inadequate to define a concept of contradiction useful for metainferential logics. Instead, we will extend the framework introducing the kind of negation for (meta)inferences Da Ré et al. [2020] use to characterise duality.

In this vein, we will divide the class of inferences and metainferences in two types: positive and negative. The negative inferences (metainferences) will be interpreted as the negation of positive inferences (metainferences). So, from now on, a regular inference will be denoted as $\Gamma \Rightarrow^{+} \Delta$ and its negation (the negative inference) will be denoted as $\Gamma \Rightarrow^{-} \Delta$[17]. We need to specify then which are the satisfaction conditions of a negative inference in a logic $\boldsymbol{L}$ [see Da Ré et al., 2020]:

A valuation v *l-satisfies* a negative inference or metainference $\Gamma \Rightarrow^{-}_{n} \Delta$ (written $v \vDash_{\mathbf{L}} \Gamma \Rightarrow^{-}_{n} \Delta$) if and only if both v l-satisfies $\Rightarrow^{+}_{n} \gamma$ for any

[17] Of course, all of the notions introduced in Section 2 can be straightforwardly adapted to these changes. For instance, a logic is a set of positive and negative inferences and metainferences.

$\gamma \in \Gamma$, and v l-satisfies $\delta \Rightarrow_n^+$ for any $\delta \in \Delta$. A negative inference is *valid* in $\boldsymbol{L}$ if and only if for every valuation v, $v \vDash_{\boldsymbol{L}} \Gamma \Rightarrow_n^- \Delta$.

Coming back to the discussion on the previous section, one might wonder in what sense this is a negation; see:

> Just like formula negation operates at the formula-level, toggling between truth and falsity, we will let inference-negation operate at the meta-level, affecting the inference satisfaction conditions. While the satisfaction conditions for a positive inference in a given valuation consist in the fact that if all premises receive a designated value, then so do some of the conclusions, the satisfaction conditions for the corresponding negative inference invert that to a relevant extent. Thus, a valuation satisfies its corresponding negative inference if and only if it assigns a designated value to each premise and it assigns a non-designated value to each conclusion. [Da Ré et al., 2020, p. 323]

In the case of classical logic, as Da Ré et al. [2020] note, it is easy to check that for every classical valuation v (boolean bivaluation), and for every pair of inferences $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$,

Exclusion: v does not *cl*-satisfy both;

Exhaustion: v *cl*-satisfies some.

and similarly for a pair of meta$_n$inferences $\Gamma \Rightarrow_n^+ \Delta$ and $\Gamma \Rightarrow_n^- \Delta$.

So, positive and negative inferences (metainferences) form contradictory pairs in classical valuations. However, as it happens in the inferential non-classical setting with the negation connective, in some metainferential logics one or both of these conditions can fail but now regarding this kind of negation for inferences. In particular, there are valuations which *st*-satisfy both a positive and a negative inference, while there are valuations which do not *ts*-satisfy none of them.[18]

Put another way, let us say that a valuation *unsatisfies* a positive inference if it makes all its premises true and all its conclusions false, and

[18] Someone might wonder why we chose to define negative inferences in this way and not, for example, using the more usual notion of invalidity. The reason for this is that the negation derived from this notion is purely classical (an inference is or is not satisfied in a valuation, and cannot be satisfied and not satisfied at the same time), and we are looking for a weaker notion. Also, Cobreros et al [2021] introduce another notion related to invalidity: antisatisfaction. Although very interesting, we won't use it in this article. However it is our hope to provide in future works a comprehensive study of the relation between these different notions and the concepts developed by Da Ré et al. [2020], which we are using here.

that it unsatisfies a negative inference if it makes some of its premises false or some of its conclusions true. Recall that De and Omori [2016] state that φ and $\neg\varphi$ are contradictory because one of them is true in a valuation if and only if the other one is false, and this applies to, for instance, **LP** because the third value is interpreted as true and false at the same time. If we keep that interpretation of **i**, we can say that $\Gamma \Rightarrow^+ \Delta$ and $\Gamma \Rightarrow^- \Delta$ are contradictory also according to, for instance, **LP** or **ST**, because one of them is satisfied by a valuation if and only if the other one is unsatisfied by it.

Of course, this is possible because we allow some valuations to both satisfy and unsatisfy inferences. The same issue with subcontrariety which arose regarding formula-negation is reinstated here, and the same answers as before are available. Also, notice that at the level of metainferences, it is not true that we can always define another rogue, non-explosive operator which hinders the definition of paraconsistency as failure of EFQ (as it happened in the inferential case). However, this is only so because the language of inferences is much (syntactically) weaker than that of formulas, so it is doubtful that this fact can be of significant importance.

Having said that, we will now exploit this concept of negative inferences to give a proper definition of metainferential paraconsistency:

A logic is *paraconsistent* if and only if either Explosion is invalid or the following rule (meta_nExplosion):

$$\text{Meta}_n \text{ Explosion} \quad \frac{\Gamma \Rightarrow^+_{n-1} \Delta \qquad \Gamma \Rightarrow^-_{n-1} \Delta}{\Sigma \Rightarrow^{+/-}_{n-1} \Pi}$$

is locally invalid, for some n with $n \geqslant 1$.

As before, we will say that a logic is $\text{metainferentally}_n$ paraconsistent if meta_n Explosion is locally invalid in it. Notice first that we are here representing the intuitive idea of ECQ in the desired way. First, the premises of the rule of meta Explosion form a pair of contradictory metainferences of the form A and *not* A, and the conclusion has the form of any inference (any inference is an instance of it[19]). Also, it is important to note that there is no mention of any formulas in this

[19] Notice that the empty set is not an instance of the conclusion of Meta_n Explosion. However, something similar happens with the inferential version of Explosion ($\varphi, \neg\varphi \Rightarrow^+ \psi$), i.e., the empty set is not an instance of ψ (and therefore, e.g., $p, \neg p \Rightarrow^+$ is not an instance of Explosion).

definition. Actually, it could be the case that the inferential logic lacks a negation for formulas, but nevertheless it is still possible for it to be metainferentially paraconsistent. In other words, the problems we found in the BPS definition can be solved in this new definition.

Now, let us apply this definition to some of the logics we have been working with. Perhaps surprisingly, a logic containing **LP**/**LP** can fail to be metainferentially paraconsistent. Actually, for every pair of inferences $\Gamma \Rightarrow^{+} \Delta$ and $\Gamma \Rightarrow^{-} \Delta$, a valuation v cannot lp-satisfy both, so the premises of meta Explosion are never lp-satisfied, and thus the metainference is locally valid in **LP**/**LP**.[20] Regarding a logic containing **ST**/**ST** it still is metainferentially paraconsistent, while **TS**/**TS** and $\mathbf{K_3}/\mathbf{K_3}$ fail to be metainferentially paraconsistent (since no valuation satisfies the premises of meta Explosion). So under this new definition none of the logics considered above is both inferentially and metainferentially paraconsistent. One might wonder, thus, whether there is *some* logic that satisfies both requirements. To this end, take a logic containing **ST**/**LP**, where the standard for the premises of the locally valid metainferences of level 1 is provided by **ST** while the standard for the conclusions is provided by **LP**. This logic is both inferentially paraconsistent and metainferentially paraconsistent, as the following locally invalid instance of meta Explosion shows:

$$\text{Meta Explosion}\ \frac{\Rightarrow^{+} p \qquad \Rightarrow^{-} p}{\Rightarrow^{+} q}$$

To see this, consider a counterexample given by a valuation v such that $v(p) = \mathbf{i}$ and $v(q) = \mathbf{f}$, which of course *st*-satisfies the premises of the metainference, but it does not lp-satisfy the conclusion.

So, this logic has a common policy regarding paraconsistency at the first two levels. Extending the idea provided by Da Ré [2019], we can give the following definition: a logic is *purely paraconsistent* if and only if it is inferentially paraconsistent and it is meta$_n$inferentially paraconsistent, for any n.

A nice feature of this property is that it changes the focus and expands the possibilities on how to understand what a paraconsistent logic is. One might be tempted to think that the characterization of paraconsistency by means of an inferential negation should be replaced with

[20] It is easy to check that as in the classical case, for every pair of inferences $\Gamma \Rightarrow^{+} \Delta$ and $\Gamma \Rightarrow^{-} \Delta$ every valuation satisfies Exclusion and Exhaustion.

the stronger requisite of being purely paraconsistent (as an anonymous referee seemed to suggest). But this would exclude *all* logics traditionally considered paraconsistent, so it is doubtful that such a move can solve any existent debates on the matter. On the contrary, this notion allows us to broaden the possible features that characterize a paraconsistent logic.

In the following section we will show how to design one logic which is purely paraconsistent and we will discuss whether the inferential and all the metainferential logics contained in it can be thought as being the same logic.

4. Pure paraconsistency and uniformity

In this section, first, we will show how to define a purely paraconsistent logic, i.e., a logic that is paraconsistent at every level. Next, we will discuss whether this policy regarding paraconsistency can be extended to all the properties of the logics, i.e., if they are uniform in the sense of being the *same* logic.

4.1. A purely paraconsistent logic

In [Barrio et al., 2020], BPS define a hierarchy of logics based on **ST** which recovers in each level more and more classical metainferences. We will use this collection of metainferential logics to define the system we are looking for (the purely paraconsistent logic). Adapting a bit the terminology, we have the following definition [Barrio et al., 2020]:

Let $\mathbf{P}_0 = \mathbf{ST}$ and in general $\mathbf{P}_j = \overline{\mathbf{P}_{j-1}}/\mathbf{P}_{j-1}$, for any $j \geqslant 1$, where $\overline{\mathbf{ST}} = \mathbf{TS}$ and $\overline{\mathbf{P}_j} = \mathbf{X}/\mathbf{Y}$ if $\mathbf{P}_j = \mathbf{Y}/\mathbf{X}$.

We will call $\mathbf{P}_\omega$ the logic which results of joining all these inferential and metainferential logics.[21] To simplify notation, the slash symbol in the logics of lesser levels is sometimes omitted. So, e.g., $\mathbf{P}_2 = \mathbf{STTS}/\mathbf{TSST}$.

Regarding the logics contained in $\mathbf{P}_\omega$, the authors prove the following (adapting a bit the terminology):

Theorem 4.1 (Barrio et al., 2020). *For any n, any set of metainferences Γ and any metainferences φ, $\vDash_{\mathbf{P}_n} \Gamma \Rightarrow_n^+ \varphi$ if and only if $\vDash_{\mathbf{CL}} \Gamma \Rightarrow_n^+ \varphi$.*

[21] It is worth mentioning that the logic $\mathbf{P}_\omega$ was first introduced in [Barrio et al., 2020] and denoted as $\mathbb{ST}$ (although it was not consider a logic itself but a collection of logics). Also, it was later studied in [Pailos, 2019b], where it is called $\mathbf{CM}_\omega$, and the author not only considers it to be a logic but he also claims that it is classical logic. See [Pailos, 2019b] for details.

So, having defined this collection of metainferential logics, let's go back to our goal: to design a purely paraconsistent logic. It is done by recursively generating a new set of metainferential logics building on the $\mathbf{P}_i$ sequence above, as follows: we put $\mathbf{L}_0 := \mathbf{LP}$ and $\mathbf{L}_k = \mathbf{P}_{k-1}/\mathbf{L}_{k-1}$, for any $k \geqslant 1$. So, for instance $\mathbf{L}_2 = \mathbf{TSST}/\mathbf{STLP}$, and $\mathbf{L}_3 = \mathbf{STTSTSST}/\mathbf{TSSTSTLP}$, where again the slashes are omitted for readability.

And actually, it's not difficult to check the following:

Fact 4.2. *The logic* $\mathbf{L}_0$ *is inferentially paraconsistent and each logic* $\mathbf{L}_n$, $n \geqslant 1$, *is metainferentially*$_n$ *paraconsistent.*

Proof. The case of $\mathbf{L}_0$ is trivial since $\mathbf{L}_0 := \mathbf{LP}$. The case of $\mathbf{L}_1 = \mathbf{ST}/\mathbf{LP}$ was illustrated at the end of Section 3. Let $\mathbf{L}_n$ be any logic with $n \geqslant 2$. Thus, we need to find a counterexample to some token of the following metainference:

$$\text{Meta}_n \text{ Explosion} \quad \frac{\varGamma \Rightarrow^{+}_{n-1} \varDelta \qquad \varGamma \Rightarrow^{-}_{n-1} \varDelta}{\varSigma \Rightarrow^{+/-}_{n-1} \varPi}$$

Take $\varGamma = \emptyset$ and $\varDelta = \emptyset \Rightarrow^{+}_{n-2} \ldots \Rightarrow^{+}_{0} p$ (if n=2, just omit the dots and one of the $\Rightarrow$). On the other hand, take $\varSigma \Rightarrow^{+/-}_{n-1} \varPi$ to be $\emptyset \Rightarrow^{+}_{n-1} \emptyset \Rightarrow^{+}_{n-2} \ldots \Rightarrow^{+}_{0} q$. Using the definitions it is routine to check that the valuation $v(p) = \mathbf{i}$ and $v(q) = \mathbf{f}$ satisfies the premises of the metainference, but not the conclusion. ⊣

So, each logic of the hierarchy $\mathbf{L}_n$ is paraconsistent in the corresponding level, and thus, the union of all of these systems is the purely paraconsistent logic we were looking for:

Corollary 4.3. *The logic* $\mathbf{L}_\omega = \bigcup_{n\in\omega} \mathbf{L}_n$ *is purely paraconsistent.*

Interestingly, since $\mathbf{L}_\omega = \mathbf{LP} \cup \bigcup_{n\geqslant 1} \mathbf{P}_{n-1}/\mathbf{L}_{n-1}$, in each level $k \geqslant 1$, we know that $\mathbf{L}_k = \mathbf{P}_{k-1}/\mathbf{L}_{k-1}$ and thus by Theorem 4.1 the metainferential logic of the premises $\mathbf{P}_{k-1}$ recovers all of the classical metainferences of level $k-1$.

4.2. Uniformity

The property of being purely paraconsistent can be regarded as part of a more general phenomenon, which amounts to a certain coherence among inferential levels. Coherence, however, can be understood in different ways.

The path we have followed so far has consisted in providing a broad, informal concept of ECQ, which is then instantiated by different formal principles, each one involving a different but related type of contradiction (between formulas, inferences, metainferences, etc). Doing so for every logical principle would require the metalanguage to be as rich as the language, allowing not only negations, but also, for instance, conditionals of any complexity.

Here, we will explore a different path, one which analyzes characterizations of coherence provided by tools already at our disposal in the literature. We will call these types of coherence *Uniformity*. This, in turn, raises two questions:

1. *Is* $\mathbf{L}_\omega$ *in some sense uniform?*

2. *Is pure paraconsistency a necessary condition for uniformity, or does it pertain to a different type of coherence altogether?*

For any proposed definitions, some boundaries are clear: **CL** has to be considered a paradigmatic case of a uniform logic, whereas we can take **ST**/**ST** as a paradigmatic non-uniform one, given that its inferential logic (**ST**) is classical but its metainferential one is not (i.e., some classically valid metainferences fail [see Cobreros et al., 2012].

The tricky part is that we need some tool to compare the different levels of a logic, because each one of them deals, in principle, with a different type of object, which has its own standard of validity. The probably most obvious way to do so are translations functions[22]. In [Barrio et al., 2020, 2015; Dicher and Paoli, 2019; Pynko, 2010], we can find several examples of such functions, although they are used to compare levels across *different* logics. In particular, the authors show how the metainferential logic **ST**/**ST** coincides with the inferential logic **LP** when we translate one into the other. These functions turn a metainference into an inference by translating premises as premises and conclusions as conclusions.

Given a certain translation function τ, when used to evaluate inferences of different levels inside the *same* multi-level logic, it can provide the following approximation to uniformity:

A logic is *τ-uniform above level m* if and only if for any meta$_n$ inference, $\Gamma \Rightarrow_n \Delta$ is valid iff $\tau(\Gamma \Rightarrow_n \Delta)$ is valid for all $n \geqslant m$.

[22] Notice that the method of translations is somewhat limited in its application, since it cannot account for metainferences with infinite premises or conclusions. This is of course only a problem for the case of logics which are not compact. How to deal with such cases is left as an open problem for a general and complete theory of uniformity.

However, there is an issue which is that, as they stand, the functions existent in the literature are incomplete. First, as expected, none of them considers negative inferences. Second, none are defined for inferences with empty premises *and* conclusions. This can be easily solved by adding a $\bot$ and a $\top$ constant to the language (which **LP** usually lacks, but which poses no problem for it). Third, only the authors in [Barrio et al., 2020] define a function for more than one metainferential level. Thus, taking all these details into account, we will provide an improved translation which is suitable for present purposes:

$$\tau_1(\Gamma \Rightarrow_0^+ \Delta) = \begin{cases} \neg\gamma_1 \vee \cdots \vee \neg\gamma_n \vee \delta_1 \vee \cdots \vee \delta_m & \text{if } \Gamma \neq \emptyset, \Delta \neq \emptyset \\ \neg\gamma_1 \vee \cdots \vee \neg\gamma_n & \text{if } \Delta = \emptyset \\ \delta_1 \vee \cdots \vee \delta_m & \text{if } \Gamma = \emptyset \\ \bot & \text{if } \Gamma = \Delta = \emptyset \end{cases}$$

$$\tau_1(\Gamma \Rightarrow_0^- \Delta) = \begin{cases} \gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\delta_1 \wedge \cdots \wedge \neg\delta_m & \text{if } \Gamma \neq \emptyset, \Delta \neq \emptyset \\ \gamma_1 \wedge \cdots \wedge \gamma_n & \text{if } \Delta = \emptyset \\ \neg\delta_1 \wedge \cdots \wedge \neg\delta_m & \text{if } \Gamma = \emptyset \\ \top & \text{if } \Gamma = \Delta = \emptyset \end{cases}$$

$$\tau_1(\Gamma \Rightarrow_n^{+/-} \Delta) = \{\tau_1(\gamma_i) : \gamma_i \in \Gamma\} \Rightarrow_{n-1}^{+/-} \{\tau_1(\delta_j) : \delta_j \in \Delta\}, \text{for } n > 0$$

When τ is τ_1, the answer to question number 2 is affirmative. The translation of Meta$_n$ Explosion is:

$$\neg \bigwedge \Gamma \vee \bigvee \Delta, \bigvee \Gamma \wedge \neg \bigvee \Delta \Rightarrow \Pi$$

which is (equivalent to) an instance of the inferential version of Explosion. Thus, τ_1-uniform logics have to be either purely paraconsistent or not paraconsistent at all. However, this partial uniformity is not enough for the case of $\mathbf{L}_\omega$:

Fact 4.4. $\mathbf{L}_\omega$ *is not* τ_1*-uniform.*

Proof. There are metainferences which are $\mathbf{L}_\omega$-invalid, but their translations are $\mathbf{L}_\omega$-valid. As a witness case, take the following metainference of level 1:

$$\frac{\Rightarrow^+ \varphi \qquad \varphi \Rightarrow^+}{\psi \wedge \neg\psi \Rightarrow^+}$$

A valuation $v(\varphi) = \mathbf{i}$ and $v(\psi) = \mathbf{t}$ st-satisfies the premises but it is an *lp*-counterexample to the conclusion, and thus the metainference is $\mathbf{L}_\omega$-invalid, but its τ_1-translation is the following **LP**-valid inference:

$$\varphi, \neg\varphi \Rightarrow^{+} \neg(\psi \wedge \neg\psi) \qquad \dashv$$

Thus, the answer to the question number 1 is negative. Of course, this result hangs on the reliability of τ_1, since other translations could categorize $\mathbf{L}_\omega$ as uniform. And there might be some reasons to be wary of its reliability. On the one hand, it seems dubious that there are *any* logics which are uniform according to τ_1 besides **CL**. On the other hand, τ_1 looks to be much too coarse for our purposes. Take **LP** as an example. Given that negated conclusions are equated with asserted premises, we get that different metainferences, with different validity status, get translated into the same inference:

$$\left.\begin{array}{l} \vDash_{\mathbf{LP}/\mathbf{LP}} \dfrac{\Rightarrow \varphi \qquad \varphi \Rightarrow}{\Rightarrow} \\ \nvDash_{\mathbf{LP}/\mathbf{LP}} \dfrac{\Rightarrow \varphi \qquad \Rightarrow \neg\varphi}{\Rightarrow} \end{array}\right. \quad \rightsquigarrow_{\tau_1} \quad \nvDash_{\mathbf{LP}} \quad \varphi, \neg\varphi \Rightarrow \bot$$

$$\left.\begin{array}{l} \vDash_{\mathbf{LP}/\mathbf{LP}} \dfrac{\Rightarrow \varphi}{\Rightarrow \neg\neg\varphi} \\ \nvDash_{\mathbf{LP}/\mathbf{LP}} \dfrac{\Rightarrow \varphi}{\neg\varphi \Rightarrow} \end{array}\right. \quad \rightsquigarrow_{\tau_1} \quad \vDash_{\mathbf{LP}} \quad \varphi \Rightarrow \neg\neg\varphi$$

This means that not only **LP** is not uniform at all, but also, that **LP/LP** (according to τ_1) is neither weaker nor stronger than **LP**, which seems odd. The root of this problem, of course, is the fact that the translation equates negated formulas with contradictions, and **LP**'s negation is too weak for that. A possible solution then is to enrich the language, by adding a stronger negation $\sim$:

	$\sim$
t	**f**
i	**f**
f	**t**

Which can then be used to define a new translation τ_2 by simply replacing $\neg$ by $\sim$ in τ_1. Notice that, in doing so, it is no longer the case that the metainferences above are conflated under translation. The second and third metainferences stay the same under τ_2, but the first one now correlates to $\varphi, \sim\varphi \Rightarrow$, which is duly lp-valid, whereas the fourth one is transformed into $\varphi \Rightarrow \sim\neg\varphi$, which is lp-invalid. In fact, we can prove the more general result that $\mathbf{LP}_\omega$ is τ_2-uniform above level 1 (that is, with respect to its metainferences, not with respect to its inferences and formulas, naturally). Namely, straightforward by induction on the level of the metainference, we obtain:

Fact 4.5. *A valuation lp-satisfies a meta$_n$inference $\Gamma \Rightarrow_n \Delta$ iff it lp-satisfies $\tau_2(\Gamma \Rightarrow_n \Delta)$, for any $n \geqslant 1$.*

Let us then come back to the questions raised in this context by the concept of uniformity, now instantiated via τ_2. With respect to the first one, nothing changes, the answer is still negative because the counterexample for $\mathbf{L}_\omega$ works just as well. However, now the answer to the second question becomes negative too. The reason is that $\mathbf{LP}_\omega$, as we have shown, is not purely paraconsistent, despite being (mostly) uniform. Given that we started out this section by considering the possibility of framing the discussion about paraconsistency as part of a broader discussion about uniformity, this mismatch seems to dismantle the project.

In any case, is τ_2 better than τ_1? On the one hand, at least we know that the concept of uniformity which arises from it has some applicability, given that not only **CL** but also $\mathbf{LP}_\omega$ (and others) are τ_2-uniform. τ_1-uniformity, on the contrary, appears to fully align with (conservative extensions of) classical logic. On the other hand, τ_2 requires an enrichment of the language, since $\sim$ is not part of the standard **LP** vocabulary. And in opposition to what happens with $\bot$ or $\top$, the impact of $\sim$ is more significant: we now have two competing negations which validate different inferences, whereas the negation definable by $\bot$ is the same as $\neg$. This poses the problem of having to choose which one to use to represent *actual* negation. Moreover, as is well known, such a negation is incompatible with one of the most canonical applications of **LP**, which is to build theories that include a naive truth predicate. Hence, which benefits outvalue the downsides is, at least for now, an unsettled issue.

Also, more radically, one may question the virtuousness of uniformity itself. On the one hand, coherence constraints are abundant in epistemology. If somebody already accepted some principles at some level, what could be the reason that they stop holding at higher ones? Even more so, if all these levels are understood as properly *logical*. On the other hand, non uniform logics can be more versatile. For instance, they can be useful to deal with paradoxical vocabulary, while at the same time maintaining the maximum possible strength at lower levels. Then, there might be a tension between uniformity, applicability and minimal mutilation, which might not be obvious how it is better resolved.

5. Conclusion

Throughout this article, we have argued that, when a logic is characterized as a set of metainferences of every level n, it comes in handy to have a concept of contradiction specific to each level, independent of the others (and especially, independent of the more traditional sense of contradiction between formulas). In particular, we expanded BPS's work on the question of what a paraconsistent logic can be by applying the (meta)inferential negation from [Da Ré et al., 2020] to propose a more general take.

As a result, a logic such as $\mathbf{LP}_\omega$ turns out to be paraconsistent because Explosion fails, yet not purely paraconsistent, for Meta_1Explosion is valid. On the other hand, $\mathbf{ST}_\omega$ turns out to be paraconsistent yet not purely paraconsistent for the contrary reasons, Explosion is valid, yet Meta_1Explosion is not. We gave an example of a purely paraconsistent logic, $\mathbf{L}_\omega$ in which no instance of Meta_n Explosion is valid at any level.

Finally, we compared the concept of pure paraconsistency with two elucidations of what it is for a logic to be uniform. As it turned out, pure paraconsistency is a prerequisite for uniformity, if the latter is defined according to some translations (τ_1 in our case), but it is not if one chooses other functions (for instance, τ_2). In any case, $\mathbf{L}_\omega$ is not uniform according to either. There are cases in favor and against of each of these functions, and there might be more worth considering. There is also the open option of defining Uniformity in some way which gets rid of translations altogether (which seems to be quite desirable). We hope to pursue this line of inquiry in future work.

Although in this article we have focused on the notion of paraconsistency, there are many routes yet to be explored. Among them, one particularly interesting is how to define or generalize the notion of paracompleteness to metainferential logics. In the inferential case, a logic is paracomplete if the law of excluded middle ($\Rightarrow \varphi, \neg\varphi$) is not valid (notice that $\mathbf{K_3}$ and $\mathbf{TS}$ are paracomplete). So, one could try to formalize this law for metainferential levels with the aid of the metainferential negation we have been using in this article (without the need of introducing any new vocabulary) and wonder which logics are metainferentially paracomplete or purely paracomplete. We think that many of the results and ideas of this article can be dualized (in some sense, pursuing again the ideas from [Da Ré et al., 2020]) to shed some light on these questions. However, a proper study of these considerations needs to wait for another occasion.

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Bruno Da Ré, Mariela Rubin and Paula Teijeiro
Department of Philosophy
University of Buenos Aires
IIF-SADAF-CONICET
Argentina
{brunohoraciodare,marubin,paulateijeiro@gmail.com