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Minimal Periodic Orbit Structure of 2-Dimensional Homeomorphisms

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Summary. We present a method for estimating the minimal periodic orbit structure, the topological entropy, and a fat representative of the homeomorphism associated with the existence of a finite collection of periodic orbits of an orientation-preserving homeomorphism of the disk D^2. The method focuses on the concept of *fold* and recurrent *bogus transition* and is more direct than existing techniques. In particular, we introduce the notion of complexity to monitor the modification process used to obtain the desired goals. An algorithm implementing the procedure is described and some examples are presented at the end.

Key words. 2-D homeomorphisms of the disk, Thurston classification theorem, minimal periodic orbit structure, topological entropy, pseudo-Anosov representative

1. Introduction

We are interested in 3-D dynamical systems (ODEs) that admit a Poincaré section Σ. Hence, a Poincaré return map can be defined on Σ and the periodic orbit structure can be understood in terms of the periodic points of the Poincaré return map $F : \Sigma \mapsto \Sigma$, which is assumed to be an orientation-preserving homeomorphism. Periodic points of F are in one-to-one correspondence with periodic orbits of the original flow (although it is clear that F admits many different suspensions that can be classified according to their *global torsion* [1]). We will focus in the case where Σ is a topological disk on the plane (an example would be a flow defined on $\mathbf{D^2} \times \mathbf{S}^1$ where the coordinate $\phi \in \mathbf{S}^1$ satisfies $\dot{\phi} > 0$).

Given a periodic orbit (or a collection of several periodic orbits) of the flow, and an order for the p intersection points of the orbit(s) on Σ, one can associate with the orbit an element of the Braid group of p strands in the following way: (a) Project Σ onto the interval $[0, 1]$ in such a way that the order among the p points is preserved, i.e., $0 \leq v_1 < v_2 < \cdots < v_p \leq 1$, (b) project analogously the images of Σ by the time evolution onto a cylindric surface spawned from $[0, 1]$, i.e., $[0, 1] \times S^1$, thus producing strings on the cylindric surface representing portions of the orbit between points v_i and $v_{(i+1) mod(p)}$, (c) keep track of the crossings among pairs of adjacent strings by checking for each t associated with a crossing point on the projection, the signed distance δ on the time-evolved Poincaré surface Σ_t between each involved point and its (common) projection (which are either "left over right" if $\delta(\textit{left}) > \delta(\textit{right})$ or otherwise "left under right"), and (d) recast the cylindric surface as the unit square.

Different choices of Poincaré sections that are equivalent up to conjugation yield conjugated braids associated with a given periodic orbit. Hence, rather than the braid itself, the object that summarizes the dynamical information of a periodic orbit is its equivalence class upon conjugation. This object is called the *braid type* [2].

In contrast with the periodic orbits of 3-D flows, periodic orbits of Poincaré maps do not carry the linking information by themselves. Let P be the set of points belonging to the periodic orbit (or collection of periodic orbits) of F under consideration. The "braid content of the orbit" actually consists of the action of F on $D^2 - P$. Considering F as a Poincaré map, we will in the sequel refer to the *braid (or dynamical) content of the set of periodic orbits* P meaning the braid type of F in $D^2 - P$.

The braid content of a collection of periodic orbits can be read directly on the Poincaré section via the action of F. Let P be the ordered set $\{v_1, v_2, \ldots, v_p\}$. Consider a Jordan curve on Σ joining (in order) the points in P as it is traveled in counterclockwise form. The image by F of this Jordan curve is called the *circle diagram* of the braid. The isotopy equivalence classes of circle diagrams are in one-to-one correspondence with the elements of the braid group quotiented with the global torsions [3]. The braid associated with $\{F, P\}$ by this procedure depends on the choice of the Jordan curve and the ordering of the points $\{v_1, v_2, \ldots, v_p\}$. However, it is not difficult to show [4] that different choices of Jordan curves and orderings of the points are associated with conjugated braids. Hence, the braid type associated with P by the action of F on $D^2 - P$ is independent of the ordering of the points of P and the choice of Jordan curve. In practical applications one uses a standardized Jordan curve obtained by (a) conjugating F so that the points of P lie on a straight line, (b) numbering these points 1 to p from left to right, and (c) choosing the curve as a straight line joining all the points and an arc joining v_p to v_1 counterclockwise.

Hall [5] considered in a similar context the *line diagram*. Given the permutation of P acted by F, there is a one-to-one correspondence between the circle diagram and the line diagram. A line diagram is obtained from the circle diagram by deleting the arc going from v_p to v_1; conversely, the circle diagram is recovered from the line diagram by closing a (topological) circle from v_p to v_1 with a counterclockwise orientation. For simplicity we will mainly use the line diagram in the sequel.

The approach we will present allows us to consider a more general class of starting diagrams—called "trees" below—than just line diagrams. The central question we address in this article is: Given a tree (or in particular a line or circle diagram), which periodic orbits of an orientation-preserving homeomorphism F are necessarily present along with those given by the vertices of the tree? In other words, we aim to obtain a 2-D homeomorphism with the least number of periodic orbits for each period, compatible with the tree.

All answers to the central question rely on Thurston's classification theorem for orientation-preserving homeomorphisms [6], which in our case reads as follows:

Theorem A (Thurston). *Let Σ be compact and P a finite F-invariant set of points. Then F is isotopic to a homeomorphism ϕ on $\Sigma - P$ such that one of the following three cases occur:*

1. *ϕ^n is the identity for some positive integer n (ϕ is said to have finite order).*
2. *ϕ is reducible, i.e., there exists a ϕ-invariant finite set of disjoint closed curves that are not boundary homotopic nor puncture homotopic in $\Sigma - P$.*
3. *ϕ is pseudo-Anosov.*

The simplest homeomorphisms of a disk are rigid rotations. A map whose irreducible components are all of finite-order type will be called a *collection of pure rotations*. The braid structure of homeomorphisms with zero topological entropy can be described as a *family of hereditarily rotation-compatible orbits*, i.e., a finite or infinite sequence of cabled rotations [7]. Gambaudo et al. have shown that the converse result is also true at least for C^1 diffeomorphisms [7].

In the reducible case, we can decompose P in a collection of two or more (irreducible) ϕ^k-invariant sets. In fact, in the case that the points of P belong to just one periodic orbit, for some k, ϕ^k maps each invariant curve onto itself and there are $l = p/k$ points of P within each curve. Hence, reducibility requires p to not be a prime number [2]. Confining ourselves to prime periods, Thurston's theorem reduces to two alternatives: finite order or pseudo-Anosov homeomorphisms. The latter case implies positive topological entropy and the existence of an infinite number of periodic orbits that are not cabled rotations. A simple test on the braid word [2] of the orbit gives a sufficient condition for its being pseudo-Anosov.

From the point of view of dynamics the last case in Thurston's theorem is the most interesting. In fact, pseudo-Anosov maps have many interesting properties that allow us to assess a number of properties of the original (dynamical) map F. For a proper definition of pseudo-Anosov maps, see e.g. [5]. For the present purposes the three properties that are relevant are

1. Let ϕ be a pseudo-Anosov homeomorphism on $D^2 - P$ that maps periodically the punctures of D^2 and let Q be a periodic orbit of ϕ with braid type γ and period q not lying completely in the border of D^2. Then, the number of periodic orbits with braid type γ and period q of any homeomorphism F in the isotopy class of ϕ is greater or equal than the corresponding number for ϕ [5]. The result is not true for orbits lying completely in the border of D^2. This means that since F and ϕ both present the same invariant set P and hence lie in the same class, F has at least the same number of periodic orbits as ϕ for each period $n \geq 1$ with the possible exception of the border orbits (which are a finite number of rigid rotations).
2. The topological entropy of ϕ, $h(\phi)$, is a lower bound to that of F.
3. Pseudo-Anosov maps admit a Markov partition from which $h(\phi)$ can be computed (it is the logarithm of the largest-modulus eigenvalue of the associated Markov matrix) [8].

Hall [5] noticed that certain line diagrams associated with maps belonging to a pseudo-Anosov class can naturally be associated with a *fat representative*, i.e., a 2-D automorphism $\widehat{\theta}$ from which the transition matrix can immediately be read. He develops the concept of *bogus transition*, meaning that some power of $\widehat{\theta}$ induces a horseshoe-like folding on the line diagram that can be removed by isotopies. If a line diagram of a p-periodic orbit does not present bogus transitions for the first $p-1$ powers of $\widehat{\theta}$, then the set of periodic orbits of $\widehat{\theta}$ differs from the corresponding set of ϕ in a finite number of orbits, and both maps have the same topological entropy and Markov matrix.

There are three published algorithms dealing with orbit implication, Markov partitions, and/or topological estimates in the context of Thurston's theorem. The best established implementation of Thurston's results can be found in the paper by Bestvina and Handel [9]. The authors start with a *marked graph* that is a homotopy equivalence of the *rose of p petals* and with a topological representative $\mathcal{F}$ of the map of interest, proceeding then to transform $\mathcal{F}$ until it becomes a *train-track map*. Although this procedure is sufficient to unravel the richness of Thurston's theorem, one may regard as a limitation the fact that the starting point of the process is fixed (what in our context would be equivalent to starting the process with one given standard diagram). In a later manuscript [10], this condition is lifted. In fact, the algorithm of Bestvina and Handel [9] is more general than ours (it can be used for any surface of negative Euler characteristic) but also more complicated since it requires "valence-2 homotopies" and the concept of "peripheral subgraph."

The algorithm by Los [11] relies on "valence-three graphs" (the concept of valence is disccussed in the next section), and moreover it lacks a systematic monitoring of its evolution: One has to test the outcome of the algorithm on a number of (conjugated) representatives $\mathcal{F}$.

Finally, the algorithm of Franks and Misiurewicz [12] is the inspiration for our work. It is worth mentioning that Franks and Misiurewicz take advantage of the work performed by Bestvina and Handel; hence, in some sense it is an elaboration of this pioneering work. The present work can also be viewed as a further elaboration of [12]. Franks and Misiurewicz developed an algorithm with about 10 steps with which from any starting diagram containing the invariant set P as vertex points one can produce an associated structure having the least topological entropy (meaning that the structure induces a Markov partition for ϕ from which the topological entropy $h(\phi)$ can be computed). Their algorithm proceeds by testing different modifications of their diagrams (adding vertices, merging adjacent segments, or splitting a vertex) until a standardized structure is obtained. The drawbacks are that it provides no systematics in the application of the individual moves and it is unclear if all moves are necessary, as the authors state in their work.

The identification of the braid content of a return map is relevant also for natural sciences. An extensive program for the characterization of experimental data and the validation of proposed models [13] has been in the process of being developed since the late 80's. For such matters, more relevant than the topological entropy is the production of a fat representative[5], [12] of the return map.

The goal of this work is to merge the approaches of [5] (after suitable generalization) and [12], producing a simpler algorithm where the steps to understanding the orbit implications given by the set P and to producing the lowest-entropy diagram are guided by the identification and elimination of a generalized type of bogus transitions. Our algorithm is simpler than that of Misiurewicz and Franks [12] in that (using their language) only "gluing," "collapsing," and homotopies are needed. We avoid the move called "dragging," which is the equivalent to the "valence-2 homotopies" in [10], as well as "splitting," which is the inverse of gluing.

The basic ideas of this manuscript were outlined in 1997. In the course of writing, rewriting, and reviewing the manuscript, we became aware of two newer articles on the subject, namely [14] and [15]. The first one presents an improvement on [12] that deals with a better understanding of their *splittings* and is therefore not directly related to this work since we avoid Franks and Misiurewicz's splittings completely. The second one has many contact points with this manuscript and with [12], since similar fat representatives, collapses, and splittings are present. We will defer a comment on it until the final section.

In Section 2 we define the main tools, in the following sections we present the supporting results and describe the algorithm, and the final sections are devoted to examples and discussion.

Reading Suggestions. For the reader who wants to use the algorithm and can leave the details of the proof for a second lecture, it might be enough to read the definitions of *fold, fat representative, crossing,* and *bogus transition* in Section 2 and those of *preimage of a fold* (*PF* and the related PF_i), *collapse of a bogus transition*, and *exhaustion* as well as Lemma 6 (Lemma 10 invoked in the algorithm is a refinement of the more intuitive Lemma 6) in Section 4 before going to the algorithm description at the end of Section 4. Those readers should note that the examples in Section 5 are a mixture of of "usage" and "proof verification." For the mathematically oriented reader interested in understanding how the procedure works, the whole manuscript is of course necessary, but the key concepts are those of step and complexity in Section 4, while the collapsing procedure is motivated by the elimination of portions of phase space discussed in Theorem 1.

2. Elements of the Description

We formalize here the relevant parts of the above discussion.

2.1. Trees and Standard Maps

Let F be an orientation-preserving homeomorphism of the disk $D^2 \subset \mathbf{R}^2$, and let $P = \{v_1, \ldots, v_p\}$ be a finite F-invariant set with a given (arbitrarily chosen) numbering of its points. After possibly conjugating F, without loss of generality we can assume that the points of P lie on a (horizontal) straight line on D^2 with the canonical ordering.

Definition. Consider a counterclockwise Jordan curve joining (in order) the points $\{v_1, \ldots, v_p\}$ by straight lines and v_p to v_1 with an arc. The image by F of such curve is called a *circle diagram*, $\mathcal{C}$.

Definition. The Jordan arc from $F(v_1)$ to $F(v_p)$ of a circle diagram (i.e., removing the image of the arc $v_p \to v_1$) is called a *line diagram*, $\mathcal{L}$.

The preimage $\mathcal{L}_0$ of the line diagram (which can be taken to be a horizontal straight line), will be of use below.

Theorem B (NS [3]). *The isotopy equivalence classes of circle diagrams are in one-to-one correspondence with the group $B_p/Z(B_p)$, i.e., the braid group of p strands quotiented with its center, $Z(B_p)$, corresponding to the full-twists or global torsions.*

This equivalence is more refined than just on braid types. The whole braid group quotiented with its center is one-to-one with the circle diagrams. Braids within a given braid type differing in a conjugation that is not a global torsion will have different diagrams.

It is clear that the circle diagram isotopy equivalence classes can be put in one-to-one correspondence with line diagram isotopy equivalence classes, so the above theorem is valid for line diagrams as well.

Definition (Tree). A tree is a connected finite 1-D CW-complex that does not contain any subset homeomorphic to a circle [12]. In simpler terms, consider a set P of periodic points. Join the points with nonintersecting straight line segments in such a way that no loops are formed. We call the resulting graph a *tree*, the points of P are called *vertices*, and the line segments are called *edges*.

The number of edges emerging from a vertex is called the *valence* of the vertex. $\mathcal{L}_0$ is a good example of a tree, having vertices of valence 2 and 1 (the endpoints).

We need to define a "standard" map that hosts the given periodic orbit and tree. Following Franks and Misiurewicz [12], we let $\pi : D^2 \mapsto T$ be a projection with the following properties:

(a) π is continuous and onto.
(b) π maps the points of P bijectively onto a subset of the vertices of T (which includes all endpoints of T).
(c) For every vertex v of T, $\pi^{-1}(v)$ is a closed disk.
(d) For every $p \in P$, $p \in Int(\pi^{-1}(\pi(p)))$.
(e) For every *open* edge (i.e., without the endpoints) e of T, there is a homeomorphism H_e of $e \times [0, 1]$ such that $\pi \circ H_e$ is the projection onto the first coordinate.
(f) If e_1, e_2 are distinct open edges of T, then the closures of $\pi^{-1}(e_1)$ and $\pi^{-1}(e_2)$ are disjoint.

There is a natural Markov partition of T taking the segments joining the points of P (edges) as units. This partition induces a corresponding transition matrix for $\pi(F(\cdot))$, which we will call M_0. The matrix element $\{M_0\}_{ij}$ is a nonnegative integer indicating the number of times the edge i is mapped over the edge j by $\pi(F(\cdot))$.

The definition of π suggests that one can recast the disk D^2 as a collection of rectangles and disks forming a thickened tree. Such disks and rectangles will be called *fat vertices*

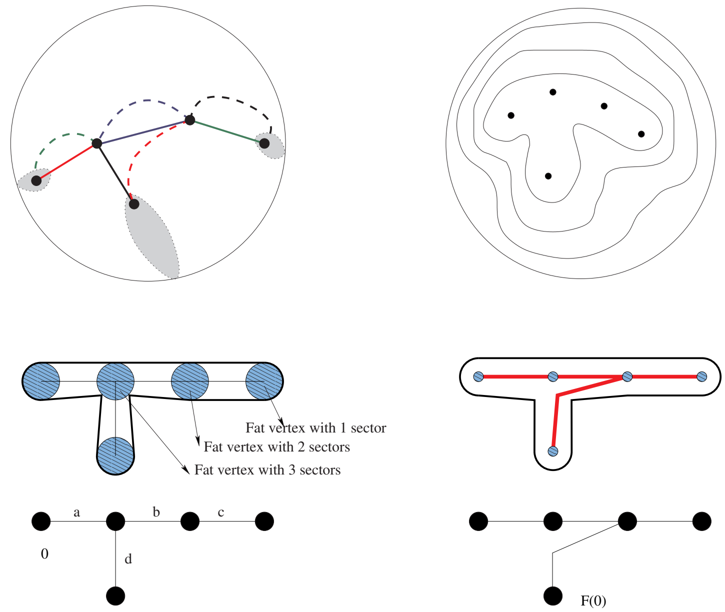

Fig. 1. A tree, its image by F, along with $\widehat{T}$, its induced partition, and its image by $\widehat{\theta}$ (see below for a definition of the map $\widehat{\theta}$). The point v_1 of P has the label 0. Sectors are illustrated as well.

and *fat edges* respectively. In this sense, $\pi : D^2 \mapsto T$ defines a *thick tree structure of* (D, P) *over* (T, P) [12].

Given for example, five periodic points on the disk, one may construct many different trees. Which one to start with is a matter of choice; it is the final result of the algorithm that provides a unique answer in terms of minimal topological entropy. Figure 1 illustrates the construction of a tree for a map of the disk with a periodic orbit of period 5. In the first row of the figure, the choice of tree is shown along with its image by F, as well as the modifications of the border of the circle along the projection π. Full lines indicate the tree and dotted lines its image by F (same colour for each edge and its respective image). The choice of endpoints is illustrated by the shaded circles, i.e., the border of the disk is partitioned via the endpoints, thus determining the labeling of the different components of the tree (see below). We will use this idea to define a standard map on the disk inheriting the properties of F.

Let $\widehat{T}$ be the topological disk obtained from T by means of a suitable choice of π^{-1}. Consider the tree T as a point set embedded in $\widehat{T}$. Everyfat vertex of valence k of $\widehat{T}$ is divided by T in k connected subsets that we will term *sectors*. (The boundary of each sector contains only one vertex in T and portion(s) of edge(s) of T at that vertex. We will consider that the boundary belongs to the sector whenever necessary.)

See the second and third rows of Figure 1 for an illustration of the concept of sector partition, for a tree with three endpoints and four fat-edges labeled a, b, c, d. We will use the same labels for edges and fat-edges and for vertices and fat vertices when no confusion arises.

Definition (Fat Representative). Let $\widehat{T}$ be the topological disk obtained from T by means of a suitable choice of π^{-1}. We define the *fat representative* $\widehat{\theta}$ of F [5] as a map $\widehat{\theta} : \widehat{T} \to \widehat{T}$ with the following properties:

1. $\widehat{\theta}$ is one-to-one and continuous.
2. $\widehat{\theta}(\widehat{T}) \subset int(\widehat{T})$.
3. $\widehat{\theta}$ coincides with F on P.
4. $\widehat{\theta}(T)$ is homotopically equivalent to $F(T)$ on $\widehat{T} - P$.
5. The image by $\widehat{\theta}$ of a fat vertex is contained in the interior of a fat vertex.
6. Given r belonging to an open edge of T, then for all t such that $\pi(t) = r$, $\pi(\widehat{\theta}(t)) = \pi(\widehat{\theta}(r))$ and, moreover, $|\widehat{\theta}(r) - \widehat{\theta}(t)| = k|r - t|$, for some positive $k < 1$. k is constant on each open edge.

The existence of a map $\widehat{\theta}$ with the proposed properties results from the following observations: First, consider $F(T)$ as a collection of segments with endpoints in P. Then, it is always possible to produce a tight model of $F(T)$, i.e., with lines parallel to the edges in T along the fat edges by applying suitable homotopies to $F|_T$ (F restricted to T). We can name such a map $\widehat{\theta}|_T : T \to \widehat{T}$. Additionally, we require that $\widehat{\theta}|_T$ never contracts. Secondly, the map $\widehat{\theta}|_T$ can be extended to the fat edges as a map of a set of disjoint rectangles that contracts uniformly by a factor k in one direction and expands as $\widehat{\theta}|_T$ along the perpendicular direction with stable and unstable foliations in coincidence with the (local) Cartesian coordinates of the rectangle. Such a map coincides with the restriction of $\widehat{\theta}$ to the union of fat edges.

Finally, in identical form, the image of a fat vertex is a fattened version of the restriction of $\widehat{\theta}|_T$ to the corresponding vertex in such a way that the "gaps" between images of edges are filled and $\widehat{\theta}$ maps $\widehat{T}$ continuously and injectively on its interior.

We define the projected map $\theta : T \to T$ as $\theta(r) = \pi(\widehat{\theta}(r))$. This map will be considered repeatedly in the rest of this paper.

In Figure 1 we show a tree T and its image by F (first row) and the corresponding $\widehat{T}$ with thickened edges and vertices (second row) along with the image of $\widehat{T}$ by $\widehat{\theta}$ (third row), which is purposely drawn within the original $\widehat{T}$.

2.2. Folds and Bogus Transitions

A central concept in our understanding of the problem is that of "folding point" or "fold."

Definition (Fold). Let v' be a vertex of T and v the vertex of T that is the unique vertex preimage of v' by $\widehat{\theta}$. We say that θ has a *fold* f at v' whenever θ is not one-to-one restricted to any small neighborhood of v. We say that $\widehat{\theta}$ has a fold at v' whenever θ has a fold at v'. We count one fold for every pair of contiguous edges at v with the same image by θ locally around v'.

These two contiguous edges at v define a unique sector $x(v)$ in $\widehat{T}$. We will call *fold* the subset of $\widehat{\theta}(\widehat{T})$ given by $\widehat{\theta}(x(v))$. The fold has a *border* in $\widehat{\theta}(T)$ given by $\widehat{\theta}(x(v) \cap T)$, and a *local interior* in $\widehat{\theta}(\widehat{T})$ that is the complement of the border in the fold (the fold "minus" its border). θ maps $\{x(v) \cap T - \{\pi(v)\}\}$ two-to-one onto a portion of one edge at v'.

Some regions in $\widehat{T}$ will stretch, fold, and map onto themselves by some power of $\widehat{\theta}$ as a consequence of the existence of the invariant set P. Some of these foldings may be unavoidable, but others may be avoided via homotopies that collapse the whole stretched and folded region and all its preimages to a point. Our next goal is to identify these regions.

Definition (Fold Preimage Set). Let $\widehat{\theta}$ have a fold f at v and let x be the sector at the preimage of v mapping onto the local interior of the fold by $\widehat{\theta}$. We define the set of *fold preimages* $PI(f)$ having sectors as elements as follows: $x \in PI(f)$, and in addition $y \in PI(f)$ iff $y \cap T$ maps (locally) one-to-one by θ^k onto $x \cap T$, for $k \geq 1$. Note that a sector cannot be associated with more than one fold, and the sector at an endpoint cannot belong to $PI(f)$ since it cannot be mapped by θ one-to-one and onto the local part of T at a valence-m vertex with $m > 1$ in the way prescribed above. We will call $PI(\widehat{\theta}) = \cup_f PI(f)$, the set of all the sectors associated with folds in the map.

Definition (Crossings). Consider an open edge e and its image by $\widehat{\theta}$. If we can divide e in three consecutive nonempty portions e_0, e_1, e_2 such that $\widehat{\theta}(e_i), i = 0, \ldots, 2$ intersect three consecutive elements (sectors or edges) of the tree, we will say that $\widehat{\theta}(e)$ **crosses** the second intersected element (the one corresponding to e_1). Notice that if $\widehat{\theta}(e)$ crosses an edge, the edge portions e_0, e_2 intersect sectors, since edges connect sectors. See for example, Figure 2. The image of the edge joining vertices 2 and 3 crosses the fat edge 2–3, two sectors at vertex 3 (labeled below as $3D$ and $3R$), and the fat edge 3–5. Also, the image of the edge 1–2 crosses the fat edge 3–5, sector $3R$, and fat edge 3–4.

The expressions "an edge maps along..." and "an edge maps over..." used above when discussing Markov partitions and edges can easily be restated in terms of crossings.

In more general terms, consider a connected region of the fat tree composed of successive sectors (or unions of consecutive sectors) and fat edges, h_i $i \in 1, \ldots, k$, then $\widehat{\theta}(e)$ crosses the region if there are adjacent nonempty portions of e, e_i $i = 0, \ldots, k+1$, such that $\widehat{\theta}(e_0)$ and $\widehat{\theta}(e_{k+1})$ cross elements of the tree (fat edges or sectors) adjacent to the region considered while $\widehat{\theta}(e_i)$ crosses the element h_i.

Definition (Bogus Transition). Consider the set of *fold crossings* $CR(\widehat{\theta})$ indicating which sectors or unions of consecutive sectors associated with the points P are crossed by the image by $\widehat{\theta}$ of an edge of T. The orbit by $\widehat{\theta}$ of the elements in CR consists of a sequence of sectors or union of consecutive sectors that could either map into one or more folds in a finite number of steps or be infinite. In the same way, the orbit by θ of the border of these sectors in T either is two-to-one after a finite number of steps (in which case we say that the orbit *terminates* in the fold) or keeps being one-to-one for any number of iterates. We say that the tree T has a *bogus transition* at all the folds lying in the forward image by $\widehat{\theta}$ of an element of $CR(\widehat{\theta})$ whose orbit *terminates*, in the present sense.

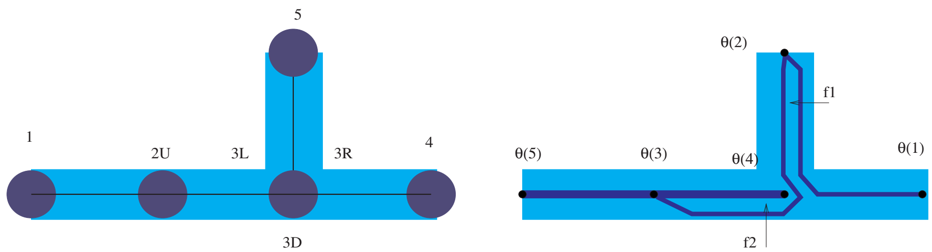

Fig. 2. A tree with bogus transitions at 5 and 2. The fold $f1$ is the image of $2U$ while $f2$ is the image of $3D$. $CR = \{3D + 3R, 3R\}$, $PI(f1) = \{3R, 2U\}$, and $PI(f2) = \{3D\}$. The orbit of $3R$ is $3R \to 2U \to f1$; hence $f1$ has a bogus transition and $BT(f1) = PI(f1)$; the orbit of $3D + 3R$ is $3D + 3R \to f2 + 2U \to f1$ and then besides $f1$, $f2$ also has a bogus transition. $BT(f2) = PI(f2)$ since $3D \cap (3D + 3R) = 3D$.

The set $PI(f)$ has a natural order given by $\widehat{\theta}$. We give to the sector x the label n (which is the cardinality of $PI(f)$) and the remaining sectors in $PI(f)$ are ordered in such a way that x^i maps by $\widehat{\theta}$ onto x^{i+1} for $i = 1, \dots, n-1$. Hence, the element x^k maps (for the first time) into the interior of the fold after $n - k + 1$ iterations of $\widehat{\theta}$.

We introduce the set $BT(f)$ for future use. For each fold with a bogus transition, $BT(f)$ is the subset of $PI(f)$, with the natural order given by $\widehat{\theta}$, that has nonempty intersection with the forward image of the elements of $CR(\widehat{\theta})$. $BT(f)$ indicates the sectors where tree modifications will be necessary. If this set is empty, there are no bogus transitions associated with f. We will abuse notation often in the sequel and regard $PI(f)$, $CR(\widehat{\theta})$, and $BT(f)$ as the sets of associated vertices rather than sectors.

We illustrate the definition of bogus transition in Figure 2. $\widehat{\theta}$ has two folds: $f1$ at vertex 5, which is the image of vertex 2, and $f2$ at vertex 2, which is the image of 3. The sector $2U$ (at vertex 2) maps on the local interior of the fold $f1$. We have that $CR = \{3D + 3R, 3R\}$, $PI(f1) = \{3R, 2U\}$, and $PI(f2) = \{3D\}$. The orbit of $3R$ is $3R \to 2U \to f1$; hence $f1$ has a bogus transition and $BT(f1) = PI(f1)$; the orbit of $3D + 3R$ is $3D + 3R \to f2 + 2U \to f1$ and then besides $f1$, $f2$ also has a bogus transition. $BT(f2) = PI(f2)$ since $3D \cap (3D + 3R) = 3D$.

Let ϕ denote the collection of irreducible components of a map conjugate to F according to Thurston's theorem (in the irreducible case, ϕ is just one map: the pseudo-Anosov or pure rotation conjugate map to F).

Definition. We say that $\widehat{\theta}$ has *minimal periodic orbit structure* in the isotopy class of F [5] whenever it has the same number of orbits as ϕ on the interior of D^2 for all braid types plus at most a finite number of orbits of the same braid type as P (same braid type as the irreducible components associated with P) and if the orbits of $\widehat{\theta}$ on ∂D^2 differ from those of ϕ in a finite number of rigid rotations.

Theorem C (Hall [5]). *For F, P, ϕ, and $\widehat{\theta}$ belonging to a pseudo-Anosov isotopy class and all defined as above, the line diagram of P has no bogus transitions if and only if $\widehat{\theta}$ has minimal periodic orbit structure. Moreover, the transition matrix of $\widehat{\theta}$ is*

Perron-Frobenius (see below for a definition) and the logarithm of its largest-modulus eigenvalue is a lower bound for the topological entropy of F.

The concept of bogus transition developed by Hall in [5] is closely related to the concept of *gluing reduction possibility* (GRP) of Franks and Misiurewicz [12, p. 83]. For P in the irreducible case, the absence of GRP in a tree is enough to warrant that the entropy is minimal (see [12]); however, some tree maps presenting bogus transitions also have minimal entropy. It is actually not difficult to find line diagrams with zero associated entropy that present bogus transitions.

2.3. Recurrence

In order to have periodic orbits, it is necessary to have some kind of recurrence in $\widehat{\theta}$ (some regions of $\widehat{T}$ that return onto themselves). A sufficient condition for recurrence is to have a transition matrix M_0 (all entries of M_0 are nonnegative) such that for every pair i, j there exists an $m \geq 1$ such that $(M_0^m)_{ij} > 0$. We call such a matrix, and the corresponding map θ, *transitive*. If some power of M_0 has all entries strictly positive, the matrix is called *Perron-Frobenius*. Such matrices have a largest-modulus eigenvalue $\lambda > 1$ with multiplicity 1. In particular, a transitive matrix with positive trace is Perron-Frobenius.

A concept related to transitivity is that of matrix reducibility. A reducible matrix (in the sense of matrices, hereafter called *matrix reducible*) implies the existence of a proper subset of edges that maps within itself. Then, M_0 can be written in such a way that it has a non-diagonal block identically zero. Matrix-reducible matrices are not transitive.

Necessary conditions for having a map with positive topological entropy are (a) recurrence, in order to have periodic orbits, and (b) expansivity, in order to have folds. Folds will eventually be involved in horseshoe-like formations in some power of $\widehat{\theta}$. By *expansive* we mean a map θ such that at least one edge maps onto two or more edges (or onto the same edge twice). In terms of M_0 at least one row has two or more nonzero entries (or some entry larger than one). A map $\widehat{\theta}$ that has an associated expansive map θ will also be called expansive.

The presence of a bogus transition indicates the possible existence of an infinite set of periodic orbits that can be removed by a suitable homotopy. The actual existence of this removable set of orbits depends on the bogus transition being recurrent. This will be the basic ingredient of Theorem 1.

Definition (Recurrent Bogus Transition). Let θ (and consequently $\widehat{\theta}$) have a fold f at v' and let v be the unique vertex preimage of v'. Further, let a and b be the consecutive edges at v that will map (via some positive power k of θ) onto the edge bt at v'. Finally let x denote the sector at v that maps onto the local interior of the fold. We say that a bogus transition is *recurrent* whenever there exists $m > 0$ such that the following three conditions hold: (i) $\widehat{\theta}^m(bt) \cap x \neq \emptyset$; (ii) $\theta^m(bt) \cap a \neq \emptyset$; and (iii) $\theta^m(bt) \cap b \neq \emptyset$, in other words: $\widehat{\theta}(bt)$ crosses x. It follows immediately that $a \subseteq \theta^m(bt) \cap a$ and $b \subseteq \theta^m(bt) \cap b$.

3. Supporting Results

Throughout this section, let θ be a tight tree map.

Lemma 1. *θ has folds if and only if θ is expansive.*

Proof. We will use throughout that θ is onto.

If θ is not expansive, each edge maps onto just one edge. No two edges can map onto the same edge, and hence θ has no folds.

We shall now prove that there is a contradiction between θ being expansive and θ having no folds.

Since $\widehat{\theta}$ is expansive, there is at least one fat edge such that its intersection with $\widehat{\theta}(T)$ consists of more than one edge portion. We also observe that if $\widehat{\theta}$ has no folds, every valence-k vertex is mapped onto a valence-k vertex and, additionally, all the edges locally at the vertex are mapped one-to-one (locally) into edges of the image.

Let e be an edge of T with several preimages; call v one of the end vertices of e and $a \neq e$ the closest edge portion of $\widehat{\theta}(T)$ crossing the fat edge E, where $\pi(E) = e$. We shall further consider the point $r \in a$ at the border of $E \cap a$ such that $\pi(r) = v$.

Since the tree is connected, there is a unique oriented path between the preimage of v in T and the preimage of r in T. The image of this path is a path in $\widehat{\theta}(T)$ that begins and ends at the same fat vertex of T. Since the image-path is almost a closed loop, there must be at least one "turning point" along it—let us call it t—and $\pi(t)$ must certainly be a vertex. Assume for the moment that t is not a vertex, since otherwise there is a fold at t.

If there are no folds, the image path on $\widehat{\theta}(T)$ must proceed from one branch at a vertex to a consecutive branch considered in the cyclic order of the edges at the vertex. Hence, the only possibility for a path to turn back into the same fat-vertex is to wind around an endpoint of the tree where the only edge that reaches the endpoint in T is also its consecutive edge.

So far we have shown that if the map is expansive and has no folds, there is an edge with one of its associated vertices being an endpoint that has several preimages. We can now proceed to draw $\widehat{\theta}(T)$ with images of simple paths of the form described above but starting from an endpoint (i.e., the endpoint rounded in the previous step). Each path will reveal the existence of at least another endpoint that does not belong to the path. Since the number of endpoints is finite, this process must terminate, but the process requires the existence of yet one more disconnected endpoint to go around. It follows that it is impossible to have a tree $\widehat{\theta}(T)$ without folds for an expansive map $\widehat{\theta}$. □

Lemma 2. *θ has no folds if and only if θ acts as a permutation on the set of edges. Furthermore, if M_0 is M-irreducible, then the permutation is cyclic.*

Proof. It is clear that a map that permutes edges cannot have folds since no column of M_0 can have in such case more than one nonzero entry as is required by expansivity. On the other hand, if the map has no folds, by Lemma 1 it is not expansive; hence each line of M_0 has only one nonzero entry, which in addition is equal to one. Considering that $\theta(T) = T$, we see that no column of M_0 can have all entries equal to zero. Hence, M_0 has as many nonzero entries as there are edges in T, these entries are equal to one, and there is exactly one nonzero entry for each column, i.e., the matrix is a permutation matrix. It is also clear that if the permutation is not cyclic, then it can be decomposed in two (or more) cyclic permutations and hence its matrix cannot be transitive. □

Lemma 3. (i) *If ${M_0}^k$ is not transitive for some $k \geq 1$, then $\widehat{\theta}$ is reducible in the sense of Thurston's classification theorem or it is a collection of pure rotations.* (ii) *If $\widehat{\theta}$ is irreducible and expansive, then M_0 is Perron-Frobenius.*

Proof. (i) Let k be the least integer such that ${M_0}^k$ is not transitive.

Then there exists at least one invariant set $Y \subset T$ consisting of unions of edges such that $\theta^k(Y) = Y$ and $Y \neq T$. We shall consider X to be the union of all such minimal (i.e., with no proper invariant subsets) invariant sets.

Consider first the case when $X \neq T$. Decompose X in connected components $\{X_i\}$, $i = 1, \ldots, n$, and it is clear that $\theta^m(X_i) \subseteq X_j$ for some $1 \leq j \leq n$ and that each X_i is the image of one and only one X_j. θ permutes the sets $\{X_i\}$. Then, $\theta^m(X_i) \cap X_i = \emptyset$ for $m = 1, \ldots, k-1$, since otherwise M_0^m would be matrix reducible for some $m < k$.

In this case, the essential curves required by the reducible case of Thurston's theorem encompass the component(s) of X. Note that the curves are not puncture homotopic since the component(s) of X are unions of edges and hence contain at least two vertices. An example of this situation can be read in Figure 3.

Secondly, consider the case when $X = T$ and decompose X in its minimal invariant subsets under θ^k: $\{X_i\}$, $i = 1, \ldots, n$. We have that $n > 1$, since otherwise T is the minimal invariant subset of θ^k, which is a contradiction (recall X is the union of sets such that $\theta^k(Y) = Y$ and $Y \neq T$). Moreover, $\theta^m(X_i) \cap X_i$ for $m = 1, \ldots, k-1$, is at most one point, since $\theta^m(X_i)$ is invariant and minimal. Hence θ cyclically permutes the sets $\theta^m(X_i)$ with $m = 0, \ldots, k-1$, and there are n/k such orbits of θ.

The sets X_i consist of unions of closed edges (including vertices), and hence they do intersect since T is connected. The orbit of the intersection point has period q, where q divides k. Moreover, $q = 1$, since if $k > q > 1$, we have that k is not minimal, and if $q = k$, then T is not a tree (since in such a case there would be a loop in T), in either case contradicting the hypothesis.

If each X_i contains just one point of P (other than the common point), then T is an *n-star* (i.e., a tree consisting of one central vertex of valence n, and n vertices of valence 1, each joined to the central vertex by a corresponding edge) and θ is a collection of rotations of period k with a common center. Otherwise, consider the set of Jordan curves obtained as curves that encompass each set X_i minus their intersection in a periodic point, we are again in the reducible case of Thurston or in the presence of a *k-star*.

Fig. 3. Reducible case. The rightmost and leftmost edges form an invariant set and ${M_0}^2$ is not transitive. Then, by Lemma 3, $\widehat{\theta}$ is reducible, being the decomposing system of closed loops homotopic to 0–2 and 1–3 in $D - \{P\}$.

(ii) Since θ is irreducible by hypothesis and is not a cyclic rotation (because of expansivity and Lemmas 1 and 2), then by part (i) ${M_0}^k$ is transitive for all $k \geq 1$. Hence, there exists l such that $Tr({M_0}^l) > 0$ and then ${M_0}^l$ is Perron-Frobenius (and therefore also M_0). □

When we are in the reducible case, by Lemma 3, there exists a minimal integer k such that θ^k leaves all X_i invariant; see Figure 3. Hence, we can induce two (or more) irreducible "sub"-trees and corresponding fat representatives in the following way: (a) the tree corresponding to the θ^k-invariant subset X_i, with map $\pi^{-1} \circ \theta^k$ restricted to X_i, and (b) the tree obtained by collapsing each X_i to a point in T (via a projection μ), with map $\pi^{-1} \circ \mu \circ \theta$.

Lemma 4. *There is a one-to-one relationship between the periodic points in $\widehat{\theta}$ and the periodic points in the fat representatives corresponding to the factor(s) of T, except for the periodic points of $\pi^{-1} \circ \mu \circ \theta$ that correspond to $\{\mu(X_i)\}$ which have no counterpart in $\widehat{\theta}$.*

Proof. No periodic orbits of $\widehat{\theta}$ belong to both $\cup_i(\pi^{-1}(X_i))$ and its complement in $\widehat{T}$. Hence they will belong to the fat representative of one of the factors. In addition, since the sets X_i are also represented (by points) in $\mu(\widehat{T})$, the map $\pi^{-1} \circ \mu \circ \theta$ will have a finite number of extra periodic points corresponding to this set of points. □

We can now extend the concept of *minimal periodic orbit structure* to the reducible case. We say that $\widehat{\theta}$ has *minimal periodic orbit structure* if the induced fat representative maps $\widehat{\theta_i}$ of each one of the irreducible factors of T have minimal periodic orbit structure.

Theorem 1. *An expansive $\widehat{\theta}$ presents no recurrent bogus transitions if and only if it has minimal periodic orbit structure.*

Proof. If $\widehat{\theta}$ is irreducible and has no recurrent bogus transitions, then it has no bogus transitions, since by Lemma 3(ii) M_0 is Perron-Frobenius and hence all bogus transitions are recurrent. Hence , by [12][Th.10.1 and corollaries, p. 108], there is a one-to-one relationship between the orbits of $\widehat{\theta}$, θ and a pseudo-Anosov map of $D^2 - P$ (except for a finite set of orbits either of the same braid type as P or lying entirely on ∂D^2). Hence, $\widehat{\theta}$ has minimal periodic orbit structure. If $\widehat{\theta}$ is reducible and has nonrecurrent bogus transitions, recall that by Lemma 4 there are no periodic orbits associated with the bogus transition, since all periodic orbits belong to the irreducible factors. The fat representative of each of the factors of T is irreducible and has no bogus transitions (otherwise $\widehat{\theta}$ would have recurrent bogus transitions). Hence, it has minimal periodic orbit structure and, by Lemma 4 again, $\widehat{\theta}$ has minimal periodic orbit structure.

We claim now that if $\widehat{\theta}$ has a recurrent bogus transition, then it does not have minimal periodic orbit structure since there is another map in its isotopy class having fewer orbits of infinitely many periods. The proof of this claim completes the proof of this theorem.

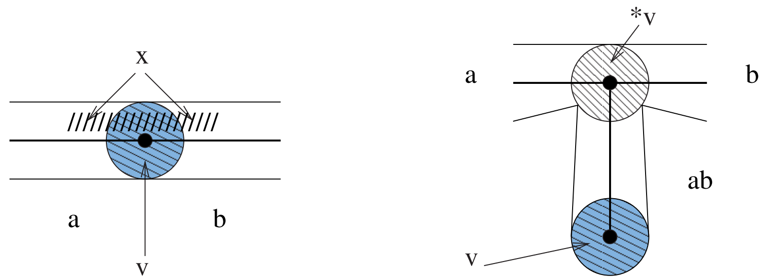

Fig. 4. The collapse process. The shaded part of fat edges a and b disappears and a new vertex is added at the end of the collapse. The corresponding parts of a and b build a new fat edge ab.

To prove the claim, note the following facts:

1. We may choose θ so that it never contracts; hence $\widehat{\theta}$ is expanding along edges.
2. The edges of T form a basis for the symbolic dynamics in T.
3. The periodic orbits of $\widehat{\theta}$ in the interior of $\widehat{T}$ are in one-to-one correspondence with the periodic orbits of θ.
4. Let a, b, bt, v, x, v', f, k, and m be as in the definition of recurrent bogus transition. Then, $\widehat{\theta}^k(x) \subset \pi^{-1}(bt)$, and $\widehat{\theta}^m(\pi^{-1}(bt))$ crosses $\pi^{-1}(a \cup b)$. This fact holds for a larger portion of $\widehat{T}$ than just x. In fact, the sector x can actually be extended along a and b to $\pi^{-1}([\alpha, \beta])$ where $\alpha \in a$, $\beta \in b$, and $\theta^k(\alpha) = \theta^k(\beta)$ is the endpoint of bt different from v'. Since $\theta^m(v') \neq v$, by the recurrence condition on the bogus transition we have that $\widehat{\theta}^{k+m}$ applied to $\pi^{-1}([\alpha, \beta])$ is a horseshoe map.
5. This horseshoe can be eliminated by identifying in T $a \cap x$ and $b \cap x$ and all their $k-1$ forward images by θ (the k-th image was already identified by θ). We illustrate this process in Figure 4. Further details will be given in Section 4.
6. After identification the new map does not have the horseshoe orbits, and it still lies in the isotopy class of the original map. Hence, the original map did not have minimal periodic orbit structure. □

The process of adding vertices in order to eliminate bogus transitions will be constructed and described in the following section. One consequence of it will be that instead of the original set of punctures P, we will in the sequel consider an extended set V consisting of P and the added vertices. Whenever the algorithm produces a reducible $\widehat{\theta}$ that decomposes T in invariant proper subsets each containing only one vertex of P and the same positive number of vertices of $V - P$, we will proceed to collapse the added sections (vertices and edges) to the associated point of P in order to free T from such somewhat artificial constructs. In this way, we will systematically avoid the existence of loops homotopic (in the sense of Thurston's theorem) to the "punctures" in $V - P$.

4. Algorithm

The relevant task to understand the periodic orbit structure associated with a map F and an F-invariant set P is to transform the initial tree T of P into a tree such that its fat representative $\widehat{\theta}$ does not have recurrent bogus transitions. In such a case $\widehat{\theta}$ is either finite order or essentially the pseudo-Anosov representative ϕ of the isotopy class of F (apart from trivial identifications in the reducible case).

The basic idea on how to proceed is given by Lemma 4 and Theorem 1. The goal is to obtain a tree without recurrent bogus transitions. The guideline for the algorithm is to detect all folds with recurrent bogus transitions and to perform continuous deformations of the fat tree identifying portions of $\widehat{T}$ for each relevant fold in order to eliminate its recurrent bogus transitions.

We first state the definitions and lemmas that contribute to the goal and then end this section by stating the algorithm.

4.1. Construction of the Algorithm

Before we proceed with the proof of several lemmas and the proof that the algorithm always ends in a finite number of steps, we need to define the notion of extension of the fold, since we will have to deal with *fold exhaustions*.

In what follows, we will be modifying, giving increasing precision, the fat representative $\widehat{\theta}$ by incorporating some periodic orbits (and *eventually periodic orbits* for intermediate steps) to the original set P. Let us define V as the set of all vertices of a tree. P is then a subset of V. In the coming figures, added vertices will be drawn in white in order to easily distinguish them from the elements of P. The notions of fat tree, fold, bogus transition, and recurrent bogus transition translate directly from replacing P by V.

4.1.1. Interior, Extension, Collapse, and Exhaustion

Definition (Preimage of a Fold). Let $\widehat{\theta}$ have a fold f at v'. The two (adjacent) folding edges at the point v, the unique vertex preimage of v', define two branches on the tree T.

Consider the sector $x(f)$ associated by $\widehat{\theta}$ with the local interior of the fold discussed in the definition of *fold*. Let $A(f)$ and $B(f)$ be the extreme points of the arc belonging to the border of the fat tree at the sector $x(f)$, $\partial\widehat{T} \cap x(f)$. Further, consider $\alpha(f) = \pi(A(f))$ and $\beta(f) = \pi(B(f))$ and the transversal arcs $A(f) - \alpha(f)$ and $B(f) - \beta(f)$. We have that $\theta(\alpha(f)) = \theta(\beta(f))$.

The connected region limited by the arc in $\partial\widehat{T}$ connecting $A(f)$ and $B(f)$ through the fat-vertex v, the transversal arcs $A(f) - \alpha(f)$, $B(f) - \beta(f)$, and the tree, T, will be called a *preimage of the fold*, $PF(f)$.

The region $PF(f)$ can be extended by montonically moving the points $A(f)$ and $B(f)$ on $\partial\widehat{T}$ in opposite directions as long as the following requirements are satisfied:

1. $\theta(\alpha(f)) = \theta(\beta(f))$.
2. $\widehat{\theta}(\partial\widehat{T} \cap PF(f))$ can be deformed into a portion of a segment transversal to the tree at $\theta(\alpha(f))$.

Any such region will also be called a preimage of the fold. In particular, we will be interested in the largest possible region of this kind, which we call $MPF(f)$, the **maximal preimage of the fold**.

Definition (Crossing a *PF*). We will say that the image of an edge e crosses $PF(f)$ whenever there are two points in e, e_A and e_B, defining a portion of an edge $e_2 = [e_A, e_B]$ and such that $\theta(e_A) = \alpha(f)$, $\theta(e_B) = \beta(f)$, and $\widehat{\theta}(e_2)$ is homotopic in $\widehat{T} - \{V\}$ to $PF \cap \partial\widehat{T}$, keeping $\theta(e_A)$ and $\theta(e_B)$ fixed in the homotopy.

Continuing with the discussion of requirement (2) above for extending the preimage of a fold, it is worth rendering its motivation clearer. Suppose that for some integer n and edge e, $\widehat{\theta}^n(e)$ crosses PF, then $\widehat{\theta}^{n+1}(e)$ will map across the fold region in the same way as $\widehat{\theta}(\partial\widehat{T} \cap PF(f))$. If this image is homotopic to a transverse arc, it will disappear via a suitable homotopy when $\widehat{\theta}^{n+1}$ is pulled tight; however, if there are "obstacles" in the form of vertices (added stars or original vertices), such homotopy cannot exist.

When the map presents a single fold, the *MPF* is easily identified. However, when more than one fold is present in a map, the folds may have adjacent prefold regions. By adjacent, we mean that $A(f) = B(f')$ or $A(f') = B(f)$, i.e., we are not considering as adjacent two regions that lie at different sides of a common edge. Under such circumstances it is possible to make further identifications considering simultaneously all the folds of the map.

Definition (Extension of the Fold). For a fold, f, whose $MPF(f)$ is not adjacent to any other *PF*, the extended preimage (or *extension) of the fold*, $Ext(f)$, coincides with $MPF(f)$. In the case where two or more folds have adjacent *PF*s, we consider the endpoints, say $A(f)$ and $B(f')$ (each one belonging to different folds adjacent at $B(f) = A(f')$), and continue to enlarge the region encompassed by them as if the adjacent folds were a single (composed) fold. We will assign to each of the folds in the composed fold the extended preimage corresponding to the largest possible *PF* of the composed fold. Note that composed folds may be adjacent to other folds and the described situation might need to be considered a finite number of times until only nonadjacent (groups of) folds are present (see Figure 5). Hence, we have that for any fold f, $MPF(f) \subset Ext(f)$. Notice that in the case of adjacent *PF*s, $Ext(f) = Ext(f')$. In particular, note that the definition of crossing $PF(f)$ above can immediately be applied to crossing $Ext(f)$.

As a final technical point, we will consider that a sector of a fat vertex is included in $Ext(f)$ if and only if it lies in between two fat edges included in $Ext(f)$; i.e., a sector at the extremes of $Ext(f)$ is hereafter explicitly excluded to facilitate the exposition.

Definition (Interior of the Fold). We will call *interior of the fold* a region $Int(f)$ homotopic to $\widehat{\theta}(Ext(f) - T)$, where the image of $\partial\widehat{T} \cap Ext(f)$ is deformed in $\widehat{T} - V$ into a portion of a transversal segment (pulled tight). The local interior of the fold is a subset of the interior of the fold.

Definition (Collapse of a Fold). Let $\widehat{\theta}$ have a fold at v'. The collapse of a fold consists in identifying points in $\widehat{T}$ in such a way that α and β coincide and *PF* has an empty interior. We call the identified *end* point $*v = \alpha = \beta$.

Since α and β exist arbitrarily close to v, one may regard the collapse as gradually increasing as long as it is "convenient." Too large a collapse may eliminate the fold at the cost of creating a new one [12], possibly with a zero net improvement from the point of view of reducing the topological entropy. Our goal, in fact, is not to eliminate folds

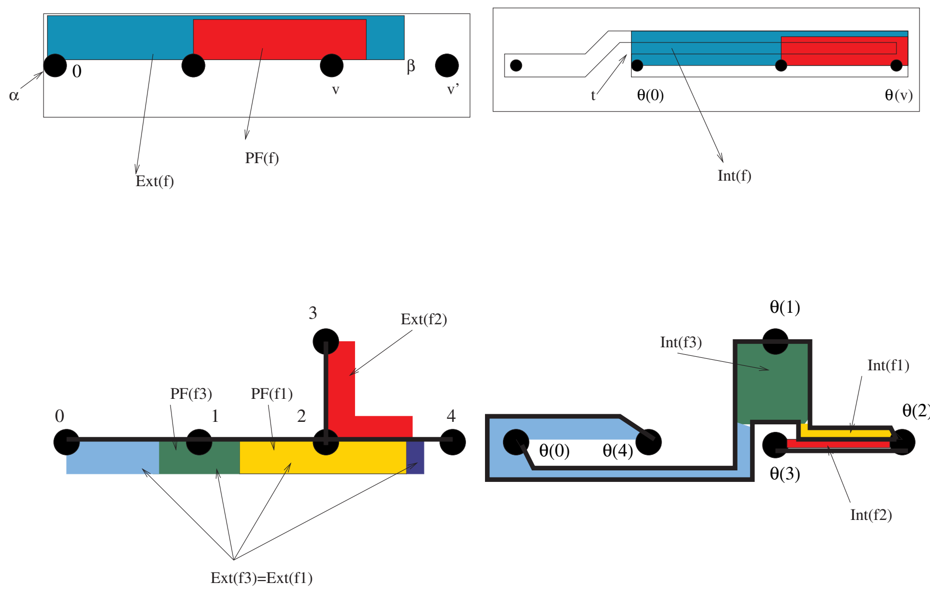

Fig. 5. Prefolds and extended preimage of a fold. In the upper line there is only one fold and $MPF(f)$ coincides with $Ext(f)$. The example in the lower line presents three folds, two of which are adjacent and present an extended preimage that is larger than the MPF of the folds involved.

but to eliminate recurrent bogus transitions without creating new ones. Hence, it may be convenient that the actual collapse stops before the whole extended preimage of the fold is completely collapsed (we say then that the fold has moved from $v' = \theta(v)$ to $\theta(*v)$). Rather than collapsing just a fold, our interest will be to collapse the fold along with all its relevant preimages. The following definition will set us on the right track.

Definition (Collapse of a Bogus Transition). Let $\widehat{\theta}$ have a fold at v'. The *collapse of a bogus transition* consists of (a) the simultaneous collapse of disjoint regions (with the exception of at most a common endpoint for adjacent regions) around all the preimage sectors of the sector at v involved in a (recurrent) bogus transition (i.e., the set BT defined in Section 2), and (b) the collapse of interior portions of the edges that map by θ^k on the collapsed regions.

We need to label the regions to be collapsed in T. We will call such regions $PF_i(f)$, defining them as follows: $PF_n(f) = PF(f)$, and for $n > i \geq 1$, $PF_i(f)$ is a preimage of the fold under $\widehat{\theta}^{n-i+1}$. $PF_i(f) - T$ is a simply connected region of $\widehat{T} - \{V\}$ that has as boundaries: (i) a piece of the tree ($[\alpha_i(f), \beta_i(f)] \subset T$) extending at both sides of a vertex preimage by $\widehat{\theta}^{n+1-i}$ of the vertex v' where the fold lies, (ii) two segments perpendicular to the tree $[\alpha_i(f), A_i(f)]$ and $[\beta_i(f), B_i(f)]$, and (iii) an arc of the boundary of the fat tree going from $A_i(f)$ to $B_i(f)$ (note that $\pi(A(_i(f))) = \alpha_i(f)$ and $\pi(B(_i(f))) = \beta_i(f)$). Finally, $\theta(\alpha_i(f)) = \alpha_{i+1}(f)$ and $\theta(\beta_i(f)) = \beta_{i+1}(f)$. Notice that by construction $\widehat{\theta}([A_i(f), B_i(f)])$ crosses $PF_{i+1}(f)$. As in the previous defini-

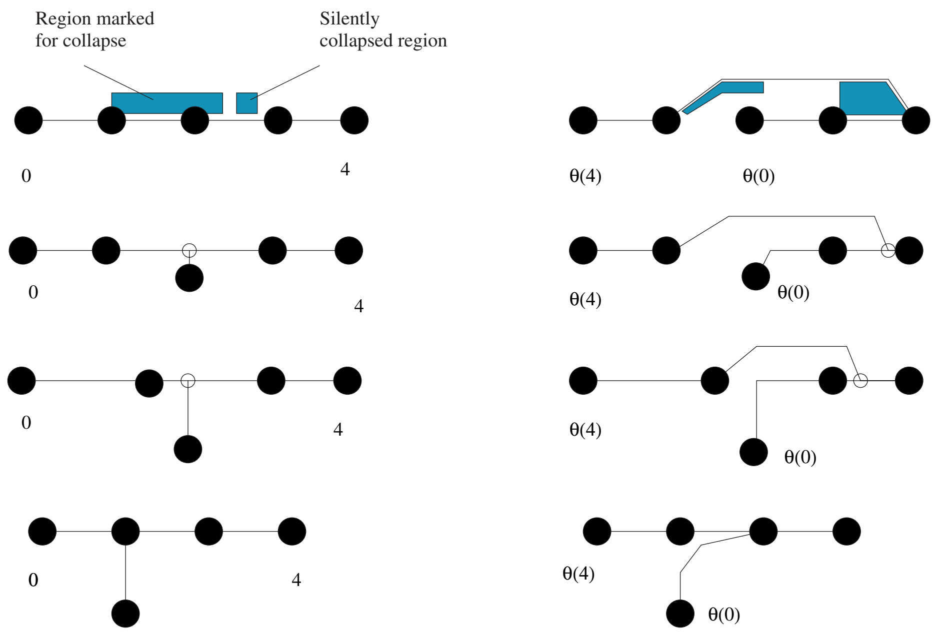

Fig. 6. The regions of explicit collapse and silent collapse are indicated in the first line. The collapse begins by adding a star in the region of the explicit collapse (second line); the collapse is increased (third line) until the star reaches a pre-existing vertex, completing the third part of the collapse (fourth line).

tion, α_i and β_i exist arbitrarily close to each corresponding vertex, and the collapse can gradually be increased as long as it is convenient.

In (a), we label the regions to be collapsed in $T\ PF_i(f)$ in such a way that after the collapse $\theta(*i) = *(i+1), i = 1, \ldots, n-1$ and we **define** $*f$ to be the image by $\widehat{\theta}$ of $*n$. The added star $*k$ appears as the result of collapsing portions of the edges around v_k up to α_k and β_k, which are identified with $*k$ at the end of the collapse, for $k = 1, \ldots, n$.

Regarding (b), the collapse of the portions of edges will be called *silent collapse*. It is introduced in order to assure that no $\widehat{\theta}$–image of an edge will spontaneously bend (fold) onto itself (in the language of [12], we want $\widehat{\theta}$ to be "tight"). Since we describe the action of θ by drawing the edges of $\widehat{\theta}(T)$ as short as possible, compatible with the underlying T, the (b) collapse is automatically performed in the action of drawing $\widehat{\theta}(T)$, without marking the regions with stars, hence the word "silent" is used as opposed to the explicitly marked regions collapsed in (a). See an illustration of the concept in Figure 6.

The two moves in a collapse of a bogus transition, labeled (a) and (b), relate to the notions of gluing and pulling tight [12], respectively. The tree, T, obtained from the explicit part, (a), of the collapse results in a new tree with part of its branches glued pairwise. The collapse emphasizes the operation on the fat tree, $\widehat{T}$, which results in the remotion of some orbits of the fat representative $\widehat{\theta}$ simply by removing part of the associated phase space at one side of the tree T.

The silent collapse of a region of phase space, move (b), produces the tightening of the image of the tree starting from the image modified according to the move (a). Once again, we emphasize the effects on phase space rather than on the image of the tree. In this case, parts of phase space centered at a preimage of f that is not a vertex of the tree are collapsed at both sides of an edge into a transversal segment. In general, we will need to keep track only of a finite number of these regions identified by its central point, those that might eventually be large enough to include a vertex point in the process of collapse of a bogus transition. The infinitely many preimages of these regions can be collapsed without requiring special consideration.

Note that when the region of collapse is gradually increased it might eventually occur that a silently collapsed region reaches a vertex v where an edge of $\widehat{\theta}(T)$ previously associated with the bogus transition begins (or ends). Under these circumstances it might be necessary to perform a new determination of the collapsing regions (if the bogus transition persists) to avoid the creation of a fold at $\widehat{\theta}(v)$. In practice, this means incorporating new sectors to the set BT.

We can think of collapsing a bogus transition as the simultaneous collapse of folds occurring in successively higher powers of $\widehat{\theta}$. If $n > 1$, a collapse around the $*n$-th region is not enough since the $(n-1)$-th region qualifies as PF for $\widehat{\theta}^2$. We proceed to collapse this fold in several steps until exhausting the set BT.

The continuous increment of the $PF(f)$ and some of its preimages required by the collapse of a bogus transition might encounter some problems derived from considering folds one by one.

While the first condition to identify a $PF(f)$ for the map $\widehat{\theta}^{n-i+1}$—namely, $\theta^{n-i+1}(\alpha_i) = \theta^{n-i+1}(\beta_i)$, $n \geq i > 1$—is satisfied as long as it is satisfied for $i = 1$, the second condition requires closer examination. If a $PF(f)$ cannot be enlarged because it is no longer possible to deform the arc $\widehat{\theta}^{n-i+1}(\partial\widehat{T} \cap PF(f))$ into a segment transversal to the tree at $\theta(\alpha(f))$, the segment and the arc enclose a region of the fat tree where a point of V lies, and such a situation will also happen for the images of the region. Hence, the collapse process will be controlled by how large the first region can be while remaining compatible with the definitions.

Our first goal is therefore to collapse the largest possible region around v_1 compatible with the definition of collapse of a bogus transition, without creating a new bogus transition. How the collapse proceeds (start, evolution, and end) will be specified by the following definition and lemmas. In the sequel we assume that whenever the collapsing process stops at any step we will revise and update the collapsing regions.

Definition (Exhaustion of a Fold). A fold is *exhausted* by a collapsing operation whenever the collapsed region in $\widehat{\theta}(T)$ reaches the extended preimage of the fold. We also say that the fold is *partially exhausted* by a collapsing operation if there is at least one collapsed region, say $j < n$, such that it is the maximal j-preimage of the fold, $PF_j(f)$. A partial exhaustion implies the presence of other folds, a fact that will be discussed in Lemmas 11 and 13.

Definition (Exhaustion Flavors). We shall call **normal exhaustion** the case when all the collapsed areas in $\widehat{T}$ end at points in T that are preexisting vertices. **Abnormal exhaustion** occurs when $\pi(*f)$ coincides with a preexisting vertex but no end point of the region collapsed in $\widehat{T}$ does. **Perfect exhaustion** happens when $\pi(*f)$ coincides with an added star in T.

Note that we can regard abnormal exhaustion as a special case of partial exhaustion.

We shall now discuss the details of the collapse of a fold in several lemmas. We begin by establishing a necessary condition to eliminate a bogus transition.

Lemma 5. *A necessary condition for a collapse to eliminate a bogus transition is that at the end of the procedure $\pi(*f)$ coincides with some vertex of (the modified) T.*

Proof. Using Theorem 1 and the concept of preimage of a fold, we can regard the addition of valence-3 stars at $*i$ as the addition of valence-2 stars at the ends α_i and β_i ($i = 1, \ldots, n$) of the collapsible segments a_i and b_i followed by the identification of these regions up to $*i$.

Increasing the collapsible region continuously, the arc of $\widehat{T}$ that goes through $\pi^{-1}(v_n)$ to produce the bogus transition will still pass through the fat vertex unless one of its endpoints maps in $\pi^{-1}(v_n)$. In such a case, $*n$ coincides either with the image of a preexisting vertex or with one of the added stars, or with $*f$, and in all cases $*f$ does not lies halfway between vertices but coincides with one vertex.

Assume then that $\pi(*f)$ does not coincide with a vertex, but rather lies halfway between $\widehat{\theta}(v_n)$ and some contiguous vertex. The fold at $\pi(*f)$ is limited by (reduced) portions of the same edges that limited the fold at $F(v_n)$ before the collapse.

Since $\theta(*n)$ is not a vertex point, the points in a neighborhood of $*n$ map into the same segment, in such a way that the fold condition is satisfied. In such a situation there is still a bogus transition at $*f$ since θ maps edges into one or several complete edges. The $\widehat{\theta}$-image of an edge passing through ab has a (respectively b) as its intersection with a (b), as required in the definition of bogus transition. □

The converse of Lemma 5 does not hold.

4.1.2. The Collapsing Process: Complexity and Steps

Preliminary Considerations. Consider a bogus transition. Then, there is at least a portion of an edge e of T that is involved in the bogus transition with the fold f. It is clear that $\widehat{\theta}(e)$ crosses $PF_j(f)$ for some $1 \leq j \leq n$. There is a region of silent collapse in e whose center is $r(e)$.

In the process of increasing PF_1, and consequently all the PF_is, there are only two situations where one may consider arresting the process and stopping the collapse:

(i) If the collapse were incremented, $\widehat{\theta}(e)$ would no longer cross the region $PF_j(f)$.
(ii) The regions PF_i cannot be further increased since either they would overlap or at least one of them would be maximal.

If none of these requirements is met, we can continue with the collapse.

In both cases, one region of silent collapse has reached a vertex (preexisting or added) and consequently its forward images have reached a vertex too.

The case (i) does not represent an obstacle to the collapse; actually, it is the kind of situation that we want to achieve and will later be called a *step*. In particular, there is no need to increase the collapse if there is no longer a bogus transition.

The case (ii) corresponds to fold exhaustion or partial exhaustion, and at least $PF_n(f)$ has reached the end of an edge, hence $*n$ corresponds to a preexisting vertex; or, if the interaction corresponds to different regions in $\{PF_j(f)\}$ becoming adjacent, there are two added stars in coincidence.

We shall then consider the situations that can be reached in the process of collapsing a bogus transition without creating new bogus transitions.

Lemma 6. *The collapse of a bogus transition can be increased without creating new folds until one of the following situations arise:*

1. *Two adjacent collapsing regions have one endpoint in common.*
2. *All added stars are in coincidence with preexisting vertices and the bogus transition no longer exists.*
3. *All added stars are in coincidence with preexisting vertices and the number of explicitly collapsed regions needs to be increased to continue the collapse.*
4. *The fold is partially exhausted.*
5. *The fold is exhausted.*

Proof. Note that a fold occurs only when local portions of two contiguous edges at a vertex are identified by θ while their preimages are not. In turn, this can happen only to silently collapsed (portions of) edges, since the sectors associated with the set BT are preimages of each other (with the possible exception of the highest preimage, which has either no sector as a preimage or no edge maps under any number of iterations into its preimage. In this later case, a fold is introduced at the beginning of the collapse but no bogus transition is created.

Hence, along the process described, no bogus transition is created.

The only situation in which a fold with an associated bogus transition could be created is if the collapse is continued beyond the point where a silently collapsed region reaches a vertex, since in such a case an arc of $\widehat{\theta}(T)$ that was not previously mapping into a collapsed region might begin to partially map into one of them.

The vertex reached can be (A) a preexisting vertex v, or (B) an added star (say $*i$).

In case (A), when two regions of collapse $PF_{i+1}(f)$ and $PF_k(f)$ become adjacent, we have that $\theta(*i) = *k$ and new regions of collapse are delimited in the form prescribed to determine the extended preimage of the fold. The fold remains at $\widehat{\theta}(*n)$ and we are in case (1).

In case (B), consider the image of the silently collapsed region under consideration to be $*k$, there are two subcases: first, we consider the situation when $k > 1$; and second, the case when $k = 1$.

In the first subcase, it is not possible to extend the regions i for $i = 1, \ldots, k-1$ since $*k \neq \widehat{\theta}(v)$ and $*k \in \pi^{-1}(\widehat{\theta}(v))$. Hence, in the terms stated in the discussion of the collapse of a bogus transition, the fold of $\widehat{\theta}^{n-k+1}$ at $*k$ has an empty interior, and we are in the presence of a partial exhaustion, case (4).

In the second subcase, $k = 1$, it is clear that all the explicitly collapsed regions have reached a vertex since $k = 1$. There are then just two possibilities: either the bogus transition no longer exists and hence case (2) holds, or at least a region around the vertex reached by the silent collapse has to be added to the preexisting regions of explicit collapse (3) since more collapse at $*1$ is needed.

If none of the situations described by (1)–(4) is reached, the collapse can be continued up to the exhaustion of the fold if needed. Hence, case (5) is achieved. □

Definition (Parts of the Collapse). The collapsing process can be regarded as consisting of three parts, which we will call *beginning*, *middle*, and *end*. The first one corresponds to the introduction of a finite set of added stars (valence three vertices) indicating the identification of certain adjacent edges as described above. The middle part corresponds to the gradual increase of the regions $PF_j(f)$ by letting the added stars move along the edges where they lie, i.e., gradually increasing the portions of adjacent edges that are identified, as long as the conditions for continuing the process are still valid (see Figure 6). The end of the collapse corresponds to one of the situations discussed in Lemma 6 above.

At the beginning of the collapse, when the added stars are introduced, sectors associated with the added star $*j$ map into sectors of $*(j+1)$ for $1 \leq j < n$ while $*n$ maps into the fold. The chain of sectors (organized by $\widehat{\theta}$) belonging to $BT(f)$ is separated and isolated from any other sectors associated with the same vertex. The mapping of sectors by $\widehat{\theta}$ might have changed at this point and the same could happen at the end of the collapse. The mapping of sectors is not modified in the middle part of the collapse, since the gradual displacement of added stars along edges can be restored by suitable homotopies.

The form in which the mapping of the preexisting sectors is altered at the beginning of the collapse is simple: Assume that $x(v)$ is a sector to be collapsed in the step under consideration and that $z(u)$ is mapped by $\widehat{\theta}$ as $\widehat{\theta}(z(u)) = x(v) \cup \cdots y(v)$. As soon as the collapse begins, the map is changed into $\widehat{\theta}(z(u)) = \cdots y(v)$.

Regarding the end of the collapse, assume that the collapse proceeds along the edge e. $x(v)$, $y(v)$ are the two sectors separated by e at v. Let us call $x(v)$ the sector that is reached by the collapsed region at $*1$, $y(v)$ the sector that lies at the other side of the collapsed region associated with e, and $z(v)$ the new sector that is incorporated with $*1$. Then, either $z(v)$ has the same preimage as $x(v)$ or the same preimage as $y(v)$.

When $x(v)$ and $z(v)$ share the same preimage, the orbit passing through $y(v)$ is not changed, while any orbit going through $x(v)$ will be changed at points $*j$ by the inclusion of the sector $z(*j)$ until the old mapping is recovered at the $n+1$ image. Any consideration of finiteness of the orbit of an element of *CR* will be the same as that before the collapse, except perhaps that the fold f may no longer have a bogus transition.

If $y(v)$ and $z(v)$ share the same preimage instead, the orbit of $x(v)$ becomes finite as well as any orbit of an element of *CR* that had $x(v)$ as an element; however this only would imply that the fold f moved by the collapse still has a bogus transition. As for the orbit that now includes the union of sectors $y(v) \cup z(v)$, it will map after n iterations into the union of elements previously in the orbit of $x(v)$ and $y(v)$ and will be finite if and only if both the orbit of $x(v)$ and $y(v)$ were finite previously; no new bogus transitions can occur, but some bogus transitions may have disappeared.

Lemma 7 (Collapsing lemma). *In the collapsing process described in Lemma 6, no bogus transitions are created.*

Proof. No new folds are created during any step in the collapsing process as shown in Lemma 6, although the fold under consideration might very well persist.

Since the presence of new bogus transitions or the disappearance of previously existing bogus transitions is associated directly with the mapping of sectors by $\widehat{\theta}$, such events can happen only at the beginning or end of the collapse but not during the middle part.

Beginning. The set *CR* is changed at the beginning of the collapse. If a sector $x(v)$ that is being collapsed is an element of *CR*, this property will be inherited by a sector at the corresponding added star. If $x(v)$ belongs to a union of consecutive sectors in *CR*, then after the collapse begins the element $x(v)$ is simply deleted from the union while its complement in the union of consecutive sectors continues to be an element of *CR*.

Hence, the orbit of any element of *CR* will be finite immediately after the changes operated at the beginning of the collapse if and only if it was finite before the collapse. Consequently, the folds with bogus transitions do not change at the beginning of the collapse except for the bogus transition at f that is inherited by $*f$.

Middle. There are no changes in the mapping of sectors in the middle part of the collapse, and consequently no bogus transition can be destroyed or created in this part.

End. The five different situations considered in Lemma 6 can be grouped for the present analysis in three groups: (A) Case (1) whether it happens in coincidence with cases (4) or (5) or not; (B) Cases (2) and (3) when added stars are in coincidence with preexisting vertices; and (C) Cases (4) and (5) when they do not happen in coincidence with other cases.

The situation (C) is the simplest since it implies no modification of the set *CR* except for those elements consisting of added stars, some (or all) of which must be removed from *CR*. Hence no bogus transition can be created at this point.

The situation (A) implies changes in the mapping of the sectors associated with the added stars where the only fold present is $*f$; hence at most the bogus transition persists, but no new bogus transitions can be created.

Finally, in the case (B) some sectors are added at preexisting vertices as a result of the collision of an added star and a vertex. These sectors map according to the mapping inherited from the added stars; hence the only effective change will occur in the mapping of sectors whose image lies in the vertex, v, that collides with $*1$.

In any case the inclusion of the sector $z(v)$ described above and the associated change of the mapping of sectors do not introduce new bogus transitions.

Hence, no new bogus transition is created at any part of the collapse, be it beginning, middle, or end.

Since we assume that after each stop dictated by Lemma 6 the collapsing regions are updated (to incorporate new vertices, if needed, when a silent collapse reaches a vertex), no bogus transition is created when further continuing the collapse beyond each stop. □

In other words, after stopping a collapse according to Lemma 6 and eventual update of the collapsing regions (be they explicit or silent), the process can continue (if necessary) without creating new bogus transitions.

Complexity of a Fold. Given a fold f at v presenting a bogus transition, there exist at least an edge e and a vertex v_q with the following properties:

1. $\widehat{\theta}^q(v_q) = v$ for some $q \geq 1$.
2. $\widehat{\theta}(e)$ crosses a sector x, at v_q.
3. $\widehat{\theta}^q(x)$ maps into the interior of the fold.
4. There is arc[3] $r_{q+1} \in Int(e)$ such that $\theta(r_{q+1}) = v_q$. We write $e(r_{q+1})$ to identify the edge e. Every arc r_{q+1} associated with the fold in this form denotes an interior portion of a silently collapsing region that we will call *primarily involved* in the bogus transition.
5. $\widehat{\theta}^q(e) \cap Ext(f) \neq \emptyset$ ($Ext(f)$ denotes the extension of the fold).

The last point is a consequence of the previous ones, but we leave it explicitly for the sake of clarity.

For every fold f we will collect all such arcs r_{q+1} for any value of q in a set $SO(f)$. The edges $e(r_{q+1})$ thus identified are those crossing the interior of the sectors in the set $BT(f)$. The cardinality of $SO(f)$ is the number of silently collapsing regions primarily involved with the fold f to produce a bogus transition. Every edge whose interior maps into the fold by $\widehat{\theta}^k$ for some k has at least one of the elements $e(r)$, $r \in SO(f)$, in its orbit. If $SO(f)$ is empty, then f has no bogus transitions.

Unfortunately, we cannot rule out the need to increase the number of explicitly collapsed regions in the process of removing a bogus transition described in Lemma 6 because of case (3). We will then seek an upper bound to the number of regions that could eventually be necessary to consider explicitly by enlarging the set $SO(f)$. Let $Ext(f)$ denote the extended preimage of the fold as previously defined. Consider the arc r_{q+1} indicating a silent collapse and the corresponding image $\widehat{\theta}^q(r_{q+1}) \subset \widehat{\theta}^q(e(r_{q+1}))$. Extending the arc r_{q+1} along its parent edge e, its image by $\widehat{\theta}^q$ will stretch along some portion of $Ext(f)$. One of the following situations can occur:

(a) $\widehat{\theta}^q(e)$ crosses $Ext(f)$, i.e., some connected portion of e containing the arc r_{q+1} has an image that crosses the whole of $Ext(f)$.
(b) $\widehat{\theta}^q(e)$ does not cross $Ext(f)$, and there is at least an arc s of e (necessarily disjoint with r_{q+1}) such that $\widehat{\theta}^q(s)$ crosses one or more sectors or fat edges that are not in $Ext(f)$. In this case $\widehat{\theta}^q(e)$ crosses some connected portion of $Ext(f)$ and also crosses other regions of the tree not related to $Ext(f)$.
(c) $\widehat{\theta}^q(e)$ does not cross $Ext(f)$ and there is at least an arc s' of e such that s' contains r_{q+1} and a vertex of e and $\widehat{\theta}^q(s')$ crosses only fat edges and sectors in $Ext(f)$. In this case the silent collapse marked by the arc r_{q+1} could eventually be extended all the

[3] Note that the labeling of the arcs follows the opposite convention to that of added stars: While it was natural to define $*1$ as the highest preimage of the fold and to study its forward images all the way until $*n = *f$, it is natural in the present context to label the set of arcs with the $\widehat{\theta}$-**pre**images of v. Hence, a lower value of the index requires fewer iterations to map into the interior of the fold than a higher one.

way up to one vertex of e in such a way that the image of the edge portion crosses a connected portion of $Ext(f)$.

The above facts are a consequence of the definitions. Since an arc along an edge has two endpoints, it might happen that situations (b) and (c) occur for one and the same edge, if e.g. one endpoint of the edge has image within $Ext(f)$, while the image of the other endpoint lies outside $Ext(f)$.

In case (a) there is no need to enlarge the set $SO(f)$. In fact, if $\widehat{\theta}^q(e(r_q))$ crosses $Ext(f)$ for every $r_q \in SO(f)$ where $Ext(f)$ is the extended preimage of the fold, the set $SO(f)$ needs no enlargement. If (a) does not occur, either (b) or (c) occurs at each endpoint of the edge e. In case (b) the silent collapse will never reach a vertex since it terminates away from the vertices of e. Hence, the only situation in which it might be necessary to enlarge $SO(f)$ is when the arc r_{q+1} can be extended up to one vertex of e according to case (c) above.

Let w now be the endpoint of e considered in case (c) and $x(w)$ a sector at w such that there exist $k > q$, an edge a, and an arc $r_{k+1} \in Int(a)$ with the following two properties:

(ca) $\widehat{\theta}^{k-q+1}(r_{k+1})$ crosses the sector $x(w)$, and
(cb) $\widehat{\theta}^{k+1}(r_{k+1})$ crosses a sector at $Ext(f)$.

(Notice that sectors in $Ext(f)$ can be crossed only in the form edge-sector-edge.)

In such a case, we enlarge $SO(f)$ by adding to it all the elements r_{k+1} associated with the vertex w of $e(r_{q+1})$. We produce in this way a new set $XSO(f)$. This process of enlargement is recursively repeated for all the elements incorporated to $XSO(f)$ until the required conditions are no longer satisfied. We shall see below that this process is finite.

We shall now introduce the notion of complexity of a fold.

Definition (Complexity). The cardinality of $XSO(f)$ is the *complexity of the fold*.

It can immediately be realized that a fold without bogus transitions has zero complexity since in that case $XSO(f)$ is empty.

Lemma 8. *The complexity of the fold is finite.*

Proof. The number of arcs in $SO(f)$ is clearly finite. By construction, every edge in T whose image crosses the local interior of the fold in $k + 1$ (recall the definition of SO) or fewer iterations is an element of $SO(f)$ and then of XSO. Additionally, any silent collapse of the map is the preimage of one of the silent collapses identified by their centers with the arcs in $SO(f)$.

Next, we observe that $\pi(\widehat{\theta}^{k-q}(a)) \cap e \in \{e, \emptyset\}$ for any $k > q$ since θ maps edges into unions of complete edges, it is one-to-one on the vertex set, and $\widehat{\theta}$ is "tight" (see above). Hence if there is a sequence of arcs such that $r_{q+1} \in e$ is a forward image of $r_{k+1} \in a$, then $\pi(\widehat{\theta}^k(a)) \supsetneq \pi(\widehat{\theta}^q(e))$ and the sequence of sets $\widehat{\theta}(\widehat{\theta}^{k-1}(e(r_{k+1}))) \cap Ext(f)$ is nondecreasing (with k) for any sequence $\{r_k\}$.

Every enlargement of the set $XSO(f)$ involves a finite increment of extension of $\widehat{\theta}(\widehat{\theta}^{k-1}(e(r_{k+1}))) \cap Ext(f)$, which is performed by including in $XSO(f)$ a finite number of terms of the corresponding sequence. That the number of terms is finite is clear from the fact that the number of edges in the tree is finite, and for each edge a requiring an enlargement according to condition (ca), there is a sector y such that $\widehat{\theta}(a)$ crosses y and

$\widehat{\theta}^l(a)$, $l = 1, \ldots, L$, crosses the sectors where the image of $\widehat{\theta}^{l-1}(y)$ lies, until the image of the sector reaches for the first time $x(w)$ for $l = L$. Since the number of sectors is finite, the number of iterations involved is also finite.

The recursive enlargement of the set *XSO*(f) will reach an end after a finite number of recursive steps, since at every required enlargement the image of the collapsed region crosses at least one fat edge more of *Ext*(f) than at the previous step, and the number of fat edges involved in *Ext*(f) is necessarily finite. □

Collapsing steps. Lemmas 5 and 6 show that we have to pay attention only to the situations in which $\pi(*f)$ coincides with an added star or a preexistent vertex after starting a collapse.

Let us consider the first possibility to begin with. We will say that two colliding regions are *at the same side of an edge* whenever they have a (transversal) common border; while when they have only one point in common (necessarily on the edge) we will refer to them as being *at opposite sides of the edge*.

Lemma 9. *When in the process described by Lemma 6 the fold persists and two or more collapsible regions on T are adjacent and have one common endpoint (case (1) of Lemma 6), then there is a q-periodic orbit ($n > q > 0$) of stars, and*

i. *If the colliding regions, say k and $(k+q)$ are at the same side of the edge, then the orbit persists under further collapse,*
ii. *If the colliding regions are at the same side of the edge and there are eventually periodic stars, the fold has a bogus transition.*
iii. *If the colliding regions are at opposite sides of an edge and the periodic orbit persists, the region* 1 *no longer has an associated bogus transition; otherwise all regions can be further collapsed, splitting the periodic orbit.*

Proof. Let n be the number of added stars, $n \geq 2$ since otherwise the conditions of case (1) of Lemma 6 are impossible to meet. As stated before, we consider the labeling of stars given by $\theta(*i) = *(i+1)$, $i = 1, \ldots, n-1$ and $\widehat{\theta}(*n) = *f$. Since after the collapse there exists i and $k < i$ such that $*k = *i$, it follows that $*n = *(k+n-i)$; hence $\theta(*n) = *(n-i+k+1)$. We define $m \equiv n-i+k+1$. Let $q = n-m+1 = i-k > 0$. The added valence-3 stars $\{*s\}$ for $s = m, \ldots, n$ form an orbit of period q of θ. Moreover, since $*n = *(m-1)$, we can assume that the periodic orbit extends from $*(m-1)$ to $*(n-1)$ (or identically from $*k$ to $*(i-1)$) and that there remain at most $n-1$ stars at the end of the process, so $q < n$ and $m \geq 2$. This proves the first statement.

Note that at this point the fold has moved from the original vertex $\widehat{\theta}(v)$ to $\widehat{\theta}(*n) = *m$.

Let $*k$ be the colliding star with lowest index. Then $m > k$, and if $k > 1$, there are eventually periodic added stars left.

Statement (i) is proved observing that the regions $*k$ and $*j$ (and their images) have become adjacent. Further collapsing, if needed, will identify points at the same side of the edge that are separated by the collapsing regions; hence a valence-3 (or higher valence) star (and its periodic orbit) will remain identified even if further collapse is needed.

Regarding statement (ii), let $*j$ be the highest eventually periodic preimage of the fold at $\widehat{\theta}(*n)$. Then, no endpoint of an edge maps into $*j$. Since θ maps edges into one or

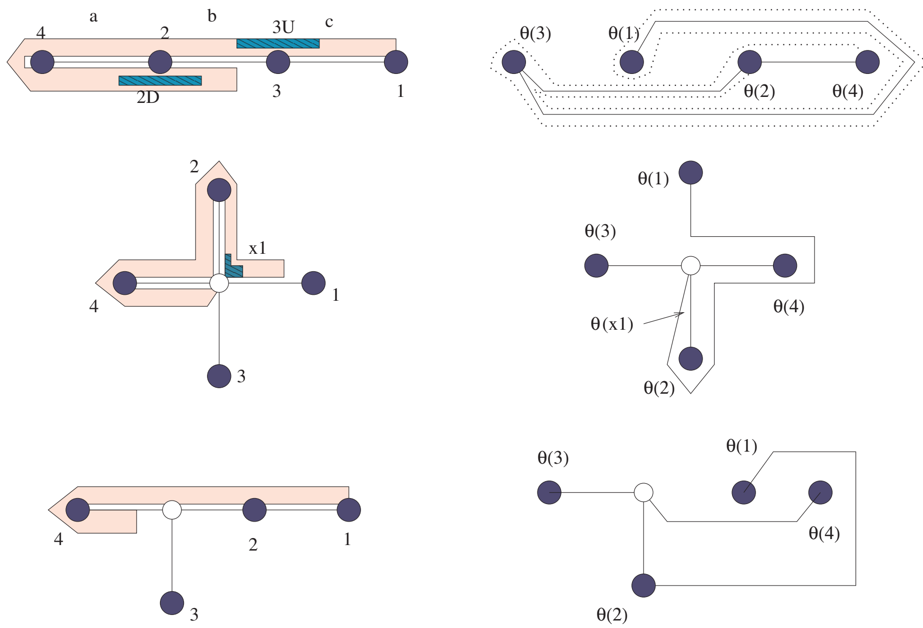

Fig. 7. Collapsing regions at opposite places at a vertex. First line, left: the extended preimage of the fold, and preimages PF_2 and PF_1 of the fold to be collapsed. Right: the image of the tree (solid line) and the image of $\widehat{\theta}^2(c)$ (dotted line). Second line: After the first collapse, the bogus transition persists, but the collapse proceeds only at the first preimage of the fold. Third line: the fold persists, but there is no longer a bogus transition. (See paragraph *Third example* for details.)

several complete edges, the portions of $\widehat{\theta}^k(T)$ (for adequate $k \geq 1$) that mapped into the j-region when the step of the collapse started will still pass around $*j$ when the collapse ends. Hence, they will be mapped (stretched and bended) inside the fold and the bogus transition will still be present.

We turn to case (iii), in which the colliding regions are at opposite sides of an edge; see Figure 7. Consider the region k in its collision with region $(k+q)$. As the colliding regions are at opposite sides of an edge, the continuation of such regions are opposed by the vertex of the valence-4 star formed in the collapse. Hence, if further collapse is needed in the lowest k colliding region, all higher regions will have to undergo further collapse and the periodic orbit will split into valence-3 vertices, preserving the former mapping. Thus, if the region 1 still has an associated bogus transition, further collapse will be required in all vertices, including those involved in collisions, and the periodic orbit will split. Otherwise, if the region 1 is no longer associated with a bogus transition, then we have reached a case of exhaustion. □

We are now in a position to give a proper definition of the notion of *step* (which we have been using intuitively up to now) in the elimination of a bogus transition.

Definition (Step). Since the enlargement of the collapsing regions proceeds continuously until the bogus transition is eliminated or until there is a need to reconsider the collapsing regions, we say that a *step* has been made in the elimination of a bogus transition when either of these two situations occur.

Lemma 10. *The steps in the elimination of a bogus transition correspond to one of the following situations:*

1. *Two adjacent collapsing regions at the same side of an edge collide.*
2. *All added stars are in coincidence with preexisting vertices and the bogus transition no longer exists.*
3. *All added stars are in coincidence with preexisting vertices and the number of explicitly collapsed regions needs to be increased to continue the collapse.*
4. *The fold is partially exhausted.*
5. *The fold is exhausted.*

Proof. The result is the direct consequence of Lemmas 6 and 9. Statement (iii) of Lemma 9 shows that the collision of collapsing regions at opposite sides of an edge either does not require a reconsideration of the collapsing regions or happens in coincidence with a (total or partial) exhaustion. ☐

Concerning examples for the different cases mentioned in the Lemma, case (1) is described in Figure 10, case (2) in the example of Figure 6 while case (4) and (5) are shown in Figure 8 (for case (4) see also Figure 9 below). Unfortunately, we could not find an example for case (3) despite the fact that we cannot rule out its occurrence.

4.1.3. The End of the Tale: Finiteness of the Procedure. We shall now turn our attention to the question of exhaustion in Lemma 6, namely when we can collapse completely (all or some) marked regions of $\widehat{T}$. The exhaustion of a fold by a collapsing operation can be produced in three slightly different ways according to the branching point reached by $\pi(*f)$ at exhaustion (added star or preexisting vertex) and whether the collapsed regions end at a star in T that coincides with a preexisting vertex.

We will start by considering "complete" exhaustion in the following lemmas, handling partial exhaustion afterwards.

Lemma 11. *After a normal exhaustion, there remain no eventually periodic stars.*

Proof. The result is evident by the definition of normal exhaustion. ☐

Lemma 12. *After a perfect exhaustion, a periodic orbit of θ is evinced by vertices and no eventually periodic stars remain.*

Proof. Consider the stars $*i$ with $i = 1 \dots n$ signaling on T the end of the collapsed regions. Since the end of the fold corresponds to a point where $\pi(*f) = \pi(\widehat{\theta}(*n)) = \theta(*n)$ is in coincidence with one of the added stars, say $*j$, then there is a periodic orbit of θ with points $*k$, $k = j \dots n$.

We shall now show that $j = 1$, i.e., that there remain no eventually periodic stars.

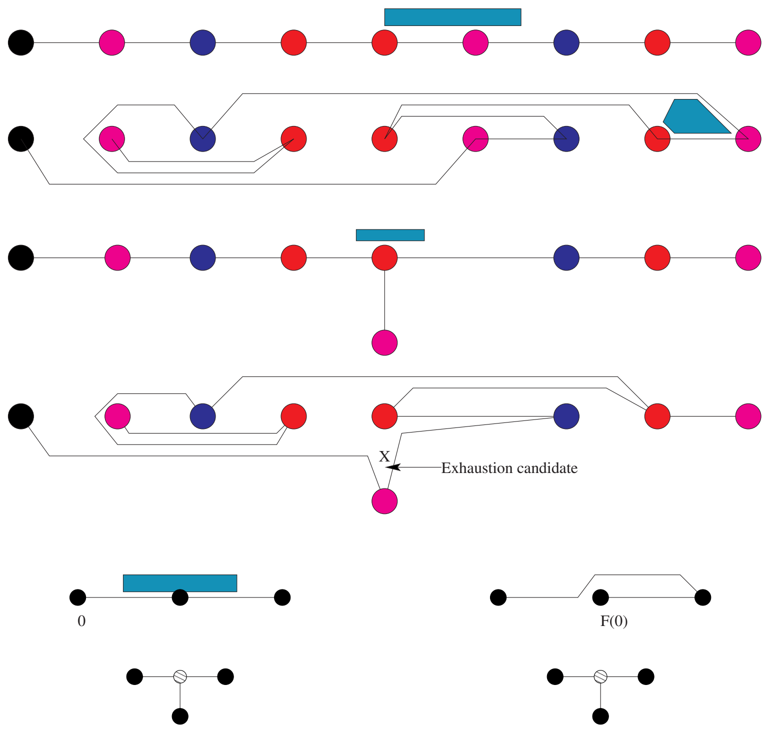

Fig. 8. Cases (4) and (5) of Lemma 10. First line: the tree and region to be collapsed. Second line: the image by $\widehat{\theta}$ of the tree and the fold. Third line: the new tree after the collapse and the region to be collapsed. Fourth line: the image of the tree. The occurrence of case (4) of Lemma 10 is indicated. Different colors correspond to different periodic orbits. Lines five and six correspond to case (5) of Lemma 10. After the first collapse, the tree presents no folds (fold exhaustion).

Assume that $j \neq 1$, $\theta(*(j-1)) = *j = \theta(*n)$. We first notice that $*i$ cannot be in coincidence with a preexisting vertex for any i, since otherwise $\pi(*f)$ would be in coincidence with a preexisting vertex, contradicting the hypothesis of perfect exhaustion.

Secondly, the exhaustion of a fold while two (or more) collapsed regions become adjacent is not possible. In such a case, the region $(j-1)$ becomes adjacent to the region n, and after the collapse the valence of $*j$ is strictly smaller than the valence of $*n$, and the fold persists.

Thirdly, by construction, θ and then $\widehat{\theta}$ are one-to-one locally around $*(j-1)$. Since no collapsed regions are adjacent, the valence of $*(j-1)$ and $*j$ is 3 and the edges emerging from $*(j-1)$ map approximately by $\widehat{\theta}$ (exactly by θ) onto the edges emerging from $*j$. The edges arising from $*j$ divide the corresponding fat vertex of $\widehat{T}$ in three sectors.

Then, since $\widehat{\theta}(T)$ maps the local part of T around $*n$ into the fat vertex at $*j$, it must map this local part into one and only one sector of the fat vertex $*j$ (otherwise $\widehat{\theta}$ would not be one-to-one). Hence, the edges emerging from $*n$ that are not in the collapsed region map into the same edge and the fold persists.

In any case, we arrive at a contradiction arising from the assumption $j \neq 1$. □

Chains of Folds. We are now left with the problem of understanding abnormal and partial exhaustion. Both of them leave eventually periodic stars behind. In order to make the exposition clearer, we will discuss first a simple result.

Let v be a vertex belonging to a period-p orbit. θ^p locally maps the vertex v onto itself, as well as the edge-germs emerging from it, thus determining a map for the sectors at v. We will simply let θ^p act on the sectors at v since this has an obvious meaning. We claim the following:

Proposition 1. *If θ^p maps a sector, say z, into two or more (adjacent) sectors x, y, then θ^p has a fold at v and x, y or $x \cup y$ are preimages of that fold of some order not necessarily 1.*

Proof. The map θ^p considered in a neighborhood of v and applied to the sectors at v can be presented as a matrix, Z, of order $m \times m$, where m is the number of sectors at v, with $Z_{ij} = 0$ if the sector x_j does not map on the sector x_i and $Z_{ij} = 1$ when x_j maps on x_i.

The following properties of Z are immediately realized:

1. Every row in Z has at least one nonzero entry (every sector has a preimage under $\widehat{\theta}^p$).
2. There is only one nonzero entry for each row (a sector cannot be the image of two different sectors).
3. The vector $(1, \dots, 1)^\dagger$ is an eigenvector of Z with eigenvalue 1 (this is an interesting fact, but it is not explicitly used in the sequel).
4. $\sum_{i,j} Z_{ij} = m$, the number of sectors at v.
5. Columns of Z corresponding to a preimage of a fold are identically zero.

According to (2) and (4), if one column has more than one nonzero entry, then there has to exist another column that is identically zero. Hence, by (5), for every sector that maps onto $n > 1$ sectors there exists at least one fold. In other words, the condition that z maps onto $x \cup y \cup \dots$ implies the existence of at least a fold at v, thus proving our first claim.

We can reorder the labeling of sectors so that sectors mapping into folds and their preimages (mapping into folds under higher iterations of θ^p) have the largest index. Then, the matrix Z can be block-diagonalized as

$$Z = \begin{bmatrix} A & 0 \\ B & N \end{bmatrix}. \tag{1}$$

The sectors eventually mapping into folds correspond to rows and columns in N, while the rows and columns in A are associated with sectors having at least part of their image onto sectors outside N. The fact that the sectors in A never map (entirely) into folds is made evident by the zero upper-right block of Z: sectors in A are not preimages of sectors in N.

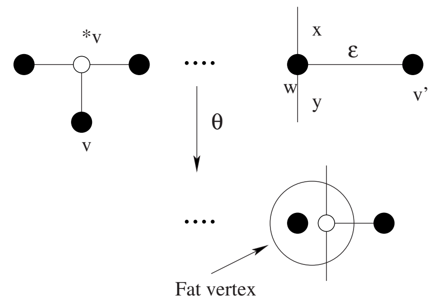

Fig. 9. An abnormal exhaustion.

It is easy to realize that A is a permutation matrix since each sector in A has to have as preimage another sector in A, and thus every column in A has only one nonzero entry.

To finish the proof, we notice that x, y are not associated with A, since in such a case A would have two equal rows and hence a zero eigenvalue, thus contradicting the fact that A is a permutation matrix. Hence at least one of x, y has entries in N and eventually maps into a fold. □

We note in passing that N is nilpotent, although this fact is not explicitly needed. The following lemma will help us to understand the situations in which eventually periodic orbits are left after a collapse.

Lemma 13.

(a) *The remaining eventually periodic stars occurring in an abnormal or a partial exhaustion are associated another fold.*
(b) *The fold associated with the eventually periodic stars had bogus transitions before the collapse originating the eventually periodic stars was initiated.*

Proof. We consider first an abnormal exhaustion. Note that since the collapse is performed at the fold and its preimages in such a way that the forward image by $\widehat{\theta}$ of a collapsed region is also a collapsed region all the way up to the fold, the phenomenon of abnormal exhaustion can be considered to happen at the fold.

Let v' be the fold point. We start the collapse by identifying corresponding portions of contiguous edges at $v = \widehat{\theta}^{-1}(v')$ in T until the images by θ of the regions beyond α and β cannot be identified. We call $*v$ the end of the identified region and $w = \theta(*v) = \pi(*f)$. The edge joining w and v' will be denoted by ε. By hypothesis, the collapse around the fold stopped before $*f$ reaches w because there are no larger identifiable regions near the preimages of v'. In other words, we have $*f = \widehat{\theta}(*v) \neq \theta(*v) = w$; see Figure 9.

First we note that there is no local edge in $\widehat{\theta}(T)$ along ε connecting to w, since that part of $\widehat{\theta}(T)$ would intersect an edge in a point that is not a vertex.

Call x and y the two sectors in $\widehat{T}$ near w having ε locally as one border. We are then under the hypothesis of Proposition 1, which proves the existence of a fold associated with at least one of x, y. Let us say that x (y) maps into a fold to fix ideas.

To prove the second claim, we simply observe that the edges of T emerging from w map precisely across the sectors x and y under the hypothesis of the theorem. Hence there is a bogus transition associated with x (y). □

Final considerations. So far we have considered partial results concerning the collapse of one fold with bogus transitions on an individual basis. We will now collect the results to show that the algorithm ends.

Lemma 14.

A. A fold f no longer has an associated bogus transition if and only if its complexity is zero.
B. A step in the collapsing process reduces the complexity of the fold by an integer value.

Proof. The proof of (A) is immediate by the definition of complexity: $SO(f)$ is empty if and only if $BT(f)$ is empty and $XSO(f)$ is empty if and only if $SO(f)$ is empty.

Regarding statement (B), this fact is assured by Lemma 10. In the case of exhaustion of the fold (case (5) of Lemma 10), the remaining complexity is zero and the same happens in case (2) of Lemma 10 since there is no longer a bogus transition associated with the fold.

In case (3) of Lemma 10 a silent collapse becomes explicit. Let r be the center of this collapsed region, $e(r)$ the edge where r lies, and q such that $\theta^{q+1}(r) = v$, where the fold lies. We notice that $r \in XSO(f)$ before the collapse, since it is a member of a sequence of preimages of the fold and moreover $\hat{\theta}^q(e(r))$ does not cross $Ext(f)$. After the collapse, the collapsed region is identified by an endpoint that in case (3) of Lemma 10 needs to be marked for further collapse. Hence, the point representative of r is no longer in $XSO(f)$, and the cardinality of $XSO(f)$ has been reduced by at least one.

Concerning case (4) of Lemma 10, the edge e in the set BT associated with silent collapses around $*1$ no longer maps into the fold and the complexity decreases by at least one.

Regarding the collision of collapsed regions at the same side of an edge (case (1) of Lemma 10), it is clear that any segment mapping in the collapsing region 1 will map along the complete segments limiting the region. Hence it will map after a suitable number of iterations into two colliding regions. Before the collision the segment had at least two different silently collapsing regions. After the collision the number of silently collapsing regions on this segment has decreased by at least one and the complexity has been reduced by at least one.

Hence, statement (B) holds for all the cases of Lemma 10. □

We now turn our attention to the elimination of eventually periodic vertices of valence 3 that might have been left in the process of eliminating the bogus transitions.

Lemma 15. *If after a number of collapsing steps a tree has eventually periodic stars, then there is still a fold with bogus transition and consequently positive complexity.*

Proof. Eventually periodic stars might be left in cases (1), (4), and (5) of Lemma 10. Regarding case (1), Lemma 12 ensures that if eventually periodic points are left, then the fold persists, and the application of Lemma 9 implies that the bogus transition persists.

If the eventually periodic stars arose in an abnormal or partial exhaustion (cases (4) and (5) of Lemma 10), then by Lemma 13 they are associated with a fold whose bogus transition is under consideration or has not been collapsed yet. We then consider this fold for collapse.

If there remain eventually periodic stars, we are again in the previous situation, but by Lemma 14 the complexity of the fold has decreased; however, this can happen only a finite number of times. □

Theorem 2.

A. *The elimination of a bogus transition does not introduce any additional folds/bogus transitions.*
B. *A (recurrent) bogus transition can always be eliminated in a finite number of collapsing steps.*
C. *The collapsing process ends in a finite number of collapsing steps.*
D. *The final tree has only periodic vertices*

Proof. Statement (A) follows from Lemmas 6 and 7. Statement (B) is a corollary of Lemmas 8 and 14 since the complexity of a fold is a finite integer and the collapse reduces the complexity in integer steps; hence the complexity can be reduced to zero in a finite number of steps. Statement (C) follows immediately since there exist only a finite number of folds with (recurrent) bogus transitions. Statement (D) is the direct consequence of Lemma 15. □

5. Algorithm and Examples of Use

In this section we will present a few examples in exhaustive form. We start by stating the algorithm that summarizes the previous results.

Algorithm.

1. Identify all folds in the map.
2. Detect folds with recurrent bogus transitions. If there are no recurrent bogus transitions, end.
3. Select the fold with eventually periodic stars associated (if there is any) or a fold with (recurrent) bogus transitions, otherwise (i.e., a fold with nonzero complexity). If no fold can be selected, end; otherwise collapse the bogus transition or fold:
   (a) Mark regions to be collapsed adding valence-3 stars at points of BT.
   (b) Perform one collapsing step (Lemma 10).
   (c) Eliminate cycles among edges with at least one star as endpoint, collapsing the edges to a point.
   (d) Go to (3).

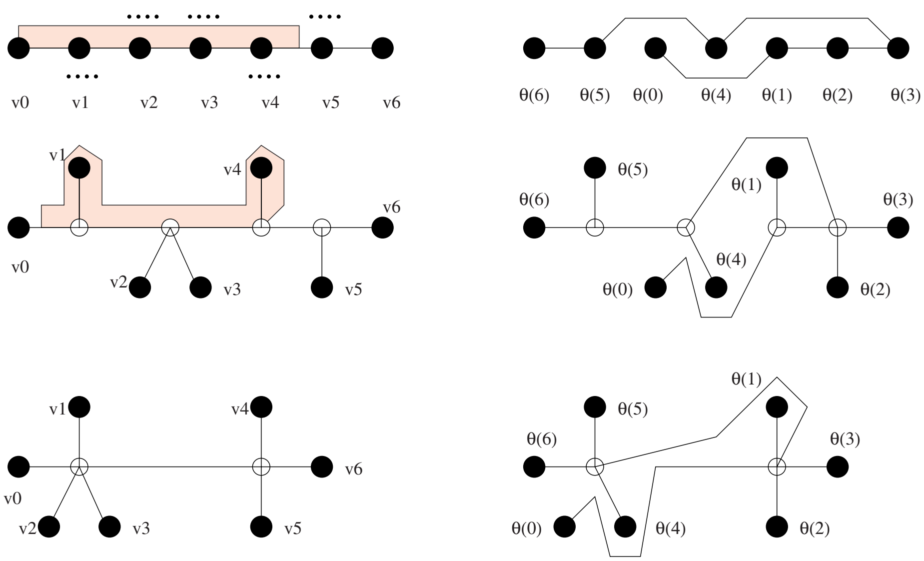

Fig. 10. A period-7 horseshoe orbit. Extension of the folds are marked with colored boxes.

Item 3(c) follows from the fact that the Markov matrix associated with the tree before operating the elimination has a zero extra-diagonal block (the edges build a subcycle), and hence the eigenvalues of the Markov matrix equal the eigenvalues of the two diagonal blocks. After elimination of the block in 3(c), we obtain a simpler tree containing the original points in P with smaller Markov matrix and the same entropy bounds. This steps restores irreducibility of the tree, whenever relevant.

Note that by Lemma 7 once the complexity of a fold is zero, it never increases (recurrent bogus transitions cannot be created by collapsing). However, after having collapsing one fold, the (nonzero) complexity of not-yet-collapsed folds may be altered in any direction. In any case, when the collapsing turn arrives to any such fold, its complexity at the end of its cycle will finally be zero.

First example. In Figure 10 we display a chaotic period-7 orbit. Each row of the figure displays T along with $\widehat{\theta}(T)$. The different rows are produced after successive applications of steps of the algorithm. Added stars are white.

Labeling the vertices v_i, $i = 0 \dots 6$ from left to right, we see from row 1 that there is one fold (hence necessarily at an end point, $v_6 = \widehat{\theta}(v_3)$). The dotted regions above (U) and below (D) points of T denote the regions that map on the fold by the iterates of $\widehat{\theta}$, the sequence defining PI is then

$$2U \to 5U \to 1D \to 4D \to 3U \to *f = v_6.$$

We have that *CR*={2U, 3D, 4U, 5U}, and since $2U$ is in *CR*, we have that $PI = BT$. The set PI contains all elements of the above sequence up to (and except) the fold v_6.

Let us label the edges of the tree with a, b, c, d, e, g from left to right. The extended preimage of the fold consists of all sectors and upper portions of the fat edges (as delimited by the tree) between v_0 and a point lying on e.

To compute the set $SO(f)$ two arcs are interesting in the first place since BT has two elements: (i) There is an interior arc r of d mapping by θ on $5U$ and reaching the fold after four extra iterations. Hence, r belongs to $SO(f)$ and no enlargement is needed since $\widehat{\theta}^4(d)$ crosses the extended preimage of the fold. (ii) An interior point s of e maps on $2U$; hence s belongs to $SO(f)$ and no enlargement is needed since $\widehat{\theta}^5(e)$ also crosses the extended preimage of the fold. Hence, the complexity of the fold is 2.

Pieces of $\widehat{\theta}(T)$ passing above or below the dotted vertices indicate the existence of bogus transitions. Collapsing around the dots produces five added stars $*2 \to *5 \to *1 \to *4 \to *3$, which eventually become four after collision of the preimage of $*f$ (namely $*3$), and its contiguous star, $*2$ (row 2). At this point, Lemmas 6 and 9 apply. The complexity of the fold is reduced to zero and the collapsing is finished. Further, the two outermost edges having added points as endpoints map onto each other and can be collapsed (row 3, item 3(c) of the algorithm). The resulting diagram still has a fold but no bogus transition.

Second example. Let us now turn to the example in the second row of Figure 5. We label the vertices from left to right as 0, 1, 2, 3, 4, letting the unaligned vertex be number 3. We label the sectors at each vertex as U, D, L, or R (up, down, left, right), as suits the natural orientation of the figure (vertex 2 has sectors L, R, and D but no U-sector). Finally, label the edges from left to right as a, b, c, d, with c being the vertical edge. $CR = \{0, 1D, 2L, 2R, 2R + 2L\}$, while $PI(f_1) = 2D$, $PI(f_2) = 2R$, $PI(f_3) = 1D$, for the three folds indicated in the figure. Among the elements of CR, only $1D$ and $2R$ have finite orbits; all others, or their images, are the only sector at an endpoint. Hence, $BT(f_2) = PI(f_2)$, $BT(f_3) = PI(f_3)$, since the corresponding PIs are subsets of the set of elements of CR having finite orbits. On the other hand, $BT(f_1) = \emptyset$, and f_1 has no bogus transition.

Regarding f_2, we have that $v = 4$. For $q = 1$, $v_1 = 2$ and $\widehat{\theta}(b)$ crosses $2R$, where b is the edge between vertices 1 and 2. For $q = 2$, $v_2 = 3$. No edge portion crosses sectors at 3 mapping into the fold, and hence the complexity is 1 since b crosses the whole of $Ext(f_1)$ and there is no need for enlargement.

As for f_3, we have that $v = 3$. For $q = 1$, $v_1 = 1$. $\widehat{\theta}(a)$ and $\widehat{\theta}(d)$ cross $2D$. For $q = 2$, $v_2 = 4$, no edge portion crosses relevant sectors, and the complexity is 2 since the conditions for enlargement are not met. It is a bit unfair to compute the complexity of f_3 before having dealt with f_2 since we cannot foresee how the modifications imposed to the tree while collapsing f_2 will affect the analysis of f_3. In fact, it turns out to be unnecessary since after collapsing f_2 up to exhausting the fold (case 5 of Lemma 10), f_3 disappears while f_1 remains, still with zero complexity. The resulting tree with zero complexity is shown in Figure 11.

Third example. Next, we consider the case of Figure 7. Vertices and interesting sectors are labeled in the the first row of the figure. Name the edges as a, b, c, from left to right. There is a fold at $v = 4$ with $CR = \{2D, 3D, 3U, 1\}$ and $PI = \{3U, 2D\} = BT$

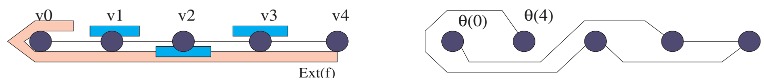

Fig. 11. The example in Figure 5 revisited.

(trivial). $Ext(f)$ consists of all the upper sectors and upper portions of edges (as delimited by the tree) plus the lower sectors and corresponding lower portions of edges up to some interior point of edge b (see marked zone in Figure 7, first row). The set SO has three elements since a portion of c crosses $3U$ for $q = 1$, while portions of b and c cross $2D$ for $q = 2$.

We have to consider the possible enlargement of SO at $2 = \widehat{\theta}(1)$. We realize that $\widehat{\theta}(1)$ is an endpoint and the sector associated with it cannot satisfy the requirement (cb) for an enlargement since the image of arcs crossing the sector at an endpoint cannot cross just one sector at a valence-m vertex with $m > 1$. Next consider the arc in b associated with the fold with $q = 2$. $\widehat{\theta}(b) = b + 2D + a$ and $\widehat{\theta^2}(b) = a + 2D + b + 3U + c$. Notice that $\widehat{\theta^2}(b)$ exits $Ext(f)$ at 2 (and "reenters" after crossing edge b); hence the part to be considered of this image is just $b + 3U + c$, and no enlargement is needed since the requirement (cb) is not met. The final possibility to be considered is an enlargement associated with the arc in c. We have that $\widehat{\theta}(c) = b+3U+c+1+c+3D+b+2D+a$ and $\widehat{\theta^2}(c) = (c+3U+b+2D+a)+(4)+(a+2D+b+3D+c+1+c+3U+b)+(2U+2D)+(b+3U+c+1+c+3D+b+2D+a)+(fold)+(a+2D+b)$ (indicated with a dotted line in Figure 7). One end point of $\widehat{\theta}(c)$ lies outside $Ext(f)$, while it is not possible to reach the other endpoint without crossing elements outside $Ext(f)$; hence there are no possibilities of enlargement. The complexity of the fold is then 3.

After collapsing we arrive at the figure shown in the second row of Figure 7, with four edges and five vertices. This first collapsing step ended with a partial exhaustion of the fold (case 4 of Lemma 10) when the two added collapsing regions at opposite sides of an edge are in contact (case (iii) of Lemma 9). Labeling the sectors at the period-1 added star that appeared in the form described by case (iii) of Lemma 9 x_1, x_2, x_3, x_4 in counterclockwise order starting from the preimage of the fold ($x1$), we can see that $CR = \{x_1, x_4, 3, 1\}$ and $PI = \{x_1\}$; hence there is just one arc in SO associated with the fold with $q = 1$ lying in the edge connecting 1 with the added star. The complexity of the fold is now 1 since there is no need of any enlargement (actually, this arc is what remains of the arc at c associated with $q = 1$).

The final step is taken collapsing at $x1$ until the bogus transition is eliminated when the region of collapse reaches 2 (case 2 of Lemma 10). The remaining tree has a fold but no bogus transition.

Last Example. We conclude the examples section by considering a case shown in [12], which we display in Figure 12.

There are two folds, one at $5 = \widehat{\theta}(4)$, which we call fold $f1$, and fold $f2$ at $4 = \widehat{\theta}(3)$. $PI(f1) = \{3L, 4U\}$, while $PI(f2) = \{3D\}$. $CR = \{3L, 3D + 3L, 5, 2D\}$ (U, D, R, L indicate above, below, right, and left respectively, as mentioned above), $BT(f1) = \{3L, 4U\}$ and $BT(f2) = \{3D\}$ (note that $3D$ is in the orbit of $2D$). The bogus transitions are eliminated after two steps, yielding the third row of the figure. In the

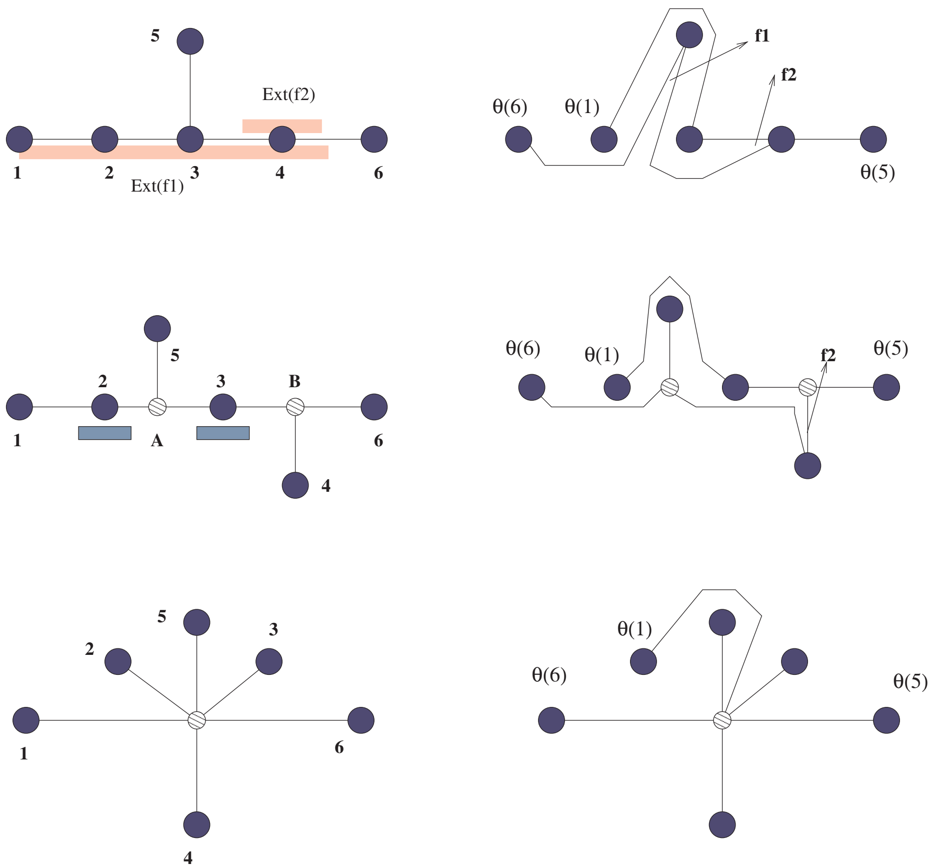

Fig. 12. The example in [12], Figure 14, p. 103, revisited. In the first step the fold $f1$ is eliminated in a perfect exhaustion identifying a period-two orbit $\{A, B\}$. In the second step the collapse is performed under 2 and 3 eliminating the bogus transition at $f2$.

first step the fold $f1$ is exhausted, leaving a period-two orbit behind (perfect exhaustion). In the second step the bogus transition at $f2$ is eliminated (Lemma 10, case 2, the fold moves all the way to A passing first through B, and an added star remains under A produced by drawing *2 and B together in the collapse). The edge connecting both stars maps onto itself and can be eliminated by item (3c) of the algorithm, leaving behind a period-1 added star. Note that in this second collapse the set $PI(f2) = \{2D, 3D\}$ contains two elements; one of them ($2D$) was not present in the previous analysis. The sets $BT(f)$ and $PI(f)$ may change after a step is performed, which explains why it is not possible to extrapolate the complexity of a map from the complexity of each fold.

6. Discussion

This article is much indebted to previous work by Hall [5] and Franks and Misiurewicz [12]. Indeed, we produce an improved version of the algorithm in [12] using concepts extended from ideas in [5]. The improvement is a consequence of focusing on the minimal periodic orbit structure rather than on the topological entropy (this being the largest entropy of the irreducible components associated with P).

The relevant differences with [12] are the following:

1. All types of invariant sets are treated on equal footing regardless of whether they are the reducible or irreducible cases, a single periodic orbit, or a link.
2. The end condition of our algorithm is the absence of (recurrent)[4] bogus transitions.
3. The regions where a collapse should be practiced are detected beforehand using the concepts of recurrent bogus transition and fold.
4. There is no need for additional algorithm moves such as "splittings" [12] (see below).

Theorem 1 establishes that the goal of an algorithm transforming a given tree into a new tree on which the fat representative $\widehat{\theta}$ presents minimal periodic orbit structure is to eliminate all recurrent bogus transitions. The guideline of the algorithm is therefore the detection of folds and the bogus transitions associated with each fold, and next to attempt to eliminate all bogus transitions associated with each fold. In this way, the folds and bogus transitions focus and organize the method.

The improved efficiency of the algorithm comes from the number of different moves required to transform the original line diagram into a suitable tree. There is essentially one move in the present algorithm consisting of eliminating regions of phase space, a property that was somehow guessed by Franks and Misiurewicz [12], who commented on the fact that in all their examples (published and unpublished) only the move called "gluing" appeared to be necessary. Gluing in [12] corresponds approximately to our collapse of a fold. In fact, [12] introduces splittings (the opposite of gluing) in order to solve two problems in their scheme, namely (a) to get rid of eventually periodic added vertices and (b) to assure that added vertices belonging to the same periodic orbit have the same valence. In our formulation, (a) is dealt with by collapses only while we dispose of (b) since it does not influence the periodic orbit structure of the map. However, no systematic efficiency comparisons were performed.

The recent paper by de Carvalho and Hall [15] deals with the possibility of *destroying dynamics* of a two-dimensional orientation-preserving homeomorphism. In fact, after constructing an object equivalent to $\widehat{T}$, they proceed to eliminate part of the dynamics in it by *prunings*, which loosely speaking are halfway between our collapses and Franks and Misiurewicz's gluings. The goal of that manuscript is to illustrate the action of pruning away part of the dynamics rather than finding the pseudo-Anosov representative (when proper) as in the present manuscript. However, the work shows that the pseudo-Anosov representative lies among the collection of pruned maps, referring to Bestvina and Handel's algorithm for its computation, improved by de Carvalho and Hall's pruning.

Apart from the different focus in both papers, the main difference with our paper lies in the way the *prunings* are done. In this respect, the introduction in the present work of silent collapses and a notion of complexity, which monitors the necessity and extent of a collapse, represents a definite advantage of the present algorithm. Indeed, prunings are performed with three basic steps, an identification step followed by a splitting step that

[4] The algorithm eliminates bogus transitions regardless of whether they are recurrent. The pertinence of an elimination should be controlled by the user.

roughly parallel the gluing and splittings in [12], and the third step is a move, resembling the dragging in [12]—consisting of the removal of added valence-2 vertices—warranting that no new periodic orbit is introduced. The authors of [15] note that the technicalities involved in the formalization and in showing the finiteness of the algorithm are "intricate, tedious" and the "effort is not worthwhile" [15, p. 328]. These inconveniences are dealt with by our method in a simpler way since the monitoring of the process via the complexity and silent collapses guarantees a finite algorithm that is free from splittings and that progresses monotonically, removing regions of the phase space.

Acknowledgments

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