Transverse, normal modes of vibration of a cantilever Timoshenko beam with a mass elastically mounted at the free end (L)

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An exact solution for the title problem is obtained by means of the classical eigenfunction approach. The natural frequencies are computed for a wide range of the intervening mechanical and geometric parameters. Normal modes of transverse vibration are plotted for some cases of practical interest. The problem is technically important in several areas of applied science and technology. © *2001 Acoustical Society of America.* [DOI: 10.1121/1.1416908]

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I. INTRODUCTION

The problem of transverse vibrations of structural elements (beams and plates) carrying masses rigidly attached to the support structure is of great technological importance in several fields of engineering: from civil engineering situations to mechanical, naval and aerospace systems passing through electronic packages which operate under severe dynamic excitations. Several studies are available on the subject matter.[1–3]

When shear and rotatory inertia effects are taken into account the problem is considerably more complex and the number of investigations published in the open literature is considerably more scarce.[4] The situation is even more critical if the mass is elastically attached to the structural element.[5]

The present investigation deals with the determination of natural frequencies and normal modes of transverse vibration of the system shown in Fig. 1.

II. ANALYTICAL SOLUTION

When the system vibrates in one of its normal modes one expresses the transverse displacement, $v(x,t)$, and the angle of rotation due to flexure $\psi(x,t)$ in the following manner:

$$v(x,t)=V(x)e^{i\omega t}, \tag{1}$$

$$\psi(x,t)=\psi(x)e^{i\omega t}, \tag{2}$$

where ω is the natural, circular frequency.

The solution of the system of differential equations of the Timoshenko beam is well known and may be expressed in the form[4]

$$V(x)=C_1 \sin \alpha x + C_2 \cos \alpha x + C_3 \sinh \beta x + C_4 \cosh \beta x, \tag{3}$$

$$L\psi(x)=-C_1\left(\frac{\delta}{\alpha L}\right)\cos \alpha x + C_2\left(\frac{\delta}{\alpha L}\right)\sin \alpha x + C_3\left(\frac{\varepsilon}{\beta L}\right)\cosh \beta x + C_4\left(\frac{\varepsilon}{\beta L}\right)\sinh \beta x, \tag{4}$$

where

$$\alpha=\sqrt{(H+\sqrt{H^2-4F})/2}, \tag{5a}$$

$$\beta=\sqrt{(-H+\sqrt{H^2-4F})/2}, \tag{5b}$$

$$\delta=\Omega^2\eta\lambda-\alpha^2L^2, \tag{6a}$$

$$\varepsilon=\Omega^2\eta\lambda+\beta^2L^2, \tag{6b}$$

$$H=[(1+\lambda)\Omega^2\eta]/L^2, \tag{7a}$$

$$F=[\Omega^2(\Omega^2\eta^2\lambda-1)]/L^4, \tag{7b}$$

$$\lambda=[2(1+\upsilon)]/k, \tag{8a}$$

$$\eta=r^2/L^2=I/A_0L^2, \tag{8b}$$

$$\Omega^2=\omega^2L^4(\rho A_0/EI), \tag{8c}$$

where k is the shear factor.

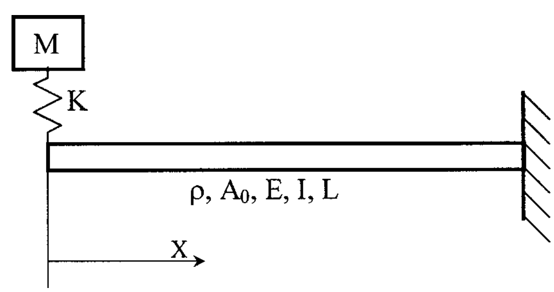

FIG. 1. Structural system under study.

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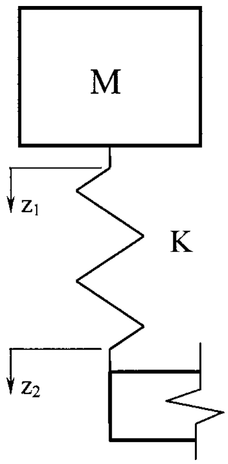

FIG. 2. Analysis of the discrete system attached to the beam tip.

The governing boundary conditions are

$$d\psi/dx|_{x=0}=0, \tag{9}$$

$$EA/\lambda\,[\psi(0)-|dV/dx|_{x=0}]=F_s, \tag{10}$$

$$V(L)=0, \tag{11}$$

$$\psi(L)=0, \tag{12}$$

where F_s is the force transmitted to the beam through the spring defined by its constant K.

Referring to Fig. 2 one determines F in the following

TABLE I. Values of frequency coefficients of the system shown in Fig. 1 for $r/L=0.01$.

K_1	m	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
⋯	0	⋯	3.512 65	21.8889	60.7409	117.516	191.18
0.1	0.2	0.695 153	3.571 3	21.898	60.7442	117.518	191.181
	0.5	0.439 825	3.569 91	21.898	60.7442	117.518	191.181
	1	0.311 043	3.569 46	21.898	60.7442	117.518	191.181
	2	0.219 939	3.569 24	21.898	60.7442	117.518	191.181
	3	0.179 611	3.569 16	21.898	60.7442	117.518	191.181
1	0.2	1.851 65	4.220 86	21.9808	60.7733	117.532	191.19
	0.5	1.205 22	4.101 43	21.9803	60.7732	117.532	191.19
	1	0.859 305	4.067 65	21.9801	60.7732	117.532	191.19
	2	0.610 001	4.0518	21.9800	60.7732	117.532	191.19
	3	0.498 694	4.046 66	21.9800	60.7732	117.532	191.19
10	0.2	2.528 59	9.3003	22.936	61.0723	117.681	191.279
	0.5	1.882 73	7.922 33	22.8719	61.0696	117.681	191.279
	1	1.418 75	7.440 48	22.8521	61.0687	117.681	191.279
	2	1.037 39	7.198 43	22.8425	61.0683	117.681	191.279
	3	0.856 732	7.117 89	22.8393	61.0681	117.681	191.279
∞	0.2	2.610 88	18.1103	52.8401	105.477	175.167	260.561
	0.5	2.015 08	16.8160	51.0214	103.425	172.997	258.340
	1	1.556 44	16.1700	50.2317	102.596	172.156	257.502
	2	1.157 60	15.7838	49.7918	102.149	171.710	257.062
	3	0.962 328	15.6438	49.6377	101.995	171.558	256.912

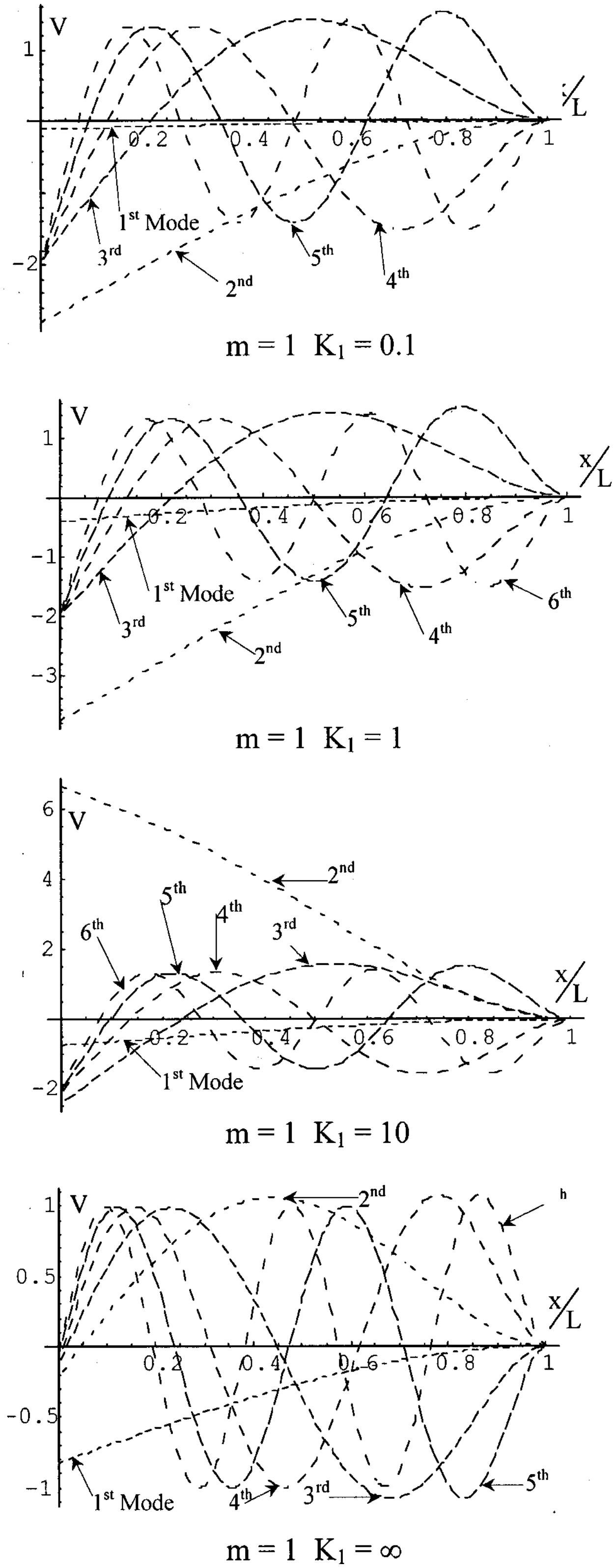

FIG. 3. Model shapes for $r/L=0.01$.

fashion.[5] Let z_1 be the displacement of the mass M and z_2 the one corresponding to the other end of the spring. Accordingly one has

$$M(d^2z_1/dt^2)=K(z_2-z_1). \tag{13}$$

Defining

TABLE II. Values of frequency coefficients of the system shown in Fig. 1 for $r/L=0.05$.

K_1	m	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
⋯	0	⋯	3.435 27	19.1036	46.6031	78.9022	113.750
0.1	0.2	0.694 873	3.493 78	19.1124	46.6062	78.9037	113.751
	0.5	0.439 659	3.492 33	19.1124	46.6062	78.9037	113.751
	1	0.310 927	3.491 86	19.1124	46.6062	78.9037	113.751
	2	0.219 873	3.491 63	19.1124	46.6062	78.9037	113.751
	3	0.179 53	3.491 55	19.1124	46.6062	78.9037	113.751
1	0.2	1.841 62	4.147 18	19.1927	46.6337	78.9174	113.759
	0.5	1.200 78	4.022 87	19.192	46.6337	78.9174	113.759
	1	0.856 542	3.987 88	19.1917	46.6337	78.9174	113.759
	2	0.608 171	3.971 48	19.1916	46.6337	78.9174	113.759
	3	0.497 234	3.966 17	19.1916	46.6337	78.9174	113.759
10	0.2	2.490 03	9.140 84	20.1613	46.9218	79.0568	113.84
	0.5	1.8601	7.772 75	20.0759	46.9174	79.0561	113.839
	1	1.403 97	7.291 43	20.0501	46.9159	79.0558	113.839
	2	1.027 52	7.0492	20.0377	46.9152	79.0557	113.839
	3	0.848 861	6.968 55	20.0336	46.9149	79.0557	113.839
∞	0.2	2.567 34	16.1768	41.6733	72.9088	107.23	143.104
	0.5	1.986 35	15.1074	40.3738	71.6557	106.056	142.035
	1	1.536 36	14.5623	39.7905	71.1294	105.58	141.608
	2	1.143 65	14.2331	39.4607	70.8408	105.323	141.379
	3	0.951 041	14.1131	39.3444	70.7405	105.234	141.300

$$z=z_2-z_1 \tag{14}$$

and replacing in (13) one obtains

$$M(d^2z_2/dt^2)=Kz+M(d^2z/dt^2), \tag{15}$$

and since

$$z_2=W(0)e^{i\omega t}, \tag{16}$$

substituting in (15) results in

$$M\,(d^2z/dt^2)+Kz=-\omega^2MW(0)e^{i\omega t}. \tag{17}$$

whose particular solutions is

$$z=[\omega^2MW(0)/\omega^2M-K]\,e^{i\omega t}. \tag{18}$$

Accordingly,

$$F_s=Kz=[\omega^2MW(0)/\omega^2M/K-1]\,e^{i\omega t}. \tag{19}$$

Substituting (3) and (4) in the governing boundary conditions and applying the nontriviality requirement to the resulting homogeneous system of equations,

$$\begin{vmatrix} 0 & \frac{\delta}{L^2} & 0 & \frac{\varepsilon}{L^2} \\ -\left(\frac{\delta}{\alpha L}+\alpha L\right) & -\frac{\Omega^2 m\eta\lambda}{1-\Omega^2 m/K_1} & \left(\frac{\varepsilon}{\beta L}-\beta L\right) & -\frac{\Omega^2 m\eta\lambda}{1-\Omega^2 m/K_1} \\ \sin\alpha L & \cos\alpha L & \sinh\beta L & \cosh\beta L \\ -\frac{\delta}{\alpha L}\cos\alpha L & \frac{\delta}{\alpha L}\sin\alpha L & \frac{\varepsilon}{\beta L}\cosh\beta L & \frac{\varepsilon}{\beta L}\sinh\beta L \end{vmatrix}=0, \tag{20}$$

where $m=M/\rho A_0 L$ and $K_1=K/(EI/L^3)$.

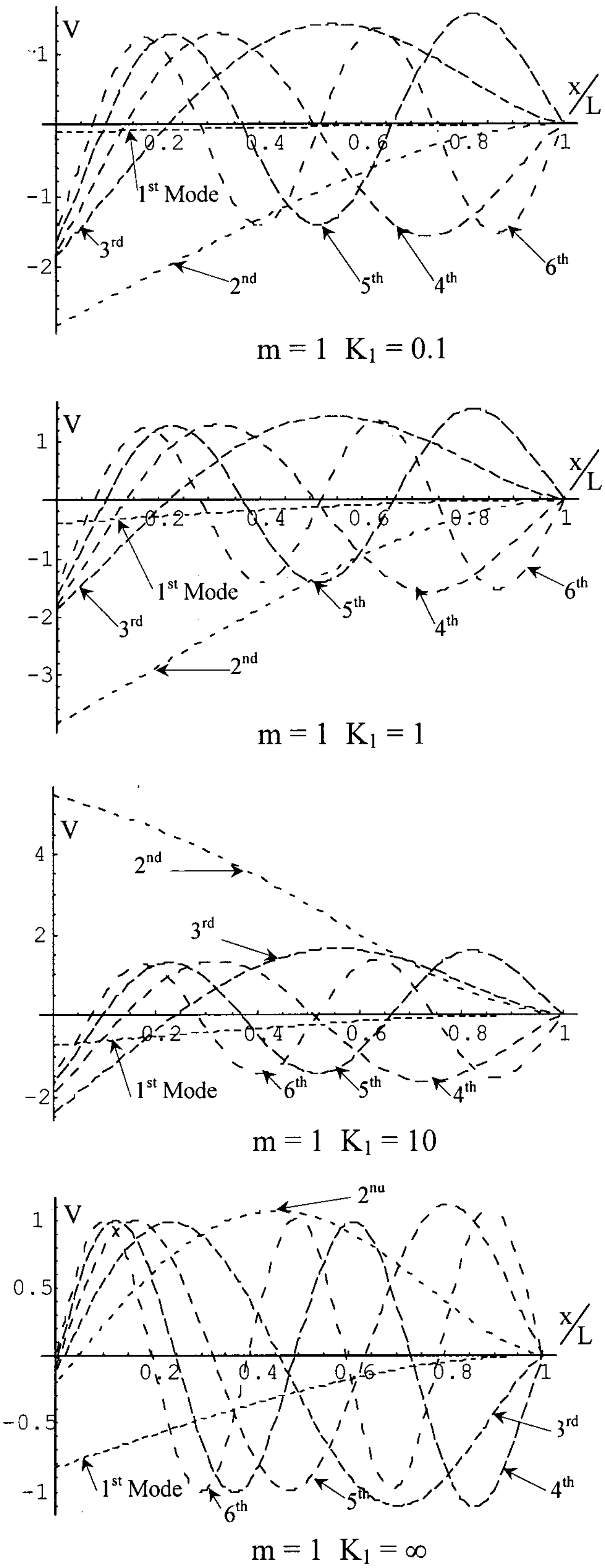

FIG. 4. Model shapes for $r/L=0.05$.

III. NUMERICAL RESULTS

The numerical determinations have been performed making Poisson's ratio (ν) equal to 0.3 and the shear factor (k) equal to $\frac{5}{6}$.

The frequency coefficients have been tabulated as a function of $K_1 = K/(EI/L^3)$ and $m = M/\rho A_0 L$.

Table I depicts values of Ω_i for $r/L = 0.01$ which corresponds to a rather thin beam. The case of $r/L = 0$ is obviously the Bernoulli–Euler situation and has been treated extensively in a previous publication.[6]

The first line of Table I corresponds to the vibrating bare beam (no spring–mass attached to the structural element). The values of Ω_i corresponding to $K_1 = 0.1$, 1, 10 and ∞ (rigidly attached) are depicted in the rest of the table for $m = 0.2$, 0.5, 1, 2, and 3. The lowest frequency determined for $K_1 = 0.1$, 1, and 10 is the frequency of the sprung system modified by the presence of the continuous structural element.

Table II shows values of Ω_i for a stiff beam ($r/L = 0.05$) and determined for the same values of K_1 and m which have been previously considered. As it was to be expected the frequencies corresponding to the spring–mass system experience minor change when compared with those obtained for $r/L = 0.01$.

Figures 3 and 4 depict the lower modal shapes of the system for $r/L = 0.01$ and 0.05, respectively.

One observes that for very small values of K_1 the beam exhibits, for both values of r/L, very small displacement amplitudes in correspondence with the first mode. On the other hand, the discrete system exhibits rather large vibration amplitudes.

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[1] T. W. Lee, ''Vibration frequency for a uniform beam with one end spring-hinged and carrying a mass at the other free end,'' J. Appl. Mech. **95**, 813–815 (1973).

[2] P. A. A. Laura, J. L. Pombo, and E. A. Susemihl, ''A note on the vibrations of a clamped-free beam with a mass at the free end,'' J. Sound Vib. **37**, 161–168 (1974).

[3] R. B. Bhat and H. Wagner, ''Natural frequencies of a uniform cantilever with a tip mass slender in the axial direction,'' J. Sound Vib. **45**, 304–307 (1976).

[4] R. E. Rossi, P. A. A. Laura, D. R. Avalos, and H. A. Larrondo, ''Free vibrations of Timoshenko beams carrying elastically mounted, concentrated masses,'' J. Sound Vib. **165**, 209–223 (1993).

[5] P. A. A. Laura, E. A. Susemihl, J. L. Pombo, L. E. Luisoni, and R. Gelos, ''On the dynamic behavior of structural elements carrying elastically mounted concentrated masses'' Appl. Acoust. **10**, 121–145 (1977).

[6] C. A. Rossit and P. A. A. Laura, ''Free vibrations of a cantilever beam with a spring–mass system attached to the free end,'' Institute of Applied Mechanics, Publication 2000-24 (Bahía Blanca, Argentina, 2000).