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Experimental Control of Simple Pendulum Model

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Abstract. This paper conveys information about a Physics laboratory experiment for students with some theoretical knowledge about oscillatory motion. Students construct a simple pendulum that behaves as an ideal one, and analyze model assumption incidence on its period. The following aspects are quantitatively analyzed: vanishing friction, small amplitude, not extensible string, point mass of the body, and vanishing mass of the string.

It is concluded that model assumptions are easily accomplished in practice, within small experimental errors. Furthermore, this way of carrying out the usual pendulum experiments promotes a better understanding of the scientific modeling process. It allows a deeper comprehension of those physical concepts associated with model assumptions (small amplitude, point mass, etc.), whose physical and epistemological meanings appear clearly related to the model context. Students are introduced to a scientific way of controlling the validity of theoretical development, and they learn to value the power and applicability of scientific modeling.

1. Introduction

The direct references of the scientific theories are not the natural phenomena (because their are so complex) but the models, i.e., intellectual constructions based on generalizations, abstractions and idealizations (Bunge 1985).

In particular, simple models, in addition to their scientific relevance, are valuable didactic tools. By means of them, students can perform activities and make decisions consistent with those accepted by the scientific community, and control the adjustment between theory and reality.

2. The Problem

Our proposal is a Physics laboratory experiment for students with some theoretical knowledge about oscillatory motion. They have to construct a simple pendulum that behaves as an ideal one, and analyze model assumptions which affect its period. The following aspects are quantitatively analyzed: vanishing friction, small amplitude, not extensible string, point mass of the body, and vanishing mass of the string.

Among the various textbooks that treat the topic at an adequate level, for easy access to the lecturer, we chose a well known one: *Physics* (Vol. I) by Resnick et al. (1992).

In particular, students must know the equation for the periods of (a) physical pendulum: any rigid body, suspended from some axis through it, that can oscillate on a vertical plane, and (b) ideal simple pendulum: a particle suspended from a light, not extensible string (Resnick et al. 1992):

(a) Physical pendulum period

$$T_p = 2\pi\sqrt{\frac{I}{mgd}} \tag{1}$$

where T_p represents the period of the physical pendulum, I the moment of inertia, m the mass of body, g the acceleration due to gravity and d the distance between the axis and the center of gravity of the system.

(b) Ideal simple pendulum period

$$T_s = 2\pi\sqrt{\frac{l}{g}} \tag{2}$$

where l is the length of the string.

Equation (1) was deduced assuming:

A_1: negligible friction (the resultant torque on the system about the horizontal axis is solely due to the weight of the body).

A_2: small oscillation amplitudes (in the equation of motion, the sine of the amplitude angle can be replaced by the angle in radians).

A_3: the pendulum is a rigid body (invariable mass distribution, constant moment of inertia).

In order to consider a simple pendulum as a particular case of a physical one, we must reformulate assumption A3. In fact, a string cannot be considered a rigid body, but the pendulum mass distribution can be considered invariable if the string keeps its length while the pendulum moves. Thus, we have:

A_3': the string must keep its length.

The system constructed by the students must satisfy these conditions, as well as two additional ones which are based on two requirements that allow us to pass from (1) to (2). These are associated with a simple expression for the moment of inertia and with the condition $d = l$. Thus,

A_4: the string mass must be negligible.

A_5: the body mass must be concentrated at a point.

In fact, under these conditions, Equation (1) can be written

$$\begin{aligned} T_p &= 2\pi\sqrt{\frac{I}{m \cdot g \cdot d}} \\ &= 2\pi\sqrt{\frac{m_p \cdot l^2}{m_p \cdot g \cdot l}} \end{aligned} \tag{3}$$

$$T_s = 2\pi\sqrt{\frac{l}{g}}$$

where m_p is the mass of the particle.

3. Analysis Of The Error Introduced By The Model Assumptions

Once the pendulum is properly built, students are asked to obtain g from (2) with a random error usually given in relative terms: $\varepsilon_g = \Delta g/g$ (suitable values for ε_g are about 5×10^{-2}). ε_g is associated with a random error in T:

$$\varepsilon_T = \varepsilon_g/2 \tag{4}$$

Equation (4) was deduced using (2) and assuming

$$\Delta l/l \lhd\lhd \Delta g/g \tag{5}$$

which easily holds because l and ε_g are arbitrary.

Students must realize that, besides the random error, there is a systematic error in T including several independent terms, introduced by the fact that assumptions A_1 to A_5 are not fulfilled.

In the following analysis we will derive equations to value these terms, namely, the contributions due to friction (ε_f), initial amplitude (ε_α), variable length of the string due to a variable tension during oscillation ($\varepsilon_{\mathcal{T}}$), mass distribution of the body (ε_b), and mass of the string (ε_s). These equations can help students to design experimental system by determining the suitable values of certain variables so that real pendulum fits the model, i.e., the system can be designed so that the systematic error is negligible. Conversely, given a specific experimental system, students can use these equations to value the effect of model systematic errors, i.e., the systematic error in any given system may be not negligible, but can be subtracted from T. In any case, these equations allow students to control the agreement between a real pendulum and an ideal simple one.

Assuming the former case (negligible systematic error), the equation for the compound error, using (4), stands

$$\varepsilon_f + \varepsilon_\alpha + \varepsilon_{\mathcal{T}} + \varepsilon_b + \varepsilon_s \lhd\lhd + \varepsilon_g/2 \tag{6}$$

3.1. VANISHING FRICTION

In a simple pendulum a friction effect may exist between the system and the medium in which it oscillates, and between the string and the oscillation axis. The latter is easier to avoid than to compute. Therefore, we suggest fastening the string to the axis firmly enough to avoid any movement of the knot. In particular, coiling must be avoided as well "loose ring" knots. If the string is coiled, during the oscillation it will coil and uncoil a fraction of turn, varying its effective length in each oscillation. On the other hand, if the knot is made as a loose ring, it will introduce an excessive friction the effect of which is difficult to calculate.

In order to evaluate the air friction, the torque, τ_R, that it produces on the system can be written:

$$\tau_R = F_R \cdot d \tag{7}$$

where d denotes the effective force arm, and F_R, the friction force. Both of them depend on the shape and size of the body, and F_R also depends on the velocity. Since the body is supposed to be small, we can assume that d is the distance between the oscillation axis and the center of gravity. Furthermore, F_R is

$$F_R = b \cdot v \tag{8}$$

which is valid for small velocities, b being the friction coefficient between the air and the body; and v, the tangential relative velocity. Including (7) and (8) in the differential equation of motion leads to

$$\alpha = \alpha_0 \cdot e^{\frac{-bt}{2m}} \cdot \cos(\omega' t + \varphi) \tag{9}$$

where α and α_0 are the displacement angle and the amplitude, respectively, the exponential is the damping factor, and ω' is equal to $2\pi f'$ where f' is the frequency of the damped pendulum.

Thus, for a damped simple pendulum, the period becomes

$$T' = 2\pi/\omega' = \frac{2\pi}{\frac{g}{l} - \left(\frac{b}{2m}\right)^2} \tag{10}$$

In Equation (10), it can be seen that ω' is different from the angular velocity of a not damped pendulum, namely, $\omega = (g/l)^{1/2}$. If the second term in the root of (10) is much smaller than he first one, the air friction will be negligible. Unless the body is made of an extremely low density material, this condition easily holds in most of the common cases. b can be computed from (9), by eliminating the cosine dependence, setting $t = nT$ in the exponential, and measuring the amplitude after an entire number, n, of oscillations.

The relative systematic error in the period, introduced by using (2) instead of (10) is

$$\frac{T' - T_s}{T_s} = \frac{\dfrac{2\pi}{\sqrt{\dfrac{g}{l} - \left(\dfrac{b}{2m}\right)^2}} - \dfrac{2\pi}{\sqrt{\dfrac{g}{l}}}}{\dfrac{2\pi}{\dfrac{g}{l}}} \tag{11}$$

After dividing numerator and denominator of (11) by $2\pi(g/l)^{-1/2}$, this equation becomes

$$\frac{T' - T_s}{T_s} = \left(1 - \frac{\left(\frac{b}{2m}\right)^2}{\frac{g}{l}}\right)^{-1/2} - 1 \tag{12}$$

Expanding the parenthesis on the right hand side as a binomial series, and neglecting third and higher order terms, we obtain

$$\frac{T' - T_s}{T_s} = \frac{1}{2}\frac{\left(\frac{b}{2m}\right)^2}{\frac{g}{l}} \tag{13}$$

Therefore, for the friction effect to be vanishing, it suffices to equate the right hand side of (13) to ε_f fitted by (6).

3.2. SMALL INITIAL AMPLITUDE

The general equation for the period, including amplitude dependency, is (Resnick et al. 1992):

$$T_\alpha = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\left(\frac{\alpha}{2}\right) + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\left(\frac{\alpha}{2}\right) + \cdots\right) \tag{14}$$

Thus, truncation on the second term, in order to compute the error, leads to

$$T_\alpha = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\left(\frac{\alpha}{2}\right)\right) \tag{15}$$

Therefore, the relative systematic error introduced by using (2) instead of (15), which depends on amplitude, is

$$\frac{T_\alpha - T_s}{T_s} = \frac{1^2}{2^2}\sin^2(\alpha/2) \tag{16}$$

from which the maximum initial amplitude allowed can be valued, after the right hand side is set equal to ε_α fitted by (6).

3.3. INEXTENSIBLE STRING

To analyze this condition we must consider two features: (i) string deformation while applying the weight statically, and (ii) string deformation during the oscillation.

(i) While applying the weight statically, the string can undergo a considerable elongation without altering the model. It suffices that the deformation remains under the elastic limit and the lineal density remains constant, in order to compute the moment of inertia.

(ii) During the oscillation, one must keep in mind that a variable tension is applied to the string, due to the radial component of the weight, and the centripetal force associated with the circular motion:

$$\mathcal{T} = W \cdot \cos\alpha + m \cdot \frac{v^2}{l} \tag{17}$$

where W is the weight of the body, and v, its velocity with respect to the earth.

Equating the maximum kinetic energy of the body to its maximum potential energy, one can compute its maximum velocity, v_m, when it passes through its equilibrium position:

$$v_m = \sqrt{2.g.l(1-\cos\alpha_0)} \tag{18}$$

A minimum tension is exerted when the pendulum is in its maximum amplitude, and a maximum tension when it passes through its equilibrium position. The difference of tension between these extreme values is

$$\Delta\mathcal{T} = 2W(1-\cos\alpha_0). \tag{19}$$

Therefore, once the initial amplitude is fixed using (16), one must choose a string strong enough (whether due to its Young coefficient and/or to its size), or a body light enough, so that the difference of tension given by (19) stretches the string an amount not greater than that fixed for the error of its length (we have assumed $\Delta l/l \ll \Delta g/g$). Hence, it is suggested that vinyl strings and textile fibers be avoided.

3.4. POINT MASS OF THE BODY

The distribution of the oscillating body mass affects its moment of inertia. The moment of inertia about an arbitrary axis may be written as follows (Resnick et al. 1992):

$$I = I_G + m \cdot d^2 \tag{20}$$

where I_G is the moment of inertia about a parallel axis through the center of mass of the body, m is the mass of the body, and d is the distance between the two axis.

The second term on the right hand side in (20) is the moment of inertia of a point mass about the suspension point of the pendulum, and we can interpret the first term as being the error due to the non-point character of the mass of the body. So, the mass of the body may be considered to behave as a "point mass" if the ratio between the first and the second terms is very small.

To know how small this quotient must be, let us consider the relative error due to the use of the simple pendulum period expression (T_s) instead of the physical pendulum period expression (T_p). We will assume that a string of vanishing mass and a body of finite size form the physical pendulum. On the other hand, in the simple pendulum period expression we will use "d" instead of "l", given their identity in the case under consideration. We obtain:

$$\frac{T_p - T_s}{T_s} = \frac{2\pi\sqrt{\dfrac{I_G + m \cdot d^2}{m \cdot g \cdot d}} - 2\pi\sqrt{\dfrac{d}{g}}}{2\pi\sqrt{\dfrac{d}{g}}}$$

$$= \frac{2\pi\sqrt{\dfrac{d}{g}\left(\dfrac{I_G}{m \cdot d^2}\right)} - 2\pi\sqrt{\dfrac{d}{g}}}{2\pi\sqrt{\dfrac{d}{g}}}$$

$$= \left(\frac{I_G}{m \cdot d^2} + 1\right)^{1/2} - 1 \qquad (21)$$

Expanding the statement between parenthesis in (21) as a binomial series and neglecting third and higher order terms, it can be seen that the mass of the body may be considered a "point mass" if the following relationship is verified:

$$\frac{1}{2}\frac{I_G}{m \cdot d^2} \leq \varepsilon_b \qquad (22)$$

where ε_b must be fitted by (6).

3.5. VANISHING MASS OF THE STRING

The mass of the string affects the oscillation period because:

– it contributes to the moment of inertia of the system
– it changes the position of the center of gravity of the system
– it changes the mass of the system.

Let us first consider how the mass of the string influences the moment of inertia of the system. The moment of inertia of the system about the point of suspension is:

$$I = I_{\text{string}} + I_{\text{body}}. \tag{23}$$

In equation (1), this effect of I_{string} appears as an increase in the oscillation period.

Let us now consider how the mass of the string influences the position of the center of gravity of the system. The distance between the oscillation axis and the gravity center is (Resnick et al. 1992):

$$d = \frac{m_s \cdot l/2 + m_b \cdot l}{m_s + m_b} \tag{24}$$

In Equation (24), it can be seen that $d < l$. So, in Equation (1), this effect of m_{string} appears as an increase in the oscillation period.

Let us finally consider how the mass of the string influences the mass of the system.

The total mass of the system is:

$$m = m_{\text{string}} + m_{\text{body}}. \tag{25}$$

In Equation (1), this effect of m_{string} appears as a decrease in the oscillation period.

Now we will insert (23), (24) and (25) in Equation (1), to find an expression which shows the quantitative difference between the period values predicted by the physical and the simple pendulum models.

We will take into account that the moment of inertia of a homogeneous string is given by the following expression (Resnick et al. 1992):

$$I_{\text{string}} = \frac{1}{3} \cdot m_{\text{string}} \cdot l^2 \tag{26}$$

and we will define:

$$k = m_s/m_b. \tag{27}$$

We obtain:

$$\begin{aligned} T_p &= 2\pi\sqrt{\frac{I}{m \cdot g \cdot d}} \\ &= 2\pi\sqrt{\frac{I_s + I_b}{(m_s + m_b) \cdot g \cdot \dfrac{m_s \cdot 1/2 + m_b \cdot l}{m_s + m_b}}} \end{aligned}$$

$$= 2\pi \sqrt{\frac{\left(\frac{1}{3}k + 1\right) \cdot m_b \cdot l^2}{g \cdot \left(\frac{1}{2}k + 1\right) \cdot m_b \cdot l}}$$

$$T_p = 2\pi \sqrt{\frac{l}{g}} \cdot \sqrt{\frac{\frac{1}{3}k + 1}{\frac{1}{2}k + 1}} \tag{28}$$

The second root in the right hand side of (28) is less than unity. Thus, $T_p < T_s$, i.e., the total influence of the mass of the string is to increase the period value predicted by the simple pendulum model.

We can write:

$$\frac{T_p - T_s}{T_s} = \sqrt{\frac{l}{g}} \cdot \sqrt{\frac{\frac{1}{3}k + 1}{\frac{1}{2}k + 1}} - 1 \tag{29}$$

According to the difference just mentioned between T_p and T_s, this expression gives a negative relative error due to a non-vanishing mass of the string. Thus, it suffices for the mass of the string effect to be vanishing that the right hand side of (29) is equal ε_s fitted by (6).

4. Conclusions

From quantitative analysis of systematic error we have shown that model assumptions are easily accomplished in practice, within small experimental errors. Considered separately, within an error of 1%:

- an initial amplitude of 23° is "small".
- a sphere, whose diameter is 30% of the length of the string, is "a point mass".
- a mass of the string equal to 10% of the mass of the body is "vanishing".
- any elastic elongation suffered by the string during the static process of loading is negligible, providing the string length is measured after the loading.
- without loosing its property of 'not extensible', the string may vary its length during oscillation (due to a variable tension), providing this variation is less than the measurement error of the string length.

This way of carrying out the usual pendulum experiments:

- promotes a better understanding of the scientific modeling process.
- allows a deeper comprehension of those physical concepts associated with model assumptions (small amplitude, point mass, etc.), whose physical and epistemological meanings appear clearly related to the model context.

– introduces students to a scientific way of controlling the validity of theoretical development, and helps them to value the power and applicability of scientific modeling.

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