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Efficient and Perfect domination on circular-arc graphs

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Abstract

Given a graph $G = (V, E)$, a *perfect dominating set* is a subset of vertices $V' \subseteq V(G)$ such that each vertex $v \in V(G) \setminus V'$ is dominated by exactly one vertex $v' \in V'$. An *efficient dominating set* is a perfect dominating set V' where V' is also an independent set. These problems are usually posed in terms of edges instead of vertices. Both problems, either for the vertex or edge variant, remains NP-Hard, even when restricted to certain graphs families. We study both variants of the problems for the circular-arc graphs, and show efficient algorithms for all of them.

Keywords: Efficient Domination, Perfect Domination, Circular-Arc graphs

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1 Introduction

Given a graph $G = (V, E)$, a **perfect dominating set** is a subset of vertices $V' \subseteq V(G)$ such that each vertex $v \in V(G) \setminus V'$ is dominated by exactly one vertex $v' \in V'$. An **efficient dominating set** is a perfect vertex dominating set V' where V' is also an independent set. Every graph G contains a perfect dominating set, for instance, take $V(G)$. But not every graph contains an efficient vertex dominating set. These problems consists in searching the sets with minimum number of vertices. All of them are NP-hard, even when restricted to certain graph families. The weighted version of these problems, where each vertex v has a weight assigned $\omega(v)$, consists on finding a perfect vertex dominating set where the sum of the weights is minimum. We denote these problems as Minimum Weighted Perfect Vertex Domination (MWPVD), Minimum Weighted Efficient Vertex Domination (MWEVD). We denote the edge-versions of these problems as Minimum Weight Perfect Edge Domination (MWPED) and Minimum Weight Efficient Edge Domination (MWEED). Efficient edge dominating sets are also known as dominating induced matchings, and denoted as DIM's. Note that for these edge-versions the dominating set consists of edges instead of vertices, hence the weights are on the edges, and the adjacency of two edges is defined as two edges that shares a vertex. We say a *pendant* vertex (also known as *leaf*) is one whose degree is exactly one. In this paper we show results for the weighted perfect domination problem, and for the efficient domination problem, restricted to circular-arc graphs. The proofs and details of this paper can be found in [6].

2 Circular-Arc graphs

The following definitions and results come from [4]

Given a circular-arc model $\mathcal{M} = (C, \mathcal{A})$ where $\mathcal{A} = \{A_1 = (s_1, t_1), \ldots, A_n = (s_n, t_n)\}$, two points $p, p' \in C$ are equivalent if $\mathcal{A}(p) = \mathcal{A}(p')$. The $2n$ extreme points from the n arcs of $\mathcal{A}$ divide the circle C in $2n$ segments of the following types: (i) (s_i, t_j) (ii) $[t_i, t_j)$ (iii) $(s_i, s_j]$ (iv) $[t_i, s_j]$. We say the segments of type (i) are *intersection segments*. It is easy to see that all points inside one of the $2n$ segments are equivalent.

Corollary 2.1 *[4]There are at most $2n$ distinct $\mathcal{A}(p)$.*

Lemma 2.2 *[4] Given a CA model $\mathcal{M} = (C, \mathcal{A})$, if there are no two or three arcs of $\mathcal{A}$ that covers the entire circle C then $\mathcal{M}$ is an HCA model.*

We consider the four variants of the mentioned problems for circular-arc graphs, we propose efficient algorithms to solve them, except for the MWEVD for which already exists a linear time algorithm. We assume that our input is a circular-arc graph G and a weight function ω over the vertices or edges depending on the problem we were solving. For simplicity, we may use the circular-arc model from G. For these cases, there is an implicit previous step which is applying a linear time algorithm [10] in order to obtain a circular-arc model from G. For MWEVD and MWEED, we consider the function ω nonnegative.

3 Minimum weighted efficient vertex domination

The minimum weighted efficient vertex domination problem (MWEVD) on a graph G can be expressed as an instance of the minimum weighted dominating set problem (MWDS) on the same graph G making some minor adjustments on the way described in [1] for unweighted version of MWEVD. There is an $O(n+m)$ time algorithm to solve MWDS for circular-arc graphs [2]. Hence, MWEVD can be solved for circular-arc graphs in linear time.

4 Minimum weighted efficient edge domination

The Minimum weighted efficient edge domination problem (MWEED) for a graph G is equivalent to either:

(i) MWEVD for the line graph $L(G)$

(ii) MWIS for the square of the line graph $L^2(G)$

If G is a circular-arc graph, then the graph $G' = L^2(G)$ is also a circular-arc graph. The graph G' has exactly m vertices and up to $O(m^2)$ edges. The algorithm from [11] solves MWIS problem in linear time for circular-arc graphs. It is known that a graph that admits a DIM is K_4-free. This property imposes a bound on the number of edges in circular-arc graphs, thus the algorithm complexity bound can be improved from $O(m^2)$ to $O(n^2)$.

Lemma 4.1 *Every K_4-free graph G where $|V(G)| \geq 2$ such that G is circular-arc graph has at most $2n$ edges.*

A linear-time algorithm to solve MWEED for general graphs given a fixed dominating set was presented in [7]. If there exists a set of at most three arcs that covers the entire circle, then there is a dominating set of size at most 3, thus the problem can be solved using the mentioned algorithm in linear time.

We assume a model $\mathcal{M}$ without a set S that covers the entire circle such that $|S| \leq 3$. Therefore $\mathcal{M}$ is a Helly circular-arc (HCA) model and the original graph G is an HCA.

We analyze the model $\mathcal{M}$ (each case can be implemented in linear time):

- **(a)** $max_{p \in C}|\mathcal{A}(p)| \geq 4$**:** Then G is not K_4-free, hence it does not admit a DIM
- **(b)** $min_{p \in C}|\mathcal{A}(p)| = 0$**:** Then $\mathcal{M}$ is an interval model. Algorithm from [9] can be applied
- **(c)** $max_{p \in C}|\mathcal{A}(p)| = 2$**:** The detailed proof can be found in [6].
- **(d)** $max_{p \in C}|\mathcal{A}(p)| = 3$**:** The detailed proof can be found in [6].

5 Minimun weighted perfect vertex Domination

An $O(n + m)$ time algorithm to solve MWPVD for interval graphs was presented in [3]. The same paper shows the only known algorithm to solve MW-PVD for circular-arc graphs in $O(n^2+nm)$ time. Note that any efficient vertex dominating set is also a perfect vertex dominating set.

Given a circular-arc graph G, a circular-arc model $\mathcal{M}$ from G can be obtained in $O(n + m)$ time and universal arcs can be identified in $O(n)$ time. If a universal arc exists, we can solve the problem using the procedure shown in [6]. Thus we assume G does not contain a universal vertex.

It is possible to solve MWEVD in linear time for circular-arc graphs. Hence we can save the best efficient vertex dominating set as a candidate solution, and search for the perfect vertex dominating sets that are not efficient vertex dominating sets. We determine in $O(n)$ time the point p such that $|\mathcal{A}(p)|$ is minimum, and according to this value a different approach can be used. (see [6] for more details)

6 Minimum weighted perfect edge domination

We give an $O(n+m)$ time algorithm to solve MWPED for circular-arc graphs. To the best of our knowledge there is no known polynomial time algorithm to solve this problem on circular-arc graphs. It is proved in [9] the NP-completeness of unweighted version of this problem for bipartite graphs.

Theorem 6.1 *[9] There is an $O(n+m)$ time algorithm to solve MWPED on chordal graphs*

Corollary 6.2 *[9] There is an $O(n+m)$ time algorithm to solve MWPED on* *interval graphs*

Definition 6.3 Given a graph $G = (V, E)$ and a perfect edge dominating set $E' \subseteq E$ from G, we denote $D = \{v \in V : vw \in E'\}$ the vertices incident to an edge from E'. We can define the following 3-coloring for the vertices of G: The black vertices $B = \{v \in D : N[v] \subseteq D\}$, the gray vertices $R = D \setminus B$ and the white vertices $W = V(G) \setminus D$.

The following properties can be easily checked:

(P1) Each gray vertex has exactly one non-white neighbor while the rest of his neighborhood are white vertices. (the gray vertex has degree at least 2)

(P2) If $v \in W$, then $N(v) \subseteq R$. Hence W is an independent set.

It is easy to see that for any 3-coloring of vertices that satisfies properties (P1) and (P2) $E' = \{vw \in E : vw \in B \cup R\}$ is a perfect edge dominating set and for any 3-coloring of G that satisfies (P1) and (P2), if it contains K_p, with $p \geq 4$, then vertices from K_p should be black.

Any efficient edge dominating set of a graph G (if it exists), is also a perfect edge dominating set from G.

Given a circular-arc graph $G = (V, E)$, we show how to solve MWPED in linear time. First, solve MWEED in $O(n)$ time. If there is a solution (DIM), we save the one with minimum weight as a candidate solution. Therefore the candidates that should be explored are the perfect edge dominating sets that are not DIM. Note that the set E is also a candidate solution.

We obtain circular-arc model $\mathcal{M} = (C, \mathcal{A})$ from G. We solve MWPED for G according to the following properties from $\mathcal{M}$. You can check [6] for the details and proofs of each case.

(a) There are 2 arcs, $A_v = (s, v, t_v), A_w = (s, w, t_w) \in \mathcal{A}$ such that $A_v \cup A_w = C$. Let $E' \neq E$ be a perfect edge dominating set which is not a DIM.

(b) It does not satisfy (a) and there are three arcs $A_v, A_w, A_z \in \mathcal{A}$ such that $A_v \cup A_w \cup A_z = C$. Let $E' \neq E$ a perfect edge dominating set such that is not a DIM.

(c) It satisfies neither (a) nor (b), in this case $\mathcal{M}$ is a Helly circular-arc model.

References

[1] A. Brandstädt, A. Leitert, and D. Rautenbach. Efficient dominating and edge dominating sets for graphs and hypergraphs. In *ISAAC'12 International Symposium on Algorithms and Computation*. LNCS 7676, pp. 267–277, 2012.

[2] M. Chang. Efficient algorithms for the domination problems on interval and circular-arc graphs. *SIAM J. Comput.* 27, pp. 1671–1694, 1998.

[3] M. Chang, and Y. Liu. Polynomial algorithms for weighted perfect domination problems on interval and circular-arc graphs. *J. Inf. Sci. Eng.* 11, pp. 549–568, 1994.

[4] G. Durán, M.C. Lin, S. Mera, and J.L. Szwarcfiter. Algorithms for clique-independent sets on subclasses of circular-arc graphs. *Discrete Appl. Math.* 154, pp. 1783–1790, 2006.

[5] M.C. Lin, R.M. McConnell, F.J. Soulignac, and J.L. Szwarcfiter. On cliques of Helly circular-arc graphs. *Electron. Notes Discrete Math.* 30, pp. 117–122, 2008.

[6] M.C. Lin, M.J. Mizrahi, and J.L. Szwarcfiter. Efficient and Perfect domination on circular-arc graphs. *CoRR*, abs/1502.01523, 2015.

[7] M.C. Lin, M.J. Mizrahi, and J.L. Szwarcfiter. Exact algorithms for dominating induced matchings. *CoRR*, abs/1301.7602, 2013.

[8] M.C. Lin, M.J. Mizrahi, and J.L. Szwarcfiter. Fast algorithms for some dominating induced matching problems. *Inf. Process. Lett.* 114, pp. 524–528, 2014.

[9] C. L. Lu, M.-T. Ko and C. Y. Tang. Perfect edge domination and efficient edge domination in graphs. *Discrete Appl. Math.* 119, pp. 227–250, 2002.

[10] R.M. McConnell. Linear-Time Recognition of Circular-Arc Graphs *Algorithmica* 37, pp. 93–147, 2003.

[11] J. Spinrad. *Efficient Graph Representations.: The Fields Institute for Research in Mathematical Sciences.* Fields Institute monographs. American Mathematical Soc., 2003.