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On normal operator logarithms

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ABSTRACT

Let X, Y be normal bounded operators on a Hilbert space such that $e^X = e^Y$. If the spectra of X and Y are contained in the strip $\mathcal{S}$ of the complex plane defined by $|\mathrm{Im}(z)| \leqslant \pi$, we show that $|X| = |Y|$. If Y is only assumed to be bounded, then $|X|Y = Y|X|$. We give a formula for $X - Y$ in terms of spectral projections of X and Y provided that X, Y are normal and $e^X = e^Y$. If X is an unbounded self-adjoint operator, which does not have $(2k+1)\pi$, $k \in \mathbb{Z}$, as eigenvalues, and Y is normal with spectrum in $\mathcal{S}$ satisfying $e^{iX} = e^Y$, then $Y \in \{e^{iX}\}''$. We give alternative proofs and generalizations of results on normal operator exponentials proved by Schmoeger.

1. Introduction

Solutions to the equation $e^X = e^Y$ were studied by Hille [1] in the general setting of unital Banach algebras. Under the assumption that the spectrum $\sigma(X)$ of X is incongruent (mod $2\pi i$), which means that $\sigma(X) \cap \sigma(X + 2k\pi i) = \emptyset$ for all $k = \pm 1, \pm 2, \ldots$, he proved that $XY = YX$ and there exist idempotents $E_1, E_2, \ldots, E_n$ commuting with X and Y such that

$$X - Y = 2\pi i \sum_{j=1}^{n} k_j E_j, \qquad \sum_{j=1}^{n} E_j = I, \qquad E_i E_j = \delta_{ij},$$

where $k_1, k_2, \ldots, k_n$ are different integers. If the hypothesis on the spectrum is removed, it is possible to find non commuting logarithms (see e.g. [1,6]). In the setting of Hilbert spaces, when X is a normal

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operator, the above assumption on the spectrum can be weakened. In fact, Schmoeger [5] proved that X belongs to the double commutant of Y provided that $E_X(\sigma(X)\cap\sigma(X+2k\pi i)) = 0, k = 1, 2, \ldots$, where E_X is the spectral measure of X. We also refer to [3] for a generalization of this result by Paliogiannis.

In this paper, we study the operator equation $e^X = e^Y$ in the setting of Hilbert spaces under the assumption that the spectra of X and Y belong to a non-injective domain of the complex exponential map. Our results include the relation between the modulus of X and Y (Theorem 3.1), a formula for the difference of two normal logarithms in terms of their spectral projections (Theorem 4.1) and commutation relations when X is a skew-adjoint unbounded operator (Theorem 5.1). The proofs of these results are elementary. In fact, they rely on the spectral theorem for normal operators. This approach allows us to give a generalization (Corollary 4.2) and an alternative proof (Corollary 3.2) of two results by Schmoeger (see [6]).

2. Notation and preliminaries

Let $(\mathcal{H}, \langle\cdot\,,\,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. The spectrum of an operator X is denoted by $\sigma(X)$, and the set of eigenvalues of X is denoted by $\sigma_p(X)$. The real part of $X \in \mathcal{B}(\mathcal{H})$ is $\mathrm{Re}(X) = \frac{1}{2}(X + X^*)$ and its imaginary part is $\mathrm{Im}(X) = \frac{1}{2}(X - X^*)$.

If X is a bounded or unbounded normal operator on $\mathcal{H}$, we denote by E_X the spectral measure of X. Recall that E_X is defined on the Borel subsets of $\sigma(X)$, but we may think that E_X is defined on all the Borel subsets of $\mathbb{C}$. Indeed, we can set $E_X(\Omega) = E_X(\Omega \cap \sigma(X))$ for every Borel set $\Omega \subseteq \mathbb{C}$. Our first lemma is a generalized version of [4, Ch. XII Ex. 25], where the normal operator can now be unbounded.

Lemma 2.1. *Let X be a (possibly unbounded) normal operator on $\mathcal{H}$ and f a bounded Borel function on $\sigma(X)$. Then*

$$E_{f(X)}(\Omega) = E_X(f^{-1}(\Omega)),$$

for every Borel set $\Omega \subseteq \mathbb{C}$.

Proof. We define a spectral measure by $E'(\Omega) = E_X(f^{-1}(\Omega))$, where Ω is any Borel subset of $\mathbb{C}$. We are going to show that $E' = E_{f(X)}$. Since f is bounded, it follows that $f(X) \in \mathcal{B}(\mathcal{H})$. Moreover, the operator $f(X)$ is given by

$$\langle f(X)\xi, \eta\rangle = \int_{\mathbb{C}} f(z)\, dE_{X\,\xi,\eta}(z)\,,$$

where $\xi, \eta \in \mathcal{H}$ and $E_{X\,\xi,\eta}$ is the complex measure defined by $E_{X\,\xi,\eta}(\Omega) = \langle E_X(\Omega)\xi, \eta\rangle$ (see [4, Theorem 12.21]). By the change of measure principle ([4, Theorem 13.28]), we have

$$\int_{\mathbb{C}} z\, dE'_{\xi,\eta}(z) = \int_{\mathbb{C}} f(z)\, dE_{X\,\xi,\eta}(z).$$

Therefore E' satisfies the equation $\int_{\mathbb{C}} z\, dE'_{\xi,\eta}(z) = \langle f(X)\xi, \eta\rangle$, which uniquely determines the spectral measure of $f(X)$ (see [4, Theorem 12.23]). Hence $E' = E_{f(X)}$. □

The following lemma was first proved in [6, Corollary 2]. See also [3, Corollary 3] for another proof. We give below a proof for the sake of completeness, which does not depend on further results of these articles.

Lemma 2.2. *Let X and Y be normal operators in $\mathcal{B}(\mathcal{H})$. If $e^X = e^Y$, then* $\mathrm{Re}(X) = \mathrm{Re}(Y)$.

Proof. The following computation was done in [6]:

$$e^{X+X^*} = e^X e^{X^*} = e^X (e^X)^* = e^Y (e^Y)^* = e^Y e^{Y^*} = e^{Y+Y^*},$$

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where the first and last equalities hold because X and Y are normal. Now we may finish the proof in a different fashion: note that the exponential map, restricted to real axis, has an inverse $\log : \mathbb{R}_+ \to \mathbb{R}$. Since $\sigma(X+X^*) \subseteq \mathbb{R}$ and $\sigma(e^{X+X^*}) \subseteq \mathbb{R}_+$, we can use the continuous functional calculus to get $X + X^* = \log(e^{X+X^*}) = \log(e^{Y+Y^*}) = Y + Y^*$. $\square$

Throughout this paper, we use the following notation for subsets of the complex plane:

- $\Omega_1 + i\Omega_2 = \{ x + iy \,:\, x \in \Omega_1,\ y \in \Omega_2 \}$, where Ω_i, $i = 1, 2$, are subsets of $\mathbb{R}$.
- For short, we write $\mathbb{R} + ia$ for the set $\mathbb{R} + i\{a\}$.
- We write $\mathcal{S}$ for the complex strip $\{ z \in \mathbb{C} \,:\, -\pi \leqslant \mathrm{Im}(z) \leqslant \pi \}$, and $\mathcal{S}^\circ$ for the interior of $\mathcal{S}$.

Lemma 2.3. *Let X, Y be normal operators in $\mathcal{B}(\mathcal{H})$ such that $\sigma(X) \subseteq \mathcal{S}$ and $\sigma(Y) \subseteq \mathcal{S}$. Then $e^X = e^Y$ if and only if the following conditions hold:*

(i) $E_X(\Omega) = E_Y(\Omega)$ *for all Borel subsets Ω of $\mathcal{S}^\circ$.*
(ii) $\mathrm{Re}(X) = \mathrm{Re}(Y)$.

Proof. Suppose that $e^X = e^Y$. Let Ω be a Borel measurable subset of $\mathcal{S}^\circ$. By the spectral mapping theorem,

$$\sigma(e^X) = \{ e^\lambda \,:\, \lambda \in \sigma(X) \} = \{ e^\mu \,:\, \mu \in \sigma(Y) \} = \sigma(e^Y).$$

It is well-known that the restriction of the complex exponential map $\exp|_{\mathcal{S}^\circ}$ is bijective. Therefore we have $\sigma(X) \cap \Omega = \sigma(Y) \cap \Omega$, and by Lemma 2.1,

$$\begin{aligned} E_X(\Omega) &= E_X(\Omega \cap \sigma(X)) = E_X(\exp^{-1}(\exp(\Omega \cap \sigma(X)))) \\ &= E_{e^X}(\exp(\Omega \cap \sigma(X))) = E_{e^Y}(\exp(\Omega \cap \sigma(Y))) = E_Y(\Omega), \end{aligned}$$

which proves (i). On the other hand, (ii) is proved in Lemma 2.2.

To prove the converse assertion, we first note that

$$\begin{aligned} E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) &= I - E_X(\mathcal{S}^\circ) = I - E_Y(\mathcal{S}^\circ) \\ &= E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi), \end{aligned}$$

since $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $E_X(\mathcal{S}^\circ) = E_Y(\mathcal{S}^\circ)$. Due to the fact that E_X and E_Y coincide on Borel subsets of $\mathcal{S}^\circ$, we find that

$$\int_{\mathcal{S}^\circ} e^z \, dE_X(z) = \int_{\mathcal{S}^\circ} e^z \, dE_Y(z).$$

Hence we get

$$\begin{aligned} e^X &= \int_{\mathcal{S}} e^z \, dE_X(z) = -\int_{\mathbb{R}+i\pi} e^{\mathrm{Re}(z)} \, dE_X(z) - \int_{\mathbb{R}-i\pi} e^{\mathrm{Re}(z)} \, dE_X(z) + \int_{\mathcal{S}^\circ} e^z \, dE_X(z) \\ &= -e^{\mathrm{Re}(X)}(E_X(\mathbb{R} + i\pi) + E_X(\mathbb{R} - i\pi)) + \int_{\mathcal{S}^\circ} e^z \, dE_X(z) \\ &= -e^{\mathrm{Re}(Y)}(E_Y(\mathbb{R} + i\pi) + E_Y(\mathbb{R} - i\pi)) + \int_{\mathcal{S}^\circ} e^z \, dE_Y(z) = e^Y. \quad \square \end{aligned}$$

Remark 2.4. We have shown that $E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) = E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi)$, whenever X, Y are normal bounded operators such that $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $e^X = e^Y$.

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Theorem 2.5. *(Kurepa [2]) Let $X \in \mathcal{B}(\mathcal{H})$ such that $e^X = N$ is a normal operator. Then*

$$X = N_0 + 2\pi i W,$$

where $N_0 = \log(N)$ and $\log$ is the principal (or any) branch of the logarithm function. The bounded operator W commutes with N_0 and there exists a bounded and regular, positive definite self-adjoint operator Q such that $W_0 = Q^{-1}WQ$ is a self-adjoint operator the spectrum of which belongs to the set of all integers.

3. Modulus and square of logarithms

Now we show the relation between the modulus of two normal logarithms with spectra contained in $\mathcal{S}$.

Theorem 3.1. *Let X be a normal operator in $\mathcal{B}(\mathcal{H})$. Assume that $\sigma(X) \subseteq \mathcal{S}$ and $e^X = e^Y$.*

(i) If Y is normal in $\mathcal{B}(\mathcal{H})$ and $\sigma(Y) \subseteq \mathcal{S}$, then $|X| = |Y|$.
(ii) If $Y \in \mathcal{B}(\mathcal{H})$, then $|X|Y = Y|X|$.

Proof. (i) We will prove that the spectral measures of $|\mathrm{Im}(X)|$ and $|\mathrm{Im}(Y)|$ coincide. Let us set $A = \mathrm{Im}(X)$ and $B = \mathrm{Im}(Y)$. Given $\Omega \subseteq [0, \pi)$, put $\Omega' = \{x \in \mathbb{R} : |x| \in \Omega\}$. Note that $\mathbb{R} + i\Omega' \subseteq \mathcal{S}^\circ$. As an application of Lemmas 2.1 and 2.3, we see that

$$E_{|A|}(\Omega) = E_A(\Omega') = E_X(\mathbb{R} + i\Omega') = E_Y(\mathbb{R} + i\Omega') = E_B(\Omega') = E_{|B|}(\Omega).$$

By Remark 2.4, we have

$$\begin{aligned} E_{|A|}(\{\pi\}) &= E_A(\{-\pi, \pi\}) = E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) \\ &= E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi) = E_{|B|}(\{\pi\}). \end{aligned}$$

Thus, we have proved $E_{|A|} = E_{|B|}$, which implies that $|A| = |B|$. On the other hand, by Lemma 2.2, we know that $\mathrm{Re}(X) = \mathrm{Re}(Y)$. Therefore

$$|X|^2 = \mathrm{Re}(X)^2 + |A|^2 = \mathrm{Re}(Y)^2 + |B|^2 = |Y|^2.$$

Hence $|X| = |Y|$, and the proof is complete.

(ii) Since X is a normal operator, $e^X = e^Y$ is also a normal operator. Then by a result by Kurepa (see Theorem 2.5), there exist operators N_0 and W such that N_0 is normal, $e^X = e^{N_0}$, W commutes with N_0 and $Y = N_0 + 2\pi i W$. In fact, N_0 can be defined using the Borel functional calculus by $N_0 = \log(e^X)$, where log is the principal branch of the logarithm. In particular, this implies that $\sigma(N_0) \subseteq \mathcal{S}$. Now we can apply i) to find that $|N_0| = |X|$. Since $N_0 W = W N_0$, we have $|N_0|W = W|N_0|$, and this gives $W|X| = |X|W$. Hence $|X|Y = Y|X|$. $\square$

Following similar arguments, we can give an alternative proof of a result by Schmoeger ([6, Theorem 3]). This result was originally proved using inner derivations. Note that a minor improvement on the assumption on $\sigma(X)$ over the boundary $\partial\mathcal{S}$ of the strip $\mathcal{S}$ can now be done. Given a set $\Omega \subseteq \mathbb{C}$, we denote by $\bar{\Omega}$ the set $\{x - iy : x + iy \in \Omega\}$.

Corollary 3.2. *Let X be a normal operator in $\mathcal{B}(\mathcal{H})$, $\sigma(X) \subseteq \mathcal{S}$, $Y \in \mathcal{B}(\mathcal{H})$ and $e^X = e^Y$. Suppose that for every Borel subset $\Omega \subseteq \partial\mathcal{S} \setminus \{-i\pi, i\pi\}$, it holds that $E_X(\bar{\Omega}) = 0$, whenever $E_X(\Omega) \neq 0$. Then $X^2Y = YX^2$.*

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Proof. We will show that $E_{X^2}(\Omega_0)$ commutes with Y for every Borel subset $\Omega_0 \subseteq \sigma(X^2)$. From the equation $e^X = e^Y$, we have $e^X Y = Ye^X$, and thus, $E_{e^X}(\Omega)Y = YE_{e^X}(\Omega)$ for any Borel set Ω. Since the set Ω is arbitrary, by Lemma 2.1 we get

(1) $E_X(\Omega')Y = YE_X(\Omega')$ for every subset $\Omega' \subseteq \mathcal{S}^\circ$.
(2) $(E_X(\Omega') + E_X(\bar{\Omega}'))Y = Y(E_X(\Omega') + E_X(\bar{\Omega}'))$, whenever $\Omega' \subseteq \partial\mathcal{S}$.

On the other hand, the image of $\mathcal{S}$ by the analytic map $f(z) = z^2$ is given by

$$f(\mathcal{S}) = \{\, u \pm i2t\sqrt{u + t^2} \;:\; u \in [-\pi^2, \infty),\; u + t^2 \geqslant 0 \,\}.$$

Let us write $f^{-1}(\Omega_0) = \Omega_- \cup \Omega_+$, where $\Omega_- = f^{-1}(\Omega_0) \cap \{\, z \in \mathbb{C} \;:\; \mathrm{Re}(z) < 0 \,\}$ and $\Omega_+ = f^{-1}(\Omega_0) \cap \{\, z \in \mathbb{C} \;:\; \mathrm{Re}(z) \geqslant 0 \,\}$. We point out that $E_{X^2}(\Omega_0) = E_X(\Omega_+) + E_X(\Omega_-)$.

Next we need to consider three cases. In the case in which $\Omega_0 \subseteq f(\mathcal{S})^\circ$, then $\Omega_+ \subseteq \mathcal{S}^\circ$ and $\Omega_- \subseteq \mathcal{S}^\circ$. By the item (1) above we have $E_{X^2}(\Omega_0)Y = YE_{X^2}(\Omega_0)$. In the case where $\Omega_0 \subseteq \partial f(\mathcal{S}) \setminus \{\, -\pi^2 \,\}$, we have that $\Omega_+ \subseteq \partial\mathcal{S} \cap \{\, z \in \mathbb{C} \;:\; \mathrm{Re}(z) > 0 \,\}$. It follows that either $E_X(\Omega_+) = 0$ or $E_X(\bar{\Omega}_+) = 0$ by our assumption on the spectral measure of X. Similarly, it must be either $E_X(\Omega_-) = 0$ or $E_X(\bar{\Omega}_-) = 0$. Therefore item (2) above reduces to the desired conclusion, i.e. $E_{X^2}(\Omega_0)Y = YE_{X^2}(\Omega_0)$. Finally, if $\Omega_0 = \{-\pi^2\}$, then $E_{X^2}(\Omega_0) = E_X(\{\, -i\pi \,\}) + E_X(\{\, i\pi \,\})$ commutes with Y by item (2), and this concludes the proof. □

4. Difference of logarithms

Let X, Y be normal operators and $k \in \mathbb{Z}$. In order to avoid lengthly formulas, let us fix a notation for some special spectral projections of these operators:

- $P_{2k+1} = E_X(\,\mathbb{R} + i(\,(2k-1)\pi,\,(2k+1)\pi\,)\,)$;
- $Q_{2k+1} = E_Y(\,\mathbb{R} + i(\,(2k-1)\pi,\,(2k+1)\pi\,)\,)$;
- $E_{2k+1} = E_X(\,\mathbb{R} + i(2k+1)\pi\,)$;
- $F_{2k+1} = E_Y(\,\mathbb{R} + i(2k+1)\pi\,)$.

As we have pointed out in the introduction, Hille showed that the difference between two logarithms in Banach algebras may be expressed as the sum of multiples of projections (see [1, Theorem 4]). In order to prove that result, the spectrum of one of the logarithms must be incongruent (mod $2\pi i$). In the case where X and Y are both normal logarithms on a Hilbert space, the spectral theorem can be used to provide a more general formula.

Theorem 4.1. *Let X and Y be normal operators in $\mathcal{B}(\mathcal{H})$ such that $e^X = e^Y$. If $\sigma(X)$ and $\sigma(Y)$ are contained in $\mathbb{R} + i[\,(2k_0+1)\pi,\,(2k_1+1)\pi\,]$ for some $k_0, k_1 \in \mathbb{Z}$, then*

$$X - Y = \sum_{k=k_0}^{k_1} \big(2k\pi i\,(P_{2k+1} - Q_{2k+1}) + (2k+1)\pi i\,(E_{2k+1} - F_{2k+1})\big).$$

Proof. We first suppose that $\sigma(X)$ and $\sigma(Y)$ are contained in the strip $\mathcal{S}$. Then we have $\mathrm{Im}(X) = \mathrm{Im}(X)(E_X(\mathcal{S}^\circ) + E_X(\,\mathbb{R} + i\pi\,) + E_X(\,\mathbb{R} - i\pi\,)\,) = \mathrm{Im}(X)P_1 + \pi E_1 - \pi E_{-1}$. Analogously, $\mathrm{Im}(Y) = \mathrm{Im}(Y)Q_1 + \pi F_1 - \pi F_{-1}$. By Lemma 2.3, we know that $\mathrm{Re}(X) = \mathrm{Re}(Y)$ and $E_X(\Omega) = E_Y(\Omega)$ for every Borel subset Ω of $\mathcal{S}^\circ$. It follows that

$$\mathrm{Im}(X)P_1 = \int_{\mathcal{S}^\circ} \mathrm{Im}(z)\, dE_X(z) = \int_{\mathcal{S}^\circ} \mathrm{Im}(z)\, dE_Y(z) = \mathrm{Im}(Y)Q_1,$$

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which implies

$$X - Y = \pi i(E_1 - F_1) - \pi i(E_{-1} - F_{-1}). \tag{1}$$

Thus, we have proved the formula in this case. For the general case, without restrictions on spectrum of X and Y, we need to consider the following Borel measurable function

$$f(t) = \sum_{k=k_0-1}^{k_1} (t - 2k\pi)\, \chi_{((2k-1)\pi,(2k+1)\pi]}(t),$$

where $\chi_I(t)$ is the characteristic function of the interval I. Set $A = \mathrm{Im}(X)$ and $B = \mathrm{Im}(Y)$. By Lemma 2.2, $\mathrm{Re}(X) = \mathrm{Re}(Y)$, and since the real and imaginary part of X and Y commute because X and Y are normal, $e^{iA} = e^X e^{-\mathrm{Re}(X)} = e^Y e^{-\mathrm{Re}(Y)} = e^{iB}$. The function f satisfies $e^{if(t)} = e^{it}$, which implies that $e^{if(A)} = e^{iA} = e^{iB} = e^{if(B)}$. Since $\sigma(f(A))$ and $\sigma(f(B))$ are contained in $[-\pi, \pi]$, we can replace in Eq. (1) to find that

$$\begin{aligned} f(A) - f(B) &= \pi\left(E_{f(A)}(\{\pi\}) - E_{f(B)}(\{\pi\})\right) \\ &= \pi \sum_{k=k_0-1}^{k_1} \left(E_A(\{(2k+1)\pi\}) - E_B(\{(2k+1)\pi\})\right) \\ &= \pi \sum_{k=k_0}^{k_1} (E_{2k+1} - F_{2k+1}). \end{aligned} \tag{2}$$

Here we have used Lemma 2.1 to express $E_{f(A)}, E_A$ and $E_{f(B)}, E_B$ in terms of E_X and E_Y respectively. In particular, note that $E_{f(A)}(\{-\pi\}) = E_{f(B)}(\{-\pi\}) = 0$. On the other hand, we have

(1) $f(A) = \sum_{k=k_0-1}^{k_1} (A - 2k\pi)\, \chi_{((2k-1)\pi,(2k+1)\pi]}(A) = A - \sum_{k=k_0}^{k_1} 2k\pi(P_{2k+1} + E_{2k+1}),$

(2) $f(B) = B - \sum_{k=k_0}^{k_1} 2k\pi(Q_{2k+1} + F_{2k+1}).$

Therefore

$$\begin{aligned} X - Y &= i(A - B) \\ &= i(f(A) - f(B)) + \sum_{k=k_0}^{k_1} \left(2k\pi i(P_{2k+1} - Q_{2k+1}) + 2k\pi i(E_{2k+1} - F_{2k+1})\right). \end{aligned}$$

Combining this with the expression in (2), we get the desired formula. □

Below we give a generalization of another result due to Schmoeger (see [6, Theorem 5]). The assumptions on the spectrum of X and Y were more restrictive in [6]: $\|X\| \leqslant \pi$, $\|Y\| \leqslant \pi$ and either $-i\pi$ or $i\pi$ does not belong to the point spectrum of one of these operators. However, these hypothesis were necessary to conclude that $X - Y$ is a multiple of a projection; meanwhile $XY = YX$ can be obtained under more general assumptions (see [6, Theorem 3], [5, Theorem 1.4] and [3, Theorem 9]).

Corollary 4.2. *Let X, Y be normal operators in $\mathcal{B}(\mathcal{H})$. Assume that $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $e^X = e^Y$. The following assertions hold:*

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(i) *If* $E_1 = 0$, *then* $XY = YX$ *and* $X - Y = -2\pi i F_1$.
(ii) *If* $E_{-1} = 0$, *then* $XY = YX$ *and* $X - Y = 2\pi i F_{-1}$.
(iii) *If* $E_1 = E_{-1} = 0$, *then* $X = Y$.

Proof. (i) Under these assumptions on the spectra of X and Y, we have established that $E_{-1} + E_1 = F_{-1} + F_1$ in Remark 2.4. On the other hand, by Eq. (1) in the proof of Theorem 4.1, we know that $X - Y = \pi i(E_1 - F_1) - \pi i(E_{-1} - F_{-1})$. Since $E_1 = 0$, we have $E_{-1} = F_1 + F_{-1}$. It follows that $X = -2\pi i F_1 + Y$. Hence X and Y commute. We can similarly conclude that (ii) holds true. To prove (iii), note that $E_1 = E_{-1} = 0$ implies that $F_1 + F_{-1} = 0$, and consequently, $F_1 = F_{-1} = 0$. Hence we get $X = Y$. □

5. Unbounded logarithms

Let X be a self-adjoint unbounded operator on $\mathcal{H}$. As before, E_X denotes the spectral measure of X. In item (i) of our next result, we will give a version of [5, Theorem 1.4] for unbounded operators (see also [3, Theorem 9]). To this end, we extend the definition given in [5] for bounded operators: a self-adjoint unbounded operator X is *generalized* 2π*-congruence-free* if

$$E_X(\sigma(X) \cap \sigma(X + 2k\pi)) = 0, \qquad k = \pm 1, \pm 2, \ldots .$$

Given $Y \in \mathcal{B}(\mathcal{H})$, the commutant of Y is the set

$$\{Y\}' = \{Z \in \mathcal{B}(\mathcal{H}) \,:\, ZY = YZ\}.$$

The double commutant of Y is defined by

$$\{Y\}'' = \{W \in \mathcal{B}(\mathcal{H}) \,:\, WZ = ZW, \text{ for all } Z \in \{Y\}'\}.$$

If X is a self-adjoint unbounded operator and $Y \in \mathcal{B}(\mathcal{H})$, recall that $XY = YX$, that is X commutes with Y, if $YE_X(\Omega) = E_X(\Omega)Y$ for every Borel subset $\Omega \subseteq \mathbb{R}$. Recall that the exponential e^{iX} of a self-adjoint unbounded operator X is a unitary operator, which can be defined via the Borel functional calculus (see e.g. [4]).

Theorem 5.1. *Let* X *be a self-adjoint operator on* $\mathcal{H}$ *and* $Y \in \mathcal{B}(\mathcal{H})$ *such that* $e^{iX} = e^{Y}$.

(i) *If* X *is generalized* 2π*-congruence-free, then* $E_X(\Omega) \in \{Y\}''$ *for all Borel subsets* Ω *of* $\mathbb{R}$. *In particular,* $XY = YX$.
(ii) *If* $\{(2k+1)\pi \,:\, k \in \mathbb{Z}\} \cap \sigma_p(X)$ *has at most one element and* Y *is normal in* $\mathcal{B}(\mathcal{H})$ *such that* $\sigma(Y) \subseteq \mathcal{S}$, *then* $XY = YX$.
(iii) *If* $(2k+1)\pi \notin \sigma_p(X)$ *for all* $k \in \mathbb{Z}$ *and* Y *is normal in* $\mathcal{B}(\mathcal{H})$ *such that* $\sigma(Y) \subseteq \mathcal{S}$, *then* $Y \in \{e^{iX}\}''$.

Proof. (i) Let $Z \in \mathcal{B}(\mathcal{H})$ such that $ZY = YZ$. It follows that $Ze^Y = e^Y Z$. Then we have $Ze^{iX} = e^{iX}Z$, and by Lemma 2.1, $ZE_X(\exp^{-1}(\Omega)) = E_X(\exp^{-1}(\Omega))Z$ for every $\Omega \subseteq \mathbb{T}$. If $\Omega' = \exp^{-1}(\Omega) \cap [-\pi, \pi]$, then

$$E_X(\exp^{-1}(\Omega)) = \sum_{k\in\mathbb{Z}} E_X(\Omega' + 2k\pi),$$

where this series converges in the strong operator topology. Suppose now that there is some $k \in \mathbb{Z}$ such that $E_X(\Omega' + 2k\pi) \neq 0$. It follows that $\sigma(X) \cap (\Omega' + 2k\pi) \neq \emptyset$, and $(\Omega' + 2l\pi) \cap \sigma(X) \subseteq \sigma(X) \cap \sigma(X + 2(l-k)\pi)$ for all $l \in \mathbb{Z}$. By the assumption on the spectral measure of X, $E_X(\Omega' + 2l\pi) \leqslant E_X(\sigma(X) \cap \sigma(X + 2(l-k)\pi)) = 0$ for $l \neq k$. Therefore for each Ω, the above series reduces to only one spectral projection corresponding to a set of the form $\Omega' + 2k\pi$. Hence Z commutes with all the spectral projections of X.

(ii) We need to consider the Borel measurable function f defined in the proof of Theorem 4.1. Since $e^{iX} = e^Y$, we have that $e^{if(X)} = e^Y$. Recall that $E_X(\{(2k+1)\pi\}) \neq 0$ if and only if $(2k+1)\pi \in \sigma_p(X)$ ([4, Theorem 12.19]). By the hypothesis on the eigenvalues of X, there is at most one $n_0 \in \mathbb{Z}$ such that $E_X(\{(2n_0+1)\pi\}) \neq 0$. According to Lemma 2.1, we get

$$E_{f(X)}(\{\pi\}) = \sum_{k\in\mathbb{Z}} E_X(\{(2k+1)\pi\}) = E_X(\{(2n_0+1)\pi\}).$$

On the other hand, $E_{f(X)}(\{-\pi\}) = 0$ for all $k \in \mathbb{Z}$ by definition of the function f. According to Corollary 4.2 *ii*), it follows that $if(X) = Y + 2\pi i F_{-1}$. By Remark 2.4, we also know that $E_X(\{(2n_0+1)\pi\}) = F_{-1} + F_1$. In order to show that Y commutes with all the spectral projections of X, we divide into two cases. If $\Omega \subseteq \mathbb{C} \setminus \{(2k+1)\pi \ : \ k \in \mathbb{Z}\}$, note that $E_X(\Omega)F_{-1} = 0$ because $F_{-1} \leqslant E_X(\{(2n_0+1)\pi\})$. Hence we get

$$E_X(\Omega)Y = E_X(\Omega)(if(X) - 2\pi i F_{-1}) = iE_X(\Omega)f(X) = if(X)E_X(\Omega) = YE_X(\Omega).$$

If $\Omega \subseteq \{(2k+1)\pi \ : \ k \in \mathbb{Z}\}$, we only need to prove that $E_X(\{(2n_0+1)\pi\})$ commutes with Y. This follows immediately, because $E_X(\{(2n_0+1)\pi\})$ is the sum of two spectral projections of Y.

(iii) As in the proof of *ii*), we have $e^{if(X)} = e^Y$. Now by the assumption on the eigenvalues of X, it follows that

$$E_{f(X)}(\{-\pi, \pi\}) = \sum_{k\in\mathbb{Z}} E_X(\{(2k+1)\pi\}) = 0. \tag{3}$$

Applying Corollary 4.2 (iii), we get $if(X) = Y$. Recall that $f(X)$ is a self-adjoint operator such that $\sigma(f(X)) \subseteq [-\pi, \pi]$.

Let $Z \in \mathcal{B}(\mathcal{H})$ such that $Ze^{iX} = e^{iX}Z$. Then we have $ZE_{e^{iX}}(\Omega) = E_{e^{iX}}(\Omega)Z$ for every Borel set $\Omega \subseteq \mathbb{T}$. We are going to show that $ZE_{f(X)}(\Omega') = E_{f(X)}(\Omega')Z$ for every $\Omega' \subseteq [-\pi, \pi]$. We need to consider two cases. If $\Omega' \subseteq (-\pi, \pi)$, there exists a unique set $\Omega \subseteq \mathbb{T} \setminus \{-1\}$ such that $\exp^{-1}(\Omega) \cap [-\pi, \pi] = \Omega'$. Therefore

$$E_{f(X)}(\Omega') = \sum_{k\in\mathbb{Z}} E_X(\Omega' + 2k\pi) = E_X(\exp^{-1}(\Omega)) = E_{e^{iX}}(\Omega).$$

If $\Omega' \subseteq \{-\pi, \pi\}$, by Eq. (3) we find that $E_{f(X)}(\Omega') = 0$. Hence we obtain that Z commutes with every spectral projection of $f(X)$. The latter is equivalent to saying that Z commute with Y, and this concludes the proof. $\square$

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