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On a generalized entropic uncertainty relation in the case of the qubit

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Abstract

We revisit generalized entropic formulations of the uncertainty principle for an arbitrary pair of quantum observables in two-dimensional Hilbert space. Rényi entropy is used as an uncertainty measure associated with the distribution probabilities corresponding to the outcomes of the observables. We derive a general expression for the tight lower bound of the sum of Rényi entropies for any couple of (positive) entropic indices (α, β). Thus, we have overcome the Hölder conjugacy constraint imposed on the entropic indices by Riesz–Thorin theorem. In addition, we present an analytical expression for the tight bound inside the square $[0\,,\ 1/2]^2$ in the α–β plane, and a semi-analytical expression on the line $\beta = \alpha$. It is seen that previous results are included as particular cases. Moreover we present a semi-analytical, suboptimal bound for any couple of indices. In all cases, we provide the minimizing states.

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1. Introduction

The uncertainty principle (UP) is a fundamental concept in physics that states the impossibility of predicting with absolute certainty and simultaneously the outcomes of measurements for pairs of noncommuting quantum observables. In its primary quantitative formulation, the principle is described by the existence of a nontrivial lower bound for the product of the variances of the operators [1–3]. However, such formulations are not always adequate due to various reasons. As an example, there exist variables with infinite variance [4], so that the second-order moment is not always convenient for describing the dispersion of a random variable. Moreover, in the case of discrete-spectrum observables, there is no universal nontrivial lower bound, and thus Heisenberg-like inequalities do not quantify the UP [5–7].

In order to overcome the potential inadequacy of the variance-based expression of the UP, many formulations based on other measures of dispersion have been proposed, for instance issuing from information theory [8–10]. The pioneering works of Hirschman [11], Bialynicki-Birula and Mycielski [12] based on important results from Beckner [13], Deutsch [14] or Maassen and Uffink [15] who proved a result conjectured by Kraus [16], have given rise to many versions based on generalized information entropies (or entropic moments) [17–25], on Fisher information [26–28], or on moments of various orders [29]. Recently, generalized versions of entropic and support inequalities in the context of variables described by frames instead of bases, have been proposed [30].

In this paper, we focus on the Rényi entropy formulation of UP in the case of discrete-spectrum operators. Specifically, we search for (tight) lower bounds for the sum of Rényi entropies associated with the outcomes of a pair of observables. In the majority of previous related studies, the entropic indices corresponding to both observables are considered to be conjugated in the sense of Hölder, since the proofs make use of Riesz–Thorin or Young–Hausdorff theorems. Extensions for nonconjugated indices exist, based on the decreasing property of Rényi entropy versus its index, leading then to suboptimal bounds [22, 30]. These bounds have been refined in the case of two-level systems (or qubits) when the entropic indices coincide and have the value $1/2$ [31] or 2 [32, 33]. Here we extend these results beyond the scope of Riesz's theorem, allowing for *arbitrary* couples of indices. We provide a semi-analytical treatment of the problem and we find significant, nontrivial inequalities expressing UP for qubits. Moreover, we supply the minimizing states for the uncertainty relations established.

The paper is organized as follows. In section 2, we begin with basic definitions and notation, and summarize the known results concerning generalized entropic uncertainty relations for N-level systems. In section 3 we state the problem for qubits and present our major results. A discussion is provided in section 4. The proofs of our results are given in the appendices.

2. Statement of the problem: notation and previous results

We consider pairs of quantum observables, say A and B, with discrete spectra on an N-dimensional Hilbert space $\mathcal{H}$. Pure states $|\Psi\rangle \in \mathcal{H}$ can be expanded onto the corresponding orthonormal eigenbases $\{|a_k\rangle\}_{k=1}^N$ and $\{|b_l\rangle\}_{l=1}^N$. In order to fix the notation, we write $|\Psi\rangle = \sum_{k=1}^N \psi_k|a_k\rangle = \sum_{l=1}^N \widetilde{\psi}_l|b_l\rangle$ where the ψ_k and $\widetilde{\psi}_l$ are complex coefficients, which we arrange in column vectors: $\psi = [\psi_1 \cdots \psi_N]^t$ and $\widetilde{\psi} = [\widetilde{\psi}_1 \cdots \widetilde{\psi}_N]^t$. From orthonormality of the bases, one has

$$\widetilde{\psi} = T\psi \qquad \text{where} \qquad T_{lk} = \langle b_l|a_k\rangle, \tag{1}$$

being T an $N \times N$ unitary matrix.

Vectors ψ and $\widetilde{\psi}$ are such that $\|\psi\|_2^2 = \sum_k |\psi_k|^2 = 1$ and similarly $\|\widetilde{\psi}\|_2^2 = 1$. A vector with components $|\psi_k|^2 = |\langle a_k|\Psi\rangle|^2$ is interpreted as a probability vector (for brevity, we denote this vector as $|\psi|^2$), where $|\psi_k|^2$ represents the probability of measuring eigenvalue a_k as the outcome for observable A when the quantum system is in the state $|\Psi\rangle$ (resp. $|\widetilde{\psi}_l|^2 = |\langle b_l|\Psi\rangle|^2$ for measurement of B).

We are interested in uncertainty relations concerning the simultaneous observations of two magnitudes, particularly their statement through the use of information-theoretic quantities [10]. The measure of ignorance or lack of information that we employ is the Rényi entropy [9] of a probability vector $p = [p_1 \cdots p_N]^t$ (with $p_k \geqslant 0$ and $\sum_{k=1}^N p_k = 1$):

$$H_\lambda(p) = \frac{1}{1-\lambda} \log \|p\|_\lambda^\lambda = \frac{1}{1-\lambda} \log \left(\sum_k p_k^\lambda \right) \tag{2}$$

where $\lambda \geqslant 0$ is the entropic index and 'log' stands for natural logarithm. The limiting case $\lambda \to 1$ is well defined and gives Shannon entropy $H_1(p) \equiv H(p) = -\sum_k p_k \log p_k$. The index λ plays the role of a magnifying glass in the following sense: when $\lambda < 1$, the contribution of the different terms in sum (2) becomes more uniform with respect to the case $\lambda = 1$; conversely, when $\lambda > 1$, the leading probabilities of the distribution are stressed in the summation. Indeed, in the extreme case $\lambda = 0$, H_0 is simply the number of nonzero components of the probability vector, regardless of the values of the probabilities; this measure is closely linked to the L^0 norm which is relevant in signal processing [30, 34, 35]. In contrast, $H_\infty = -\log(\max_k p_k)$ only takes into account the maximum component of the probability vector, and is known as min-entropy due to the nonincreasing property of H_λ versus λ for a given probability distribution. Another relevant property is that Rényi entropy H_λ is concave for $\lambda \in [0\,,1]$, or even when $\lambda \in [0\,,\lambda_*(N)]$ where the upper limit depends on the dimension of the probability vector [e.g. $\lambda_*(2) = 2$] [36, p 57]. Rényi entropies appear naturally in several contexts, as signal processing (Chernoff bound, Panter–Dite formula) [10, 37–39 and references therein], multifractal analysis [40–42], or quantum physics (collision entropy, purity, informational energy, Gini–Simpson index, index of coincidence, repeat rate) [25, 42–45 and references therein].

One can easily verify that Rényi entropies are positive and that in the N-states case they are upper-bounded by $\log N$: $0 \leqslant H_\lambda(p) \leqslant \log N$. The lower bound is achieved when the probability distribution is a Kronecker-delta, $p_k = \delta_{k,i}$ for certain i, and the upper bound corresponds to the uniform distribution, $p_k = 1/N$.

In this contribution we will consider the Rényi entropies of the probability vectors $|\psi|^2$ and $|\widetilde{\psi}|^2$, associated with the measurement of observables A and B, respectively. The fact that the sum of both entropies is lower bounded has given rise to the so-called entropic uncertainty relations, which are of the type

$$H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant \overline{\mathcal{B}}_{\alpha,\beta;N} \tag{3}$$

for any couple of (positive) entropic indices (α, β), where the bound $\overline{\mathcal{B}}_{\alpha,\beta;N}$ is nontrivial, i.e. nonzero, and universal in the sense of being independent of the state $|\Psi\rangle$ of the quantum system. The ultimate goal is to find the tightest bound, which by definition is obtained by minimization of the left-hand side, thus

$$\mathcal{B}_{\alpha,\beta;N} \equiv \min_{|\Psi\rangle} \left(H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2)\right) \geqslant \overline{\mathcal{B}}_{\alpha,\beta;N}. \tag{4}$$

It turns out that the tight bound $\mathcal{B}_{\alpha,\beta;N}$ only depends on the transformation matrix T in which an important characteristic is the so-called overlap (or coherence) between the eigenbases, given by

$$c = \max_{k,l} |\langle b_l | a_k \rangle|.$$

From the unitarity property of matrix T, the overlap is in the range $c \in [1/\sqrt{N}\,,\ 1]$. The case $c = 1/\sqrt{N}$ corresponds to observables A and B being complementary meaning that maximum certainty in the measure of one of them implies maximum ignorance about the other. In contrast $c = 1$ corresponds to the observables A and B share (at least) an eigenvector; this situation happens, for example, when the observables commute.

The problem has been addressed in various contexts, and in some cases numerical and/or analytical bounds have been found. Several results correspond to conjugated indices (in the sense of Hölder[3], i.e. $\frac{1}{2\alpha} + \frac{1}{2\beta} = 1$) as they are based on the Riesz–Thorin theorem [46];

[3] More rigorously, 2α and 2β are Hölder-conjugated. Here we employ this terminology for α and β, by misuse of language.

however there exist few results for nonconjugated indices. Before summarizing the known results in the literature, let us define the following regions in the α–β plane:

$$\mathcal{C} = \left\{(\alpha,\beta) : \beta = \frac{\alpha}{2\alpha-1} \quad \text{with} \quad \alpha > 1/2,\ \beta > 1/2\right\}$$
$$\underline{\mathcal{C}} = \{(\alpha,\beta) : 0 \leqslant \alpha \leqslant 1/2,\ \beta \geqslant 0\} \cup \left\{(\alpha,\beta) : \alpha > 1/2,\ 0 \leqslant \beta < \frac{\alpha}{2\alpha-1}\right\}$$
$$\overline{\mathcal{C}} = \left\{(\alpha,\beta) : \alpha > 1/2,\ \beta > \frac{\alpha}{2\alpha-1}\right\} \tag{5}$$

$\mathcal{C}$ is called the *conjugacy curve*, while $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are referred to as 'below the conjugacy curve' and 'above the conjugacy curve', respectively. Now a summary of results available in the literature follows:

- For $(\alpha,\beta) \in \mathcal{C}$ (conjugacy curve), N-level systems and any overlap c: the bound $\overline{\mathcal{B}}_{\alpha,\beta;N} = -2\log c$ can be deduced from the paper by Maassen and Uffink [15]. In general, this bound is not tight. Particular cases have been studied.
  - * $c = 1/\sqrt{N}$ (complementary observables): the bound $\mathcal{B}_{\alpha,\beta;N} = \log N$ is tight [20, 22].
  - * $\alpha = \beta = 1$ (Shannon entropies): Deutsch [14] found the bound $\overline{\mathcal{B}}_{1,1;N} = -2\log(\frac{1+c}{2})$, which has been improved by Maassen and Uffink [15, 16]. A further improvement has been given by de Vicente and Sanchez-Ruiz [47, 48] in the range $c \in [c^*, 1]$ with $c^* \simeq 0.834$, by using Landau–Pollak inequality linking $\max_k |\psi_k|^2$ and $\max_l |\widetilde{\psi}_l|^2$. These bounds are not tight, except for complementary observables (see also [20, 22]) or for $N = 2$ (qubits) [49]. The Shannon entropic uncertainty relation in the case of the qubit has been treated by Garret and Gull [50] and by Sánchez-Ruiz [51].
- For $(\alpha,\beta) \in \underline{\mathcal{C}}$ (below the conjugacy curve), N-level systems, and for any c: the Deutsch and Maassen–Uffink bounds remain to be valid due to the decreasing property of the Rényi entropy versus the entropic index [46]. For $c = 1/\sqrt{N}$, the bound $\mathcal{B}_{\alpha,\beta;N} = \log N$ is tight [20, 22].
- For $(\alpha,\beta) \in \overline{\mathcal{C}}$ (above the conjugacy curve), N-level systems, and for any c: Maaseen and Uffink obtained a suboptimal relation minimizing the sum of min-entropies restricted to the Landau–Pollak inequality, where the bound is the same as given by Deutsch. For $c = 1/\sqrt{N}$, a known bound is $\overline{\mathcal{B}}_{\alpha,\beta;N} = 2\log\left(\frac{2\sqrt{N}}{1+\sqrt{N}}\right)$ [15]. However, this bound is not tight; indeed for the particular case $\alpha = \beta = 2$ it has been improved by Luis [52] to the value $2\log\left(\frac{2N}{N+1}\right)$.
- For $\beta = \alpha$, N-level systems, and any overlap c, using Schur convexity arguments Puchała *et al* derived recently the bound $\overline{\mathcal{B}}_{\alpha,\alpha;N} = \frac{1}{1-\alpha}\log\{(\frac{1+c}{2})^{2\alpha} + [1-(\frac{1+c}{2})^2]^\alpha\}$ [53]. In general, this bound is not tight.
- For $\alpha = \beta = 1/2$, $N = 2$, and arbitrary c: the optimal bound $\mathcal{B}_{\frac{1}{2},\frac{1}{2};2} = \log[1 + \sqrt{4c^2(1-c^2)}]$ was obtained by Rastegin [31].
- For $\alpha = \beta = 2$ (collision entropies), $N = 2$, and arbitrary c: the tight bound was found by Bosyk *et al* [32], with $\mathcal{B}_{2,2;2} = -2\log(\frac{1+c^2}{2})$.
- For $\alpha = \beta \in (0, 2]$, $N = 2$ and $c = 1/\sqrt{2}$, the optimal bound was analyzed in the context of the Mach–Zehnder interferometric setting [33] with $\mathcal{B}_{\alpha,\alpha;2} = \log 2$ if $\alpha \leqslant \alpha^\dagger \approx 1.43$ and $\mathcal{B}_{\alpha,\alpha;2} = \frac{2}{1-\alpha}\log[(\frac{1+1/\sqrt{2}}{2})^2 + (\frac{1-1/\sqrt{2}}{2})^2]$ otherwise.

3. General Rényi entropic uncertainty relations for qubits

In this contribution we deal with the problem of generalizing the last three developments summarized in the preceding section and those by Sánchez-Ruiz [51] and Ghirardi *et al* [49] as well, i.e. for qubits ($N = 2$) and any overlap c, to the case of arbitrary Rényi entropy indices (α, β) to measure uncertainty. We seek the minimum of the entropies' sum in this general situation and also study those states that saturate the bound. Our major results are given by the following propositions.

Proposition 1. *Let us consider a pair of quantum observables A and B acting on a two-dimensional Hilbert space, and the corresponding eigenbases* $\{|a_1\rangle, |a_2\rangle\}$ *and* $\{|b_1\rangle, |b_2\rangle\}$*. Consider a quantum system in the qubit pure state* $|\Psi\rangle$ *described by the projections* $\psi = [\psi_1 \quad \psi_2]^t$ *or* $\widetilde{\psi} = [\widetilde{\psi}_1 \quad \widetilde{\psi}_2]^t = T\psi$ *on those bases respectively, where* $T_{lk} = \langle b_l|a_k\rangle$ *for* $k, l = 1, 2$*. Then, for any couple of Rényi entropic indices* $(\alpha, \beta) \in \mathbb{R}_+^2$*, the following uncertainty relation holds:*

$$H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant \mathcal{B}_{\alpha,\beta;2}(c) \tag{6}$$

where the tight lower bound for the sum of Rényi entropies is obtained as

$$\mathcal{B}_{\alpha,\beta;2}(c) = \min_{\theta\in[0\,,\,\gamma]} \left(\frac{\log \mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log \mathcal{D}_\beta(\gamma-\theta)}{1-\beta} \right) \tag{7}$$

with

$$c \;\equiv \max_{k,l=1,2} |T_{lk}| \;\in \left[\frac{1}{\sqrt{2}}\,,\, 1 \right], \qquad \gamma = \arccos c \tag{8}$$

and

$$\mathcal{D}_\lambda(\theta) \equiv (\cos^2\theta)^\lambda + (\sin^2\theta)^\lambda. \tag{9}$$

Furthermore, for any pair of two-dimensional observables we advance the minimizing solution.

Proposition 2. *Under the conditions of proposition 1, let us parameterize the matrix T in the form [54, 55]*

$$T = \Phi(\boldsymbol{u})\, V(\gamma_T)\, \Phi(\boldsymbol{v}) \tag{10}$$

where

$$\Phi(\cdot) = \exp(\imath\, \mathrm{diag}(\cdot)) \quad \text{and} \quad V(\gamma_T) = \begin{bmatrix} \cos\gamma_T & \sin\gamma_T \\ -\sin\gamma_T & \cos\gamma_T \end{bmatrix}, \tag{11}$$

in terms of $\gamma_T \in [0\,,\, \frac{\pi}{2}]$ *and the 2D real vectors* $\boldsymbol{u}, \boldsymbol{v}$*. Denote by* $\{\theta_{\mathrm{opt}}^{(i)}\}_{i\in\mathcal{I}}$ *the set of arguments that minimize the expression in equation (7), where* $\mathcal{I}$ *lists all the different possible solutions. Then the bound is achieved for the qubits whose projections onto the A-eigenbasis are*

$$\psi_{\mathrm{opt}}^{(i,n,\varphi)} = \mathrm{e}^{\imath\varphi}\, \Phi(-\boldsymbol{v}) \begin{bmatrix} \cos\left(\varepsilon_T \theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \\ \sin\left(\varepsilon_T \theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \end{bmatrix} \tag{12}$$

with

$$\varphi \in [0\,,\, 2\pi), \quad \varepsilon_T = \mathrm{sign}\left(\frac{\pi}{4} - \gamma_T\right) \quad \text{and} \quad n = 0, 1. \tag{13}$$

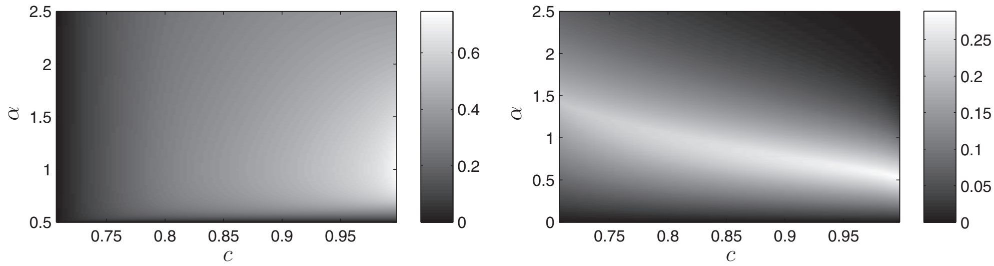

Figure 1. Relative difference $\frac{\mathcal{B}_{\alpha,\beta;2}(c)-\overline{\mathcal{B}}_{\alpha,\beta;2}(c)}{\mathcal{B}_{\alpha,\beta;2}(c)}$ between our bound $\mathcal{B}_{\alpha,\beta;2}(c)$, equation (7) and: (left) the Maassen–Uffink bound $\overline{\mathcal{B}}_{\alpha,\beta;2}(c) = -2\log c$ for conjugated entropic indices $\beta = \frac{\alpha}{2\alpha-1}$; (right) the Puchała *et al* bound $\overline{\mathcal{B}}_{\alpha,\alpha;2}(c) = \frac{1}{1-\alpha}\log\left\{\left(\frac{1+c}{2}\right)^{2\alpha} + \left[1-\left(\frac{1+c}{2}\right)^2\right]^\alpha\right\}$ for equal entropic indices $\beta = \alpha$.

We now concentrate on discussing some derivations of our approach, and postpone the proofs of the propositions to the appendices. To begin with, we make a connection with the so-called Landau–Pollak uncertainty inequality [56]. Although our proofs do not rely on this uncertainty relation, we can link *a posteriori* both results when the inequalities are saturated. For that purpose, let us introduce the probability vectors P^A and P^B respectively issued from the optimal states $\psi_{\text{opt}}^{(i,n,\varphi)}$ and $\widetilde{\psi}_{\text{opt}}^{(i,n,\varphi)} = T\psi_{\text{opt}}^{(i,n,\varphi)}$, namely

$$P^A = \begin{bmatrix} \cos^2\left(\varepsilon_T\theta_{\text{opt}}^{(i)} + n\frac{\pi}{2}\right) \\ \sin^2\left(\varepsilon_T\theta_{\text{opt}}^{(i)} + n\frac{\pi}{2}\right)\end{bmatrix} \quad \text{and} \quad P^B = \begin{bmatrix} \cos^2\left(\gamma_T - \varepsilon_T\theta_{\text{opt}}^{(i)} - n\frac{\pi}{2}\right) \\ \sin^2\left(\gamma_T - \varepsilon_T\theta_{\text{opt}}^{(i)} - n\frac{\pi}{2}\right)\end{bmatrix}.$$

A rapid inspection of the different cases for γ_T and n allows us to obtain

$$\arccos\sqrt{\max_{k=1,2} P_k^A} + \arccos\sqrt{\max_{l=1,2} P_l^B} = \arccos c \tag{14}$$

where $\arccos c = \gamma = \min(\gamma_T, \frac{\pi}{2} - \gamma_T) = \frac{\pi}{4} - |\frac{\pi}{4} - \gamma_T| \in [0, \frac{\pi}{4}]$. This corresponds precisely to the equality in the Landau–Pollak relation. This relation is explicitly used by Maassen and Uffink [15] when recovering the bound of Deutsch [14], and by de Vicente and Sanchez-Ruiz [47] to obtain their inequality.

Our bound (7) is better than (or at least equal to) all bounds that can be found in the literature dealing with the qubit, which is obvious since we are solving the optimization problem here for any couple of indices (α, β). Moreover, we stress that here we do obtain wavevectors (12) that saturate the inequality. As an illustration of the improvement provided in this contribution, we show the relative difference between our bound (7) and the Maassen–Uffink (figure 1 (left)) and Puchała *et al* (figure 1 (right)) bounds in some particular situations. It can be seen from these density plots that the differences are always positive and, in the cases shown, they can grow up to 77% or 28%, respectively.

We mention also that in the general case of arbitrary Rényi entropy indices to measure uncertainty, the bound $\mathcal{B}_{\alpha,\beta;2}$, equation (7), has to be sought numerically. However, for indices in some regions of the α–β plane, we are able to obtain analytical or semi-analytical results. These are presented in the following corollaries.

Corollary 1. *In the context of propositions 1 and 2, if the entropic indices lie within the square*

$$(\alpha, \beta) \in [0\,,\ 1/2]^2, \tag{15}$$

there exists an analytical expression for the bound in the form

$$\mathcal{B}_{\alpha,\beta;2}(c) = \frac{\log[(c^2)^\lambda + (1-c^2)^\lambda]}{1-\lambda} \quad \textit{where} \quad \lambda = \max(\alpha, \beta). \tag{16}$$

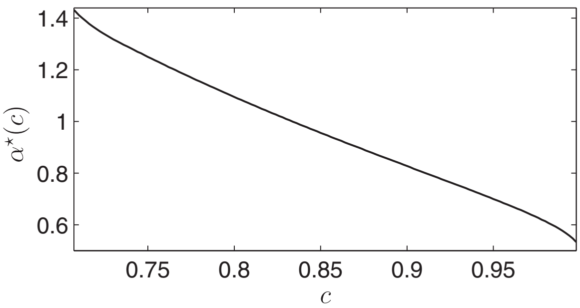

Figure 2. $\alpha^\star(c)$ versus $c \in \left[\frac{1}{\sqrt{2}}\,,\,1\right)$.

Moreover, the wavevectors that saturate the inequality correspond to: $\theta_{\mathrm{opt}} = 0$ *if* $\alpha < \beta$, $\theta_{\mathrm{opt}} = \gamma$ *if* $\alpha > \beta$, *and both solutions if* $\alpha = \beta$.

First, one can observe a transition in terms of entropic indices at $\alpha = \beta$, since only in this situation do both angles lead to wavefunctions that saturate the inequality. We notice that corollary 1 includes some of the situations discussed at the end of section 2 as particular cases. On the one hand, when c is fixed to $1/\sqrt{2}$, the optimal bound of [20, 22] is recovered, and if $\alpha = \beta$ this bound coincides with that given in [33]. On the other hand, when c is unrestricted and if $\alpha = \beta = 1/2$, one recovers the bound obtained in [31]. We stress that these results have been proven analytically and extended the scope for any c and for any couple (α, β) in the square $[0\,,\,\frac{1}{2}]^2$.

On the line $\beta = \alpha$, we obtain a semi-analytical result as follows:

Corollary 2. *In the context of propositions 1 and 2, if the entropic indices are equal* ($\beta = \alpha$), *the bound can be expressed as*

$$\mathcal{B}_{\alpha,\alpha;2}(c) = \begin{cases} \frac{\log\left[(c^2)^\alpha + (1-c^2)^\alpha\right]}{1-\alpha} & \text{if} \quad \alpha \in \left[0\,,\,\frac{1}{2}\left(1-\delta_{c,\frac{1}{\sqrt{2}}}\right) + \alpha^\dagger\,\delta_{c,\frac{1}{\sqrt{2}}}\right) \\ \min\limits_{\theta\in\left(0\,,\,\frac{\gamma}{2}\right]} \frac{\log D_\alpha(\theta) + \log D_\alpha(\gamma-\theta)}{1-\alpha} & \text{if} \quad \alpha \in \left[\frac{1}{2}\left(1-\delta_{c,\frac{1}{\sqrt{2}}}\right) + \alpha^\dagger\,\delta_{c,\frac{1}{\sqrt{2}}}\,,\,\alpha^\star(c)\right] \\ \frac{2\,\log\left[\left(\frac{1+c}{2}\right)^\alpha + \left(\frac{1-c}{2}\right)^\alpha\right]}{1-\alpha} & \text{if} \quad \alpha \in (\alpha^\star(c), +\infty) \end{cases} \tag{17}$$

where $\gamma = \arccos c$, $\alpha^\dagger \approx 1.43$ *is the unique solution of* $\frac{2}{1-\alpha}\log[(\frac{2+\sqrt{2}}{4})^\alpha + (\frac{2-\sqrt{2}}{4})^\alpha] = \log 2$, $\alpha^\star(c)$ *is shown in figure 2 for* $c \in [\frac{1}{\sqrt{2}}, 1)$, *and* $\delta_{c,\frac{1}{\sqrt{2}}} = 1$ *when* $c = 1/\sqrt{2}$, *otherwise it is 0. Moreover, the bound is achieved for* $\theta_{\mathrm{opt}}^{(i)} = \frac{\gamma}{2} + \mathrm{i}(\frac{\gamma}{2} - \theta_{\mathrm{opt}})$ *with* $i \in \mathcal{I} = \{-1, 1\}$ *and* $\theta_{\mathrm{opt}} = 0$ *in the first interval,* θ_{opt} *is the (unique, numerical) solution of the minimization in the second interval, and* $\theta_{\mathrm{opt}} = \gamma/2$ *in the third interval (thus the two solutions reduce to only one).*

From this corollary one can observe the following facts.

- When $c = 1/\sqrt{2}$, one has $\alpha^\star(\frac{1}{\sqrt{2}}) = \alpha^\dagger$. Thus, the second interval in (17) reduces to one point and the bound there takes the value $\log 2$; this is also the value acquired in the first interval. There is a transition in the behavior of the bound with α at $\alpha^\dagger$. This can be seen from the minimizers, since optimal values are $\theta_{\mathrm{opt}} = 0$, or 0 and $\gamma/2$, or $\gamma/2$, depending on whether α is smaller than, equal to, or larger than $\alpha^\dagger$ respectively. These observations agree with the results in [33]. Besides, the transition value $\alpha^\dagger$ has already been shown graphically in [6] as the index that vanishes the second derivative versus θ of $\frac{\log \mathcal{D}_\alpha(\theta) + \log \mathcal{D}_\alpha(\gamma-\theta)}{1-\alpha}$ at $\theta = \gamma/2$.

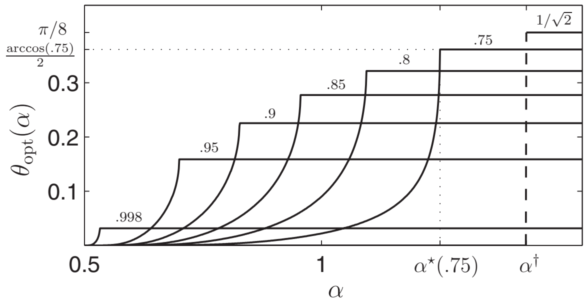

Figure 3. Optimal angle θ_{opt} as a function of $\alpha > 1/2$ for some given values of c. As an illustrative example, we specify the situation when $c = 0.75$.

- For any $c \in (1/\sqrt{2}, 1)$, the situation corresponding to the first interval in (17) is included in corollary 1; the lengths of the second and third intervals depend on c, as $\alpha^\star(c)$ decreases from $\alpha^\dagger$ to $1/2$ (see figure 2). Above $\alpha = 1/2$, the optimal angle θ_{opt} increases with α continuously from 0 to $\gamma/2$, as shown in figure 3. There is no transition in the value of the bound.
- When $c = 1$, then $\gamma = 0$ and one obtains the trivial bound $\mathcal{B}_{\alpha,\alpha;2}(1) = 0$.

The case $\beta = \alpha$ is precisely that treated by Puchała *et al* in [53], for N-level systems. However, as already pointed out, in the two-dimensional case their bound is not optimal (recall figure 1 (right)). Notice that some situations discussed at the end of section 2 are included in corollary 2 as particular cases. For instance, the de Vicente–Sanchez-Ruiz bound [47] is recovered by taking $\alpha \to 1$; the bound is optimal for qubit systems, although in [47] it has been calculated treating separately ψ and $\widetilde{\psi}$ without taking into account the relation between them, except through the Landau–Pollak inequality. Furthermore, the tight bound obtained in [33] is recovered for $\alpha = 2$. We stress that here we extend previous results along all the line $\beta = \alpha$, giving a semi-analytical expression for the bound.

Finally, using the fact that Rényi entropy H_λ decreases with λ [10, 46], one obtains the suboptimal result.

Corollary 3. *In the context of proposition 1, for any couple* $(\alpha, \beta) \in \mathbb{R}_+^2$*, the entropies' sum is lower-bounded in the form*

$$H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant \overline{\mathcal{B}}_{\lambda,\lambda;2}(c) \quad \textit{with} \quad \lambda = \max(\alpha, \beta) \tag{18}$$

where $\overline{\mathcal{B}}_{\lambda,\lambda;2}(c)$ *is given in (17).*

Notice that taking arbitrarily large entropic indices in corollaries 2 and 3, we recover the Deutsch bound and obtain the Maassen–Uffink suboptimal relation

$$H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant H_\infty(|\psi|^2) + H_\infty(|\widetilde{\psi}|^2) = -2\log\left(\frac{1+c}{2}\right) \tag{19}$$

for any couple of indices.

4. Discussion

In this contribution we deal with the most general entropic formulation of the uncertainty principle in terms of the sum of Rényi entropies associated with any given pair of 2D quantum observables, in the case of pure states of the qubit system. In this context, the UP is expressed

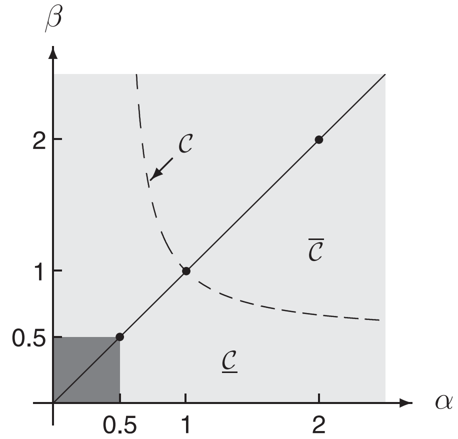

Figure 4. Sketch in the α–β plane of the uncertainty relation obtained for the entropies' sum. The tight bound is analytically known within the square $[0\ ,\ 1/2]^2$ (dark gray region, corresponding to corollary 1) and semi-analytically on the line $\beta = \alpha$ (solid line, corollary 2); otherwise the optimal bound we obtained is only calculable numerically while suboptimal bound are analytically known (light gray region, proposition 1 and corollary 3). Previous results in the literature correspond to the points $(\alpha, \beta) = (1/2, 1/2), (1, 1)$ and $(2, 2)$ (dots), as well as those on the conjugacy curve $\mathcal{C}$ (dashed line). The regions 'above' ($\overline{\mathcal{C}}$) and 'below' ($\underline{\mathcal{C}}$) the conjugacy curve, as defined in equation (5), are also indicated.

by means of inequalities of the form $H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant \overline{\mathcal{B}}_{\alpha,\beta;2}(c)$ where c is the overlap of the transformation between the eigenspaces of the observables. We search for the tightest lower bound of the entropies' sum as well as for the minimizing states. In contrast to many results in the literature, for that purpose we do not make use of Riesz–Thorin theorem, thus avoiding the Hölder conjugacy constraint on indices α and β. The bound obtained here is valid for any couple of indices, tight, and universal in the sense that it does not depend on the state of the system. This is the main result of the paper, given in propositions 1 and 2. In general the bound and minimizers are obtained numerically. Notice that in some domains of the α–β plane we are able to solve the problem in an analytical or semi-analytical way. In effect, in corollary 1 we present an analytical expression for the tight bound within the square $(\alpha, \beta) \in [0\ ,\ 1/2]^2$; whereas in corollary 2 we show a semi-analytical expression on the line $\beta = \alpha$. As particular cases, we recover optimal results given in the literature for the points $(\alpha, \beta) = (1/2, 1/2)$, $(1, 1)$ and $(2, 2)$. Notice also that in corollary 3 a semi-analytical bound is given for any couple of entropic indices, although this result is suboptimal. In particular, we recover a suboptimal relation derived by Maassen and Uffink. Figure 4 is a sketch in the α–β plane of the kind of result derived in this paper. Extension of these results to N-level systems is currently under study [57]. We have preliminary results where a reduction to the two-level case by means of the Landau–Pollak inequality is employed.

It is easy to see that the bounds given in proposition 1 and the corollaries remain valid in the case of mixed states when $(\alpha, \beta) \in [0, 2]^2$, by using the concavity property of Rényi entropy. Indeed for N-level systems, if one has a universal relation $H_\alpha(|\psi|^2) + H_\beta(|\widetilde{\psi}|^2) \geqslant \overline{\mathcal{B}}_{\alpha,\beta;N}$ valid for pure states, then assuming $(\alpha, \beta) \in [0\ ,\ \lambda_*(N)]^2$ one has $H_\alpha(\sum_m \mu_m|\psi^{(m)}|^2) + H_\beta(\sum_m \mu_m|\widetilde{\psi}^{(m)}|^2) \geqslant \sum_m \mu_m(H_\alpha(|\psi^{(m)}|^2) + H_\beta(|\widetilde{\psi}^{(m)}|^2)) \geqslant \sum_m \mu_m\overline{\mathcal{B}}_{\alpha,\beta;N} = \overline{\mathcal{B}}_{\alpha,\beta;N}$ for a mixture since $\sum_m \mu_m = 1$. In other words, any entropic uncertainty relation for pure states is also valid for mixed states in the domain $[0\ ,\ \lambda_*(N)]^2$. The bound for mixed states when the entropic indices lie outside this square remains to be studied.

Acknowledgments

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Appendix A. Proof of proposition 1 and corollaries 1 and 2

We use an approach similar to that of [50], but starting from the most general unitary transformation T and state $|\Psi\rangle$. Our approach allows us not only to evaluate the optimum entropic bounds, but also the minimizing states.

A.1. Simplification of the minimization problem

The vector ψ is such that $\|\psi\|_2 = 1$ and can thus be written under the form

$$\psi = \Phi(\boldsymbol{\varphi})\, \boldsymbol{s} \tag{A.1}$$

where $\boldsymbol{s}$ is a unit 2D vector and $\Phi(\boldsymbol{\varphi})$ is a diagonal matrix given by

$$\Phi(\boldsymbol{\varphi}) = \exp(\imath\, \mathrm{diag}(\boldsymbol{\varphi})) \tag{A.2}$$

in terms of the 2D phase vector $\boldsymbol{\varphi}$. We parameterize the unitary matrix T as the product of three unitary matrices [54, 55]

$$T = \Phi(\boldsymbol{u})\, V(\gamma_T)\, \Phi(\boldsymbol{v})\,, \qquad \text{where} \qquad V(\gamma_T) = \begin{bmatrix} \cos\gamma_T & \sin\gamma_T \\ -\sin\gamma_T & \cos\gamma_T \end{bmatrix}, \tag{A.3}$$

in terms of $\gamma_T \in [0, \frac{\pi}{2})$ and the 2D phase vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ (other possible angles can be taken into account playing with phases). This parameterization is also known as '$Z{-}Y$ decomposition for a single qubit' [58, Th. 4.1]. Notice that the overlap $c \equiv \max_{k,l} |T_{lk}| = \max_{k,l} |V_{lk}(\gamma_T)|$ does not depend on the phases. Combining equations (1) and (A.1)–(A.3), we obtain

$$\widetilde{\psi} = \Phi(\boldsymbol{u})\, V(\gamma_T)\, \Phi(\boldsymbol{v} + \boldsymbol{\varphi})\, \boldsymbol{s}. \tag{A.4}$$

Our goal is to solve the minimization problem

$$\mathcal{B}_{\alpha,\beta;2} = \min_{\boldsymbol{\varphi},\boldsymbol{s}} (H_\alpha(|\Phi(\boldsymbol{\varphi})\, \boldsymbol{s}|^2) + H_\beta(|\Phi(\boldsymbol{u})\, V(\gamma_T)\, \Phi(\boldsymbol{v} + \boldsymbol{\varphi})\, \boldsymbol{s}|^2)) \tag{A.5}$$

for given entropic indices α and β, and transformation matrix T. Recall that the argument of each entropy is a probability vector, and we use the notation $|\psi|^2 \equiv [|\psi_1|^2 \quad |\psi_2|^2]^t$ (and similarly for $|\widetilde{\psi}|^2$).

The problem simplifies due to numerous invariances and symmetries.

- Invariance under phase shifts applied to the wavevectors (multiplication by a matrix Φ):

  $$H_\alpha(|\Phi(\boldsymbol{\varphi})\boldsymbol{s}|^2) + H_\beta(|\Phi(\boldsymbol{u})V(\gamma_T)\Phi(\boldsymbol{v} + \boldsymbol{\varphi})\boldsymbol{s}|^2) = H_\alpha(|\boldsymbol{s}|^2) + H_\beta(|V(\gamma_T)\Phi(\boldsymbol{v} + \boldsymbol{\varphi})\boldsymbol{s}|^2).$$

  Also, since $\boldsymbol{\varphi} \to \boldsymbol{v} + \boldsymbol{\varphi}$ is an isomorphism, the minimization in (A.5) reduces to

  $$\mathcal{B}_{\alpha,\beta;2} = \min_{\boldsymbol{\varphi},\boldsymbol{s}} (H_\alpha(|\boldsymbol{s}|^2) + H_\beta(|V(\gamma_T)\, \Phi(\boldsymbol{\varphi})\, \boldsymbol{s}|^2)) \quad \text{where} \quad \gamma_T \in [0\,,\ \pi/2) \tag{A.6}$$

  and one can notice that the bound depends only on γ_T. Note that this fact is mentioned in [53] and, in a sense, in [51]. We show below that even if phase $\boldsymbol{v}$ has no effect on the bound, it does appear in the minimizing states.

- Invariance under permutation of the components: playing with the phases one sees that

  $$H_\alpha(|\boldsymbol{s}|^2) + H_\beta(|V(\gamma_T)\Phi(\boldsymbol{\varphi})\boldsymbol{s}|^2) = H_\alpha(|\boldsymbol{s}|^2) + H_\beta\left(\left|V\left(\frac{\pi}{2} - \gamma_T\right)\Phi(\boldsymbol{\varphi} - [\pi \quad 0]^t)\boldsymbol{s}\right|^2\right)$$

  and thus the minimization problem (A.6) reduces a step further,

  $$\mathcal{B}_{\alpha,\beta;2} = \min_{\boldsymbol{\varphi},\boldsymbol{s}}(H_\alpha(|\boldsymbol{s}|^2) + H_\beta(|V(\gamma)\,\Phi(\boldsymbol{\varphi})\,\boldsymbol{s}|^2)) \tag{A.7}$$

  where

  $$\gamma \equiv \min\left(\gamma_T\ ,\ \frac{\pi}{2} - \gamma_T\right) \in [0\ ,\ \pi/4]. \tag{A.8}$$

  This proves that the bound $\mathcal{B}_{\alpha,\beta;2} = \mathcal{B}_{\alpha,\beta;2}(c)$ depends *only* on the overlap $c = \cos\gamma$. In principle, this is not the case for N-level systems with $N > 2$. Note that the invariance under permutation of the entropy sum is mentioned in [53], suggesting that one can restrict the search for the bound to $\gamma_T \in [0\ ;\ \pi/4]$. As we show below, one can again notice that the full form of T ($\gamma_T \in [0\ ;\ \pi/2]$) has an impact on the minimizing states.
- Symmetries and periodicities on $\boldsymbol{s}$: we parameterize $\boldsymbol{s}(\theta) = [\cos\theta \quad \sin\theta]^t$. Then
  * π-periodicity:

    $$\begin{aligned} H_\alpha(|\boldsymbol{s}(\theta)|^2) + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2) &= H_\alpha(|\boldsymbol{s}(\theta+\pi)|^2) \\ &\quad + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta+\pi)|^2) \end{aligned}$$

    so that one can restrict the search to $\theta \in [-\frac{\pi}{2}\ ,\ \frac{\pi}{2}]$.
  * $\frac{\pi}{2}$-symmetry: playing with the permutations and phases, it can be shown that

    $$\begin{aligned} H_\alpha(|\boldsymbol{s}(\theta)|^2) + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2) &= H_\alpha(|\boldsymbol{s}(\theta+\pi/2)|^2) \\ &\quad + H_\beta(|V(\gamma)\Phi(J\boldsymbol{\varphi})\boldsymbol{s}(\theta+\pi/2)|^2) \end{aligned}$$

    where $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, allowing one to restrict a little bit further the interval $\theta \in [-\frac{\pi}{4}\ ,\ \frac{\pi}{4}]$.
  * opposite angle: finally we note that $\boldsymbol{s}(-\theta) = \Phi([0 \quad \pi]^t)\boldsymbol{s}(\theta)$ so that

    $$\begin{aligned} H_\alpha(|\boldsymbol{s}(\theta)|^2) + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2) &= H_\alpha(|\boldsymbol{s}(-\theta)|^2) \\ &\quad + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi} + [0 \quad \pi]^t)\boldsymbol{s}(-\theta)|^2) \end{aligned}$$

    allowing for a further restriction to $\theta \in [0\ ,\ \pi/4]$.

  From these symmetries and invariances, the minimization problem simplifies to

  $$\mathcal{B}_{\alpha,\beta;2}(c) \ = \min_{\boldsymbol{\varphi},\theta\in[0\ ,\ \pi/4]}(H_\alpha(|\boldsymbol{s}(\theta)|^2) + H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2)) \tag{A.9}$$

  where

  $$\gamma = \min\left(\gamma_T\ ,\ \frac{\pi}{2} - \gamma_T\right). \tag{A.10}$$

A.2. Trivial case $c = 1$

In this case, $\gamma = 0$ and $V(0) = I$. Clearly choosing $\boldsymbol{s} = [1 \quad 0]^t$, one obtains $H_\alpha(|\boldsymbol{s}|^2) = 0$ and $H_\beta(|V(0)\,\Phi(\boldsymbol{\varphi})\,\boldsymbol{s}|^2) = H_\beta(|\boldsymbol{s}|^2) = 0$. Thus, the solution is trivial:

$$\mathcal{B}_{\alpha,\beta;2}(1) = 0. \tag{A.11}$$

A.3. General nontrivial case $\frac{1}{\sqrt{2}} \leqslant c < 1$

We proceed in two successive steps: first we fix $\boldsymbol{s}$ (i.e. θ) and minimize the entropies' sum over phase $\boldsymbol{\varphi}$; second, for the optimal phase that depends (in principle) on θ, we determine the value of θ that minimizes the entropies' sum.

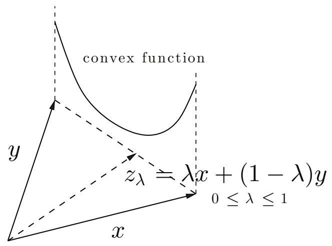

Figure A1. Vector $z = |V(\gamma)\,\Phi(\boldsymbol{\varphi})\,\boldsymbol{s}(\theta)|^2$ lives in the segment (the convex set) that links vectors $x = [\cos^2(\gamma-\theta) \quad \sin^2(\gamma-\theta)]^t$ and $y = [\cos^2(\gamma+\theta) \quad \sin^2(\gamma+\theta)]^t$. The typical behavior of a convex function $\mathbb{R}^2 \to \mathbb{R}$ is represented by the solid line, illustrating that the maximum of this function is attained at the boundaries of the convex set.

A.3.1. Minimization over phase $\boldsymbol{\varphi}$. Note that phase $\boldsymbol{\varphi}$ plays a role only in the term $H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2)$, which is invariant under multiplication of the argument of the modulus by the scalar $\exp(\imath\varphi')$, i.e. by shifting both components of $\boldsymbol{\varphi}$ by the same phase. Without loss of generality[4], we write $\boldsymbol{\varphi} = [-\varphi_2 \quad \varphi_2]^t$ and thus

$$z \equiv |V(\gamma)\,\Phi(\boldsymbol{\varphi})\,\boldsymbol{s}(\theta)|^2 = \cos^2\varphi_2 \begin{bmatrix} \cos^2(\gamma-\theta) \\ \sin^2(\gamma-\theta) \end{bmatrix} + \sin^2\varphi_2 \begin{bmatrix} \cos^2(\gamma+\theta) \\ \sin^2(\gamma+\theta) \end{bmatrix}. \tag{A.12}$$

The mappings $\varphi_2 \mapsto -\varphi_2$ and $\varphi_2 \mapsto \pi - \varphi_2$ leave the Rényi entropy unchanged so that it is enough to consider $\varphi_2 \in [0\,,\,\pi/2]$ and the solutions are modulo π. Note that

$$\begin{cases} \beta > 1 \Rightarrow \dfrac{1}{1-\beta} < 0: \ \arg\min \dfrac{1}{1-\beta} \log ||x||_\beta^\beta = \arg\max ||x||_\beta^\beta = \arg\max \dfrac{||x||_\beta^\beta}{\beta-1} \\ \beta < 1 \Rightarrow \dfrac{1}{1-\beta} > 0: \ \arg\min \dfrac{1}{1-\beta} \log ||x||_\beta^\beta = \arg\min ||x||_\beta^\beta = \arg\max \dfrac{||x||_\beta^\beta}{\beta-1}. \end{cases}$$

Thus, minimization over phase $\boldsymbol{\varphi}$ reduces to the maximization problem

$$\max_{\varphi_2\in[0\,,\,\pi/2]} \frac{\||V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2\|_\beta^\beta}{\beta-1}. \tag{A.13}$$

Now notice that (A.12) is a convex combination of the vectors $x = [\cos^2(\gamma-\theta) \quad \sin^2(\gamma-\theta)]^t$ and $y = [\cos^2(\gamma+\theta) \quad \sin^2(\gamma+\theta)]^t$. The mapping $z \mapsto \frac{\|z\|_\beta^\beta}{\beta-1}$ is a convex function[5] of z [46, 59], then the maximum in (A.13) is attained at the boundaries of the convex set defined by $\{z = \mu x + (1-\mu)y : \mu = \cos^2\varphi_2, \varphi_2 \in [0, \pi/2]\}$ (see figure A1), namely

$$\max_{\varphi_2\in[0\,,\,\pi/2]} \frac{\||V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2\|_\beta^\beta}{\beta-1} = \max\left(\frac{\mathcal{D}_\beta(\gamma-\theta)}{\beta-1}\,,\,\frac{\mathcal{D}_\beta(\gamma+\theta)}{\beta-1}\right) \tag{A.14}$$

where

$$\mathcal{D}_\lambda(\theta) \equiv (\cos^2\theta)^\lambda + (\sin^2\theta)^\lambda. \tag{A.15}$$

The maximum corresponds to $\varphi_2 = 0$ or $\varphi_2 = \pi/2$.

We now compare $\frac{\mathcal{D}_\beta(\gamma-\theta)}{\beta-1}$ and $\frac{\mathcal{D}_\beta(\gamma+\theta)}{\beta-1}$, by considering the sign of the difference

$$\Delta(\theta,\gamma) = \frac{\mathcal{D}_\beta(\gamma-\theta) - \mathcal{D}_\beta(\gamma+\theta)}{\beta-1} \tag{A.16}$$

[4] In related literature the phase is generally chosen as $[0 \quad \varphi]^t$. Our choice allows us to determine the optimal phase without calculations.

[5] When $\beta > 1$, $z \mapsto \|z\|_\beta^\beta$ is convex and $\beta - 1 > 0$ ensures the convexity of the mapping $z \mapsto \frac{\|z\|_\beta^\beta}{\beta-1}$; while when $\beta < 1$, $z \mapsto \|z\|_\beta^\beta$ is concave and $\beta - 1 < 0$ ensures the convexity of the mapping $z \mapsto \frac{\|z\|_\beta^\beta}{\beta-1}$ as well.

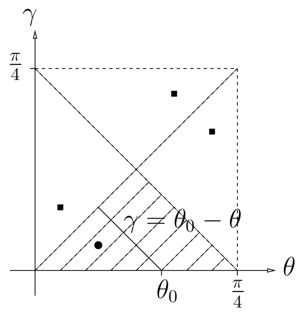

Figure A2. Illustration of the symmetries $\Delta(\theta,\gamma) = \Delta(\gamma,\theta) = \Delta(\pi/4-\theta,\pi/4-\gamma) = \Delta(\pi/4-\gamma,\pi/4-\theta)$. For instance, the point represented by a circle has three symmetric points (represented by squares) for which Δ has the same value. Thus, it suffices to study (the sign of) function Δ inside the triangle (dashed region) given in (A.17). The solid line represents the segment $\gamma = \theta_0 - \theta$ inside this triangle.

where $(\theta,\gamma) \in [0\,,\,\pi/4]^2$. The arguments that follow justify that $\Delta(\theta,\gamma) \geqslant 0$, i.e. $\frac{\mathcal{D}_\beta(\gamma-\theta)}{\beta-1} \geqslant \frac{\mathcal{D}_\beta(\gamma+\theta)}{\beta-1}$.

Making use of the symmetries $\Delta(\theta,\gamma) = \Delta(\gamma,\theta) = \Delta(\pi/4-\theta,\pi/4-\gamma) = \Delta(\pi/4-\gamma,\pi/4-\theta)$, we restrict the study of $\mathrm{sign}(\Delta)$ to the triangle

$$\{(\theta,\gamma) \in [0\,,\,\pi/4]^2 : 0 \leqslant \theta \leqslant \min(\gamma,\pi/4-\gamma)\} \tag{A.17}$$

as illustrated in figure A2. Noticing that $\Delta(\theta,0) = 0$, we study the variation of Δ on the segment $\gamma = \theta_0 - \theta$ inside the triangle, for each θ_0 between 0 and $\pi/4$, as shown in the figure. A simple derivation leads to

$$\frac{\partial\Delta(\theta,\theta_0-\theta)}{\partial\theta} = \frac{2\beta\sin(4\theta-2\theta_0)\{[\sin^2(2\theta-\theta_0)]^{\beta-1} - [\cos^2(2\theta-\theta_0)]^{\beta-1}\}}{\beta-1}. \tag{A.18}$$

Since $\theta_0/2 \leqslant \theta \leqslant \theta_0$, one has $0 \leqslant 2\theta - \theta_0 \leqslant \pi/4$; thus $\sin(4\theta-2\theta_0) \geqslant 0$ and $\sin^2(2\theta-\theta_0) \leqslant \cos^2(2\theta-\theta_0)$, therefore (A.18) is positive proving that inside the triangle, $\Delta(\theta,\theta_0-\theta)$ increases with θ and that it is non-negative.

Finally, minimization over phase $\boldsymbol{\varphi}$ gives

$$\min_{\varphi_2\in[0\,,\,\pi/2]} H_\beta(|V(\gamma)\Phi(\boldsymbol{\varphi})\boldsymbol{s}(\theta)|^2) = \frac{\log\mathcal{D}_\beta(\gamma-\theta)}{1-\beta} \tag{A.19}$$

which is attained at $\varphi_{2,\mathrm{opt}} = 0$. Note that due to the invariance of the entropies' sum under the multiplication of the wavevector by a scalar $\exp(\imath\varphi')$, in some sense there exists a 'unique' phase $\boldsymbol{\varphi}$ minimizing the entropies' sum when $\boldsymbol{s}$ is fixed. Moreover, 'this' phase does not depend on $\boldsymbol{s}$.

A.3.2. Minimization over the angle θ. Now the minimization problem (A.10) has been reduced to

$$\mathcal{B}_{\alpha,\beta;2}(c) = \min_{\theta\in[0\,,\,\pi/4]}\left(\frac{\log\mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log\mathcal{D}_\beta(\gamma-\theta)}{1-\beta}\right). \tag{A.20}$$

We have

$$\frac{\partial}{\partial\theta}\left(\frac{\log\mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log\mathcal{D}_\beta(\gamma-\theta)}{1-\beta}\right) = \frac{\mathcal{D}'_\alpha(\theta)}{(1-\alpha)\,\mathcal{D}_\alpha(\theta)} - \frac{\mathcal{D}'_\beta(\gamma-\theta)}{(1-\beta)\,\mathcal{D}_\beta(\gamma-\theta)} \tag{A.21}$$

where

$$\mathcal{D}'_\lambda(\theta) \equiv \frac{\partial\mathcal{D}_\lambda}{\partial\theta} = \lambda\sin(2\theta)\,[(\sin^2\theta)^{\lambda-1} - (\cos^2\theta)^{\lambda-1}]. \tag{A.22}$$

Since $\theta \in [0\,,\,\pi/4]$, one has both $\sin(2\theta) \geqslant 0$ and $\sin^2\theta \leqslant \cos^2\theta$; thus the first term in the right-hand side of (A.21) is positive. Moreover, $\gamma - \theta \in [-\pi/4\,,\,\pi/4]$ and thus by the same reasoning the second term in (A.21) has the same sign as the factor $\sin[2(\gamma-\theta)]$. Therefore, for $\theta \in (\gamma\,,\,\pi/4]$ the entropies' sum is increasing. Necessarily the minimum of the entropies' sum corresponds to $0 \leqslant \theta \leqslant \gamma$, reducing the interval where θ has to be sought, i.e.

$$\mathcal{B}_{\alpha,\beta;2}(c) = \min_{\theta\in[0\,,\,\gamma]}\left(\frac{\log \mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log \mathcal{D}_\beta(\gamma-\theta)}{1-\beta}\right). \tag{A.23}$$

At this step, the minimum of (A.23) has to be sought numerically. We have proved proposition 1 for the general case (α,β). In the sequel we go a step further for special cases.

A.4. Analytical result when $(\alpha,\beta) \in [0\,,\,1/2]^2$

The second derivative with respect to θ of the function to be minimized is

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{\log \mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log \mathcal{D}_\beta(\gamma-\theta)}{1-\beta}\right) = \frac{\mathcal{K}_\alpha(\theta)}{[\mathcal{D}_\alpha(\theta)]^2} + \frac{\mathcal{K}_\beta(\gamma-\theta)}{[\mathcal{D}_\beta(\gamma-\theta)]^2} \tag{A.24}$$

where $\mathcal{K}_\lambda \equiv (\mathcal{D}''_\lambda\mathcal{D}_\lambda - \mathcal{D}'^2_\lambda)/(1-\lambda)$ is given by

$$\mathcal{K}_\lambda(\theta) = \frac{2\lambda}{1-\lambda}\left[(2\lambda-1)\left(\frac{\sin^2(2\theta)}{4}\right)^{\lambda-1} - (\cos^2\theta)^{2\lambda-1} - (\sin^2\theta)^{2\lambda-1}\right]. \tag{A.25}$$

This expression is strictly negative when $\lambda \leqslant 1/2$, so that the function $\frac{\log\mathcal{D}_\alpha(\theta)}{1-\alpha} + \frac{\log\mathcal{D}_\beta(\gamma-\theta)}{1-\beta}$ is concave with respect to $\theta \in [0\,,\,\gamma]$. Thus, the minimum is attained at the borders of the segment, i.e. either for $\theta = 0$ or $\theta = \gamma$, or both if $\alpha = \beta$. The function at these extremal points takes the value $\frac{\log\mathcal{D}_\beta(\gamma)}{1-\beta}$ or $\frac{\log\mathcal{D}_\alpha(\gamma)}{1-\alpha}$. Comparing these values by using that Rényi entropy H_λ is decreasing with λ and recalling that $c = \cos\gamma$, we prove corollary 1.

A.5. Semi-analytical results when $\alpha = \beta$

The function to be minimized in (A.23) is

$$\mathcal{F}_\alpha(\theta) \equiv \frac{\log\mathcal{D}_\alpha(\theta) + \log\mathcal{D}_\alpha(\gamma-\theta)}{1-\alpha}. \tag{A.26}$$

Noting the symmetry $\mathcal{F}_\alpha(\theta) = \mathcal{F}_\alpha(\gamma-\theta)$, the minimization problem reduces to find

$$\mathcal{B}_{\alpha,\alpha;2}(c) = \min_{\theta\in[0\,,\,\gamma/2]}\mathcal{F}_\alpha(\theta) \tag{A.27}$$

The case $\alpha = \beta \in [0\,,\,1/2]$ has been treated in the preceding subsection, thus we concentrate in the case $\alpha = \beta > 1/2$. The derivative of $\mathcal{F}_\alpha$ with respect to θ is given by

$$\mathcal{F}'_\alpha(\theta) \equiv \frac{\partial}{\partial\theta}\mathcal{F}_\alpha(\theta) = \frac{1}{1-\alpha}\left(\frac{\mathcal{D}'_\alpha(\theta)}{\mathcal{D}_\alpha(\theta)} - \frac{\mathcal{D}'_\alpha(\gamma-\theta)}{\mathcal{D}_\alpha(\gamma-\theta)}\right). \tag{A.28}$$

This expression clearly vanishes at $\theta = \frac{\gamma}{2}$, thus giving a possible solution. Other solutions arise, depending on the value of c, as we show below.

A.5.1. Case $c = 1/\sqrt{2}$.

In this case, $\gamma = \pi/4$. As seen, $\theta = \pi/8$ is a possible solution. The other one is $\theta = 0$ since $\mathcal{D}'_\alpha(0) = \mathcal{D}'_\alpha(\frac{\pi}{4}) = 0$. As already observed in [33], there are different behaviors below, at or above $\alpha^\dagger \approx 1.430$, which is the unique solution of $\frac{2\log[(\frac{2+\sqrt{2}}{4})^\alpha + (\frac{2-\sqrt{2}}{4})^\alpha]}{1-\alpha} = \log 2$. We summarize the results:

- when $\alpha < \alpha^\dagger$, $\theta = 0$ is the unique solution to minimization problem, leading to the bound $\log 2$;
- at $\alpha = \alpha^\dagger$, both $\theta = 0$ and $\theta = \pi/8$ give the bound $\log 2$;
- when $\alpha > \alpha^\dagger$, $\theta = \pi/8$ is the unique solution, leading to the bound $\frac{2\log[(\frac{2+\sqrt{2}}{4})^\alpha + (\frac{2-\sqrt{2}}{4})^\alpha]}{1-\alpha}$.

A.5.2. Case $c \in (1/\sqrt{2}\,,\ 1)$. In this case, $\gamma \in (0\,,\ \pi/4)$. It can be seen that $\theta = 0$ does not solve $\mathcal{F}'_\alpha(\theta) = 0$, since $\mathcal{D}'_\alpha(0) = 0$ but $\frac{\mathcal{D}'_\alpha(\gamma)}{1-\alpha} > 0$. All the possible solutions are then in $\theta \in (0\,,\ \frac{\gamma}{2}]$. We observe numerically the following behavior: for any fixed c, there exists a value $\alpha^\star(c)$ such that

- when $1/2 < \alpha < \alpha^\star(c)$, the optimal angle $\theta_{\mathrm{opt}}(\alpha)$ is to be found numerically within $(0\,,\ \frac{\gamma}{2})$. We find that $\theta_{\mathrm{opt}}(\alpha)$ increases continuously from 0 to $\gamma/2$ (see figure 3). The bound is then numerically expressed as well.
- when $\alpha \geqslant \alpha^\star(c)$, $\theta = \gamma/2$ is the unique minimum, leading to the bound $\frac{2\log[(\frac{1+c}{2})^\alpha + (\frac{1-c}{2})^\alpha]}{1-\alpha}$.

This subsection, together with the expressions for the minimizers in the last paragraph of appendix B, completes the proof of corollary 2.

Appendix B. Proof of proposition 2

We denote by $\{\theta_{\mathrm{opt}}^{(i)}\}_{i\in\mathcal{I}}$ the set of arguments that minimize the entropies' sum, where $\mathcal{I}$ lists the different possible solutions. These minimizing angles belong to the interval $[0, \gamma]$ as shown in (A.23), where γ is either γ_T or $\pi/2 - \gamma_T$ as given in (A.7), being $\gamma_T \in [0, \pi/2]$ one of the parameters that characterizes the unitary matrix T, equation (A.3). The solutions for both situations are as follows.

- $\gamma_T = \gamma \in [0\,,\ \pi/4]$: recalling that we have redefined the phase as $\boldsymbol{\varphi} \to \boldsymbol{v} + \boldsymbol{\varphi}$, the minimizing vectors ψ take the form $\Phi(-\boldsymbol{v})\boldsymbol{s}(\theta_{\mathrm{opt}}^{(i)})$. Symmetries $\theta \to -\theta$, $\theta + \pi/2$ and $\theta + \pi$ lead to the final form for the minimizing vectors:

$$\psi_{\mathrm{opt}}^{(i,n)} = \Phi(-\boldsymbol{v}) \begin{bmatrix} \cos\left(\theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \\ \sin\left(\theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \end{bmatrix} \quad \text{with} \quad i \in \mathcal{I} \quad \text{and} \quad n = 0, 1,$$

  up to a global phase factor. In figure B1(*a*) we present a sketch of all the optimal angles for a given case (circles).
- $\gamma_T = \pi/2 - \gamma \in [\pi/4\,,\ \pi/2]$: the optimal phase vector changes to $-\boldsymbol{v} \to -\boldsymbol{v} - [0 \quad \pi]^t$. Using the same symmetries as above, the minimizing vectors take the form

$$\psi_{\mathrm{opt}}^{(i,n)} = \Phi(-\boldsymbol{v}) \begin{bmatrix} \cos\left(-\theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \\ \sin\left(-\theta_{\mathrm{opt}}^{(i)} + n\frac{\pi}{2}\right) \end{bmatrix} \quad \text{with} \quad i \in \mathcal{I} \quad \text{and} \quad n = 0, 1,$$

  up to a global phase factor. In figure B1(*a*) we present a sketch of all the optimal angles for a given case (crosses).

Noting that in the last two expressions the sign before the angle $\theta_{\mathrm{opt}}^{(i)}$ is the same as the sign of $\pi/4 - \gamma_T$, we can unify both situations, thus leading to the expression given in proposition 2.

Case $\alpha = \beta$. One has seen numerically the existence of a unique optimal angle $\theta_{\mathrm{opt}} \in [0, \gamma/2]$ (with $\gamma = \arccos c$) leading to the lower bound of the entropies' sum. Moreover, we have seen that the entropies' sum is invariant under the transformation $\theta \to \gamma - \theta$. This leads to the possible angles represented in figure B1(*b*), respectively for $\gamma_T \in [0\,,\ \pi/4]$ (circles) and$\gamma_T \in [\pi/4\,,\ \pi/2]$ (crosses). In conclusion,

$$\theta_{\mathrm{opt}}^{(i)} = \frac{\gamma}{2} + \mathrm{i}\left(\frac{\gamma}{2} - \theta_{\mathrm{opt}}\right) \quad \text{with} \quad \mathrm{i} \in \mathcal{I} = \{-1, 1\},$$

leading to the minimizers given in corollary 2.

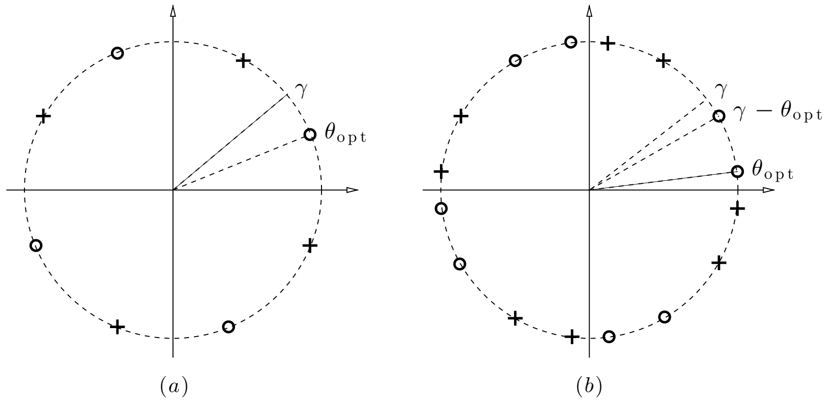

Figure B1. (*a*) Sketch of all the optimal angles derived from a given $\theta_{\rm opt}$, when $\gamma_T \in [0\ ,\ \pi/4]$ (circles) and $\gamma_T \in [\pi/4\ ,\ \pi/2]$ (crosses). For the illustration, we assume $\theta_{\rm opt}$ is unique. (*b*) The same as in (*a*), when $\alpha = \beta$, taking into account the symmetries in this special case.

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