Integr. equ. oper. theory 37 (2) 143-168
0378-620X//020143-26 $ 1.50+0.20/0

Integral Equations
and Operator Theory

PROJECTIVE SPACES OF A C^*-ALGEBRA

E. ANDRUCHOW, G. CORACH AND D. STOJANOFF *

Based on the projective matrix spaces studied by B. Schwarz and A. Zaks, we study the notion of projective space associated to a C*-algebra A with a fixed projection p. The resulting space $\mathcal{P}(p)$ admits a rich geometrical structure as a holomorphic manifold and a homogeneous reductive space of the invertible group of A. Moreover, several metrics (chordal, spherical, pseudo-chordal, non-Euclidean - in Schwarz-Zaks terminology) are considered, allowing a comparison among $\mathcal{P}(p)$, the Grassmann manifold of A and the space of positive elements which are unitary with respect to the bilinear form induced by the reflection $\varepsilon = 2p - 1$. Among several metrical results, we prove that geodesics are unique and of minimal length when measured with the spherical and non-Euclidean metrics.

1 Introduction

There are several papers ([15], [2], [24]) treating the topological and metric properties of the space $P = P(A)$ of selfadjoint projections of a C*-algebra A. These properties are used to obtain invariants for the algebra A (see [24] for instance). Many of these invariants have to do with problems concerning the length of curves in P. There are other papers ([10], [16], [23]) studying P as a differentiable manifold (in fact a complemented submanifold of A). From this viewpoint, problems concerning length of curves - e.g. characterization of curves of minimal length among the curves joining the same endpoints - can be treated using variational principles, in an infinite dimensional setting. Since the norms considered in the tangent bundle of P do not arise from inner products, this analysis does not proceed as in the riemannian case, and requires new methods.

In a series of papers [18]-[22], B. Schwarz and A. Zaks have studied what they call "matrix projective spaces". Their papers inspired our treatment of the space P as a "one dimensional" projective space of A. Most of the features introduced by Schwarz and Zaks for the algebra $M_{2n}(\mathbb{C})$ can be carried over in the general C*-algebra case and we can construct an identification between the projective space of a C*-algebra and its Grassmann manifold

*Partially supported by UBACYT TW49 and TX92, PIP 4463 (CONICET) and ANPCYT PICT 97-2259 (Argentina)

of selfadjoint projections. It should be mentioned that, instead of merely generalizing to the general case the ideas of Schwarz and Zaks, we define a projective space depending on a fixed projection, which allows us to deal simultaneously with all "higher dimensional" projective spaces in the terminology of Schwarz and Zaks. Many problems that we study here are not considered in their papers. On the other hand, many questions treated by Schwarz and Zaks in $M_n(\mathbb{C})$, particularly those concerning Moebius transformations, will be studied for general C*-algebras in a forthcoming paper.

What one gains by taking this standpoint, is the possibility of considering questions and mathematical objects related to P, which come up naturally in the projective space setting, and give interesting information concerning P and A. Among these, several metrics for P and the problem of characterizing their short curves, and the group of projectivities of A. Moreover, the natural complex structure of the projective space induces a complex structure on P. Such structure turns out to be the same that Wilkins obtained by other means in [23].

Let A be a C*-algebra and $p \in A$ a projection. Denote by $G = G_A$ the group of invertibles of A and $\mathcal{U} = \mathcal{U}_A$ the unitary group of A. The orbits $S(p) = \{gpg^{-1} : g \in G_A\}$ and $\mathcal{U}(p) = \{upu^* : u \in \mathcal{U}_A\}$ have rich geometrical and metric properties, studied, among others, in the papers [16], [23], [5], [6], [10], [2] and [15]. The orbit $\mathcal{U}(p)$ can be seen as a Grassmann manifold of A.

We denote by $\mathcal{P}(p)$ the projective space of A determined by p. It can be defined as the quotient of the set $\mathcal{K}_p$ of partial isometries of A with initial space p by the following equivalence relation: two elements $v, w \in \mathcal{K}_p$ verify $v \sim_2 w$ if there exists $u \in \mathcal{U}_{pAp}$ such that $v = wu$. In this case we denote by $[v] = [w]$ the class in $\mathcal{P}(p)$ (see (2.8) for a more detailed definition).

The paper [5] contains a geometrical study of the set

$$S_r = \{(a, b) \in A \times A : ar = a, rb = b, ba = r\}$$

where r is an idempotent element of the Banach algebra A, and a corresponding study of the selfadjoint part R_r of S_r if A is a C*-algebra and r is supposed to be selfadjoint. There is an obvious relation between the spaces $\mathcal{K}_p$ and S_r, and this paper may be seen as a kind of continuation of [5]. Many constructions done in this paper can be generalized to the Banach algebra setting. We choose the C*-algebra case in order to keep the paper into a reasonable size.

We define a natural C^∞ manifold structure on $\mathcal{P}(p)$, and the chordal and spherical metrics generalizing [19]. We show (Theorem 3.5) that the spherical metric has curves of minimal length, which are in fact the geodesics determined by the C^∞ homogeneous reductive structure induced on $\mathcal{P}(p)$ by the natural action of $\mathcal{U}_A$, given by left multiplication.

We show that the projective space $\mathcal{P}(p)$ is diffeomorphic to the Grassmann manifold $\mathcal{E}_p = \{$ projections $q \in A : q \sim p\}$, where $\sim$ denotes the usual equivalence of projections (see Theorem 2.12). Via this diffeomorphism, we characterize the chordal metric of $\mathcal{P}(p)$ as the metric induced by the norm on $\mathcal{E}_p$. Also the spherical metric of $\mathcal{P}(p)$ is identified with the geodesic metric defined in $\mathcal{E}_p$ by its natural Finsler structure (see (2.15)). Note that $\mathcal{E}_p$ is a discrete union of unitary orbits of projections of A. Then $\mathcal{E}_p$ has the same local geometrical structure as $\mathcal{U}(p)$. We show (Proposition 3.7) that there exists a unique geodesic of minimal length joining any two points of $\mathcal{P}(p)$ which have spherical distance less than $\pi/2$. This result was unknown for the Grassmann manifolds.

We define the group of projectivities of $\mathcal{P}(p)$, using the action of G_A on $\mathcal{P}(p)$ by left multiplication. The set of "finite points" $\mathcal{P}_f(p)$ of $\mathcal{P}(p)$ is characterized in terms of the chordal and spherical metrics. For example, it is shown that $\mathcal{P}_f(p)$ is exactly the set of points whose spherical distance to $[p]$ is less than $\pi/2$ (see Proposition 4.6). A consequence of this fact is that $\mathcal{P}_f(p)$ is homeomorphic to the linear manifold $H_p = (1-p)Ap$ (see Proposition 4.6). The Moebius maps are defined and their domains are characterized (4.7). They are of particular interest in the case of the algebra $A = B^{2\times 2}$ for a C*-algebra B, when $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (see (4.5)).

A holomorphic structure is defined on $\mathcal{P}(p)$ (Theorem 5.2) via the local homeomorphisms mentioned before. Also a homogeneous reductive structure is introduced, using the natural action of the Lie group G_A given by the projetivities (see (5.2)).

Finally we consider the pseudo-chordal and non-Euclidean metrics (generalizing the definitions of [19]) on the unit disc $\Delta^+(p)$, defined as the orbit of $[p]$ in $\mathcal{P}(p)$ by the action of the group of ε-unitaries $\mathcal{U}_\varepsilon(A)$ by left multiplication, where ε is the symmetry $\varepsilon = 2p-1$. The disc $\Delta^+(p)$ is characterized in several ways (Propositions 6.5 and 6.7) and the pseudo-chordal and non-Euclidean metrics are showed to be the translation of the natural metrics of the space $\mathcal{U}_\varepsilon(A)^+ = A^+ \cap \mathcal{U}_\varepsilon(A)$ studied in [8], [9] and [10] (see Theorem 6.11). Also a C^∞ manifold structure is defined on $\Delta^+(p)$ and the natural action of $\mathcal{U}_\varepsilon(A)$ converts $\Delta^+(p)$ a homogeneous reductive space with a Finsler metric. We show that the geodesics become curves of minimal length, and therefore the non-Euclidean metric is rectifiable.

The connections between the work by L. G. Brown, G. K. Pederesen [3], [4], G. K. Pedersen [14] and S. Zhang [24] and our work is not completely understood, but they may have interesting consequences in our context. We expect to deal with these matters in a future work. However, we should warm the reader about the completely different use of the word "projective" done by us and by those who deal with projective C*-algebras.

2 The projective space.

Let A be a C*-algebra, G_A the group of invertibles of A and $\mathcal{U}_A$ the unitary group of A. Let $p = p^2 = p^* \in A$ be a fixed projection. If C is a subset of A, Cp denotes the set $\{cp : c \in C\}$.

Usually, one regards the space of projections $\mathcal{E}_p = \{q : q \sim p\}$ as an homogeneous space (i.e. quotient of) the unitary group of A. Here we propose an alternate view of $\mathcal{E}_p$, considering another natural action, of the analytic Lie group G_A. Using the fixed projection p, one can regard the elements of A as 2×2 matrices. We shall consider the set of matrices with second column equal to zero, and introduce there a equivalence relation. It will be readily clear that G_A acts on the quotient $\mathcal{P}(p)$ (by left multiplication), and that this space $\mathcal{P}(p)$ is homeomorphic to $\mathcal{E}_p$.

Definition 2.1 Let A be a C*-algebra and $p \in A$ a projection. We consider the following subsets of A

$$\mathcal{L}_p = \{a \in Ap : \text{ there exists } b \in pA \text{ with } ba = p\}$$

and

$$\mathcal{K}_p = \{v \in Ap : v^*v = p\}.$$

Note that $\mathcal{K}_p$ consists of the partial isometries of A with initial space p.

Remark 2.2 If $A = M_n(\mathbb{C})$ and $p \in A$ is a projection, then

$$\mathcal{L}_p = G_A \cdot p = \{ a \in Ap : rank(a) = rank(p) \}.$$

Analogously $\mathcal{K}_p = \mathcal{U}_A \cdot p$. In general, the inclusions $G_A \cdot p \subset \mathcal{L}_p$ and $\mathcal{U}_A \cdot p \subset \mathcal{K}_p$ are strict (e.g. $A = \mathcal{B}(H)$).

Definition 2.3 Let $p \in A$ a projection. Let us state the following equivalence relations:

1. The relation $\sim_1$ in $\mathcal{L}_p$: $a_1p \sim_1 a_2p$ if there exists $h \in G_{pAp}$ such that $a_1ph = a_2p$.
2. The relation $\sim_2$ in $\mathcal{K}_p$: $v_1p \sim_2 v_2p$ if there exists $w \in \mathcal{U}_{pAp}$ such that $v_1pw = v_2p$.

Remark 2.4 If we write the elements of A as 2×2 matrices using p, then

$$\mathcal{L}_p = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in A : \text{ there exists } \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in A \text{ such that } \begin{pmatrix} ca+db & 0 \\ 0 & 0 \end{pmatrix} = p\}$$

with an analogous description for $\mathcal{K}_p$. The equivalence relation $\sim_1$ is given by

$$\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} \sim_1 \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \text{ if } \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2h & 0 \\ b_2h & 0 \end{pmatrix}, \tag{1}$$

for some $h \in G_{pAp}$ and analogously for $\sim_2$. Note that $\sim_2$ is just the restriction of $\sim_1$ to $\mathcal{K}_p$.

Remark 2.5 It is easy to see that, for $a \in Ap$,

$$a \in \mathcal{L}_p \Leftrightarrow a^*a \in G_{pAp}.$$

Therefore, if $a \in \mathcal{L}_p$, the unitary part of the right polar decomposition of a is $u = a|a|^{-1} \in A$, where $|a|^{-1}$ is the inverse of $|a| = (a^*a)^{1/2}$ in $pAp \subseteq A$. Note that, by construction, one gets that $a \sim_1 u$ and $u \in \mathcal{K}_p$.

Corollary 2.6 *If $a \in \mathcal{L}_p$, then there exists $u \in \mathcal{K}_p$ such that $a \sim_1 u$ in $\mathcal{L}_p$.*

Corollary 2.7 *If $g \in Gp$, then there exists $v \in \mathcal{U}$ such that $gp \sim_1 vp$.*

Proof. Suppose that A is faithfully represented in a Hilbert space H. Then $gp \in \mathcal{L}_p$ and $g(1-p) \in \mathcal{L}_{1-p}(A)$ and therefore there exist partial isometries $v_1, v_2 \in A$

$$v_1 : p \to Im(gp) = M \text{ and } v_2 : 1-p \to Im(g(1-p)) = N,$$

such that $gp = v_1|gp|$, $g(1-p) = v_2|g(1-p)|$, $|gp| \in G_{pAp}$ and $|g(1-p)| \in G_{(1-p)A(1-p)}$. Since $g \in G$, then $M \oplus N = H$. Let q_1 be the orthogonal projection onto M and q_2 be the orthogonal projection onto N. Then $q_1 = v_1v_1^* \in A$ and $q_2 = v_2v_2^* \in A$. Moreover, it is easy to see that that $\|q_1q_2\| < 1$. Hence $\|q_2q_1q_2\| < 1$, and $q_2 - q_2q_1q_2 = q_2(1-q_1)q_2 \in G_{q_2Aq_2}$. Therefore $(1-q_1)q_2 \in \mathcal{L}_{q_2}(A)$ and its polar decomposition is $(1-q_1)q_2 = u|(1-q_1)q_2|$, with $u \in A$ a partial isometry $u : q_2 \to 1 - q_1$.

Let $v_3 = uv_2 \in A$. Then v_3 is a partial isometry from $1-p$ to $1-q_1$, $v = v_1 + v_3$ is a unitary element of A and $gp = v_1|gp| = vp|gp|$, hence $vp \sim_1 gp$. □

Definition 2.8 Let A be a C*-algebra and $p \in A$ a projection. We define the projective space of A determined by p:

$$\mathcal{P}(p) = \mathcal{L}_p / \sim_1 \quad \text{and} \quad \mathcal{P}_0(p) = Gp / \sim_1 .$$

Remark 2.9 The previous results prove that the inclusion map $\mathcal{K}_p \hookrightarrow \mathcal{L}_p$ induces the bijection

$$\mathcal{K}_p / \sim_2 \to \mathcal{L}_p / \sim_1 = \mathcal{P}(p).$$

Analogously the inclusion map $\mathcal{U}_A \cdot p \hookrightarrow G_A \cdot p$ induces the bijection

$$\mathcal{U}_A \cdot p / \sim_2 \to G_A \cdot p / \sim_1 = \mathcal{P}_0(p).$$

In both sets we shall consider the quotient topology induced by the norm topology of A. It will be shown that these bijections are in fact homeomorphisms.

Definition 2.10 Denote $\mathcal{E}_p = \{q \in A : q^* = q = q^2 \text{ and } q \sim p\}$, where $\sim$ denotes the (Murray-von Neumann) equivalence relation for projections of a C*-algebra, i.e. for two projections $p, q \in A$, $p \sim q$ means that there exist $v \in A$ such that $vv^* = q$ and $v^*v = p$.

Remark 2.11

1. The group G_A acts on $\mathcal{P}(p)$ and $\mathcal{P}_0(p)$ by left multiplication. Namely, if $g \in G_A$ and $[a] \in \mathcal{P}(p)$, put $g \times [a] = [ga]$. The same definition works in $\mathcal{P}_0(p)$. Occasionally, we shall consider the restriction of this action to $\mathcal{U}_A$.

2. $\mathcal{U}_A$ acts also on $\mathcal{E}_p$, by means of $(u, q) \mapsto uqu^*$. The orbits of this action lie at distance greater or equal than 1 (computed with the norm of A) - it is a standard fact that projections at distance less than 1 are unitarily equivalent with a unitary element in the connected component of 1 (see [17], for example). Therefore each one of these orbits consists of a discrete union of connected components of the space of projections of A (also called the Grassmannians of A). These are well known spaces, which have rich geometric structure as homogeneous reductive spaces and C^∞ submanifolds of A (see [16] and [10]).

   Therefore $\mathcal{E}_p$ is a submanifold of the Grassmannians of A. If, additionally, $\mathcal{U}_A$ is connected, then each component of $\mathcal{E}_p$ is the unitary orbit of a projection.

We shall see that the space $\mathcal{P}(p)$ endowed with the quotient topology is homeomorphic to $\mathcal{E}_p$, therefore inheriting the differentiable structure of the Grassmannians. The mapping $\mathcal{K}_p \to \mathcal{E}_p$ given by $v \mapsto vv^*$ is clearly continuous and surjective. Clearly it defines a continuous surjective map

$$\varrho_p : \mathcal{P}(p) \to \mathcal{E}_p \quad \varrho_p([v]) = vv^*. \tag{2}$$

Theorem 2.12 *Let A be a C^*-algebra and $p \in A$ a projection. Then ϱ_p is a homeomorphism. Moreover, if $[v] \in \mathcal{P}(p)$ and $q = \varrho_p([v])$, then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{U}_A \arrow[r, "\pi_{[v]}"] \arrow[dr, "\pi_q"'] & \mathcal{P}(p) \arrow[d, "\varrho_p"] \\ & \mathcal{E}_p \end{tikzcd}$$

where $\pi_{[v]}(u) = [uv]$ and $\pi_q(u) = uqu^*$, for $u \in \mathcal{U}_A$.

Proof. Let us to prove that ϱ_p is one to one. Suppose that $v_1, v_2 \in \mathcal{K}_p$ with $v_1v_1^* = v_2v_2^*$ and let $w = v_2^*v_1$. Note that $w \in \mathcal{U}_{pAp}$ and that

$$v_2w = v_2v_2^*v_1 = v_1v_1^*v_1 = v_1p = v_1,$$

that is, $v_1 \sim_2 v_2$. Straightforwrad computations show that the diagram commute. The map $\pi_{[v]}$ is continuous and π_q has continuous local cross sections (see [16]). Using the diagram, these facts imply that ϱ_p is an open mapping. □

At the beginning of the section we noted that the sets $\mathcal{L}_p/\sim_1$ and $\mathcal{K}_p/\sim_2$ are in a bijective correspondence. Now we shall prove that their respective quotient topologies also coincide.

Proposition 2.13 *If $\mathcal{L}_p/\sim_1$ and $\mathcal{K}_p/\sim_2$ are endowed with their quotient topologies, then the inclusion map $\mathcal{K}_p \hookrightarrow \mathcal{L}_p$ induces the homeomorphism*

$$\mathcal{K}_p/\sim_2 \to \mathcal{L}_p/\sim_1 .$$

Proof. Suppose that the algebra A is represented on a Hilbart space H. It suffices to prove that the mapping

$$\mathcal{L}_p/\sim_1 \to \mathcal{E}_p \text{ given by } [a] \mapsto P_{a(H)} \quad , \quad a \in \mathcal{L}_p,$$

is continuous, where $P_{a(H)}$ denotes the projection onto the range of a. As shown before, $a \in \mathcal{L}_p$ implies $|a| \in G_{pAp}$. Now, if $(a^*a)^{-1}$ is the inverse of a^*a in pAp, then $P_{a(H)} = a(a^*a)^{-1}a^*$, since $a|a|^{-1}$ is a partial isometry with initial space p and final space $P_{a(H)}$. The result follows reasoning as in the previous theorem. □

Corollary 2.14 *Let A be a C^*-algebra, $p \in A$ a projection and $[a] \in \mathcal{P}(p)$.*

1. *The orbits of* $[a]$ *by the action of* $\mathcal{U}_A$ *and* G_A *coincide. That is,*

$$\mathcal{U}_{[a]} := \{[ua] : u \in \mathcal{U}_A\} = \{[ga] : g \in G_A\} := S_{[a]}.$$

2. *The connected component of* $[a]$ *in* $\mathcal{P}(p)$ *is contained in* $S_{[a]}$.

3. *If* G_A *(or equivalently,* $\mathcal{U}_A$*) is connected, then the connected component of* $[a]$ *in* $\mathcal{P}(p)$ *is exactly* $S_{[a]}$.

Proof. It is well known (see [16] or [10]) that 2 and 3 are true in $\mathcal{E}_p$. So they are also true in $\mathcal{P}(p)$ using the homeomorphism ϱ_p. We know that 1 is true for $a \sim_1 p$, by (2.9). For any other $[a] \in \mathcal{P}(p)$ denote by $q = \varrho_p([a])$. Then the result follows applying (2.9) to $\mathcal{P}(q)$. □

Remark 2.15 $\mathcal{P}(p)$ has C^∞ differentiable, homogeneous and (unitary) reductive structure induced by the homeomorphism with the space $\mathcal{E}_p$ which, as pointed out before, has rich geometric structure studied in [16] and [10]. Let us recall some facts:

1. The space of projections of A, with the norm topology is a discrete union of unitary orbits of projections. Each orbit is a C^∞ submanifold of A, and a C^∞ homogeneous space under the action of Lie-Banach group $\mathcal{U}_A$. The tangent space at a given projection p identifies with the 2×2 matrices (in terms of p) which are selfadjoint and have zeros in the diagonal.

2. The space of projections of A admits a natural reductive structure, which induces a linear connection. The invariants of this connection can be explicitely computed. It is torsion free and the curvature tensor is given by

$$R(x,y)z = \Big[\ [z,p],\ [x,y]\ \Big],$$

   where $[a,b] = ab - ba$ for $a, b \in A$.

3. The geodesic curves of this connection can be computed. The unique geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = x$ is given by

$$\gamma(t) = e^{t[x,p]}p\,e^{-t[x,p]}.$$

4. There is a natural invariant Finsler metric on the space of projections of A, namely the usual norm of A in every tangent space. This metric has remarkable properties which will be recalled later.

3 The chordal and spherical metrics on $\mathcal{P}(p)$.

In this section we introduce the two metrics on $\mathcal{P}(p)$ referred in the title. They are the operator theoretic analogues of the metrics considered in [19] for projective (finite dimensional) matrix spaces.

Definition 3.1 If $[a], [b] \in \mathcal{P}(p)$ for $a, b \in \mathcal{K}_p$, the chordal distance between $[a]$ and $[b]$ is

$$d_c([a], [b]) = \|\varrho_p([a]) - \varrho_p([b])\| = \|aa^* - bb^*\|.$$

Remark 3.2

1. This is the metric given by the natural (norm) metric of $\mathcal{E}_p$, and therefore induces the already considered quotient topology on $\mathcal{P}(p)$.
2. If two elements lie in different connected components of $\mathcal{P}(p)$, then their chordal distance is greater or equal than 1.
3. This metric is invariant under the action of $\mathcal{U}_A$.
4. If $a \in \mathcal{K}_p$, $u \in \mathcal{U}_A$ and $q = aa^*$, then

   $$d_c([ua], [a]) = \|uq - qu\| = \max\{\|qu(1-q)\|, \|(1-q)uq\|\} \leq 1.$$

   In particular, this shows that our definition agrees with the chordal distance considered in [19] for the case $A = M_n(\mathbb{C})$.

By means of the chordal metric we can compute length of curves in $\mathcal{P}(p)$. Given a curve $\gamma : [0,1] \to \mathcal{P}(p)$ consider the length $\ell(\gamma)$ as

$$\ell(\gamma) = \sup_{\pi} \sum_{i=1}^{n} d_c(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions π of $[0,1]$.

Definition 3.3 If $[a], [b] \in \mathcal{P}(p)$ lie in the same connected component, then

$$d_r([a], [b]) = \inf\{\ell(\gamma) : \quad \gamma : [0,1] \to \mathcal{P}(p) \text{ with } \gamma(0) = [a] \text{ and } \gamma(1) = [b]\},$$

where the infimum is taken over all rectifiable curves (i.e. $\ell(\gamma) < \infty$). Define $d_r([a], [b]) = \infty$ if $[a]$ and $[b]$ lie in different connected components. d_r is called the rectifiable metric.

Remark 3.4 The differentiable structure of $\mathcal{P}(p)$ allows one to compute d_r using C^1 curves. In this case

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt.$$

This fact means that d_r is the translation of the rectifiable (or geodesic) metric in $\mathcal{E}_p$ by means of the diffeomorphism ϱ_p. This metric has been extensively studied ([16], [10], [2], [15]). The following theorem collects some of its properties:

Theorem 3.5 *Let $p \in A$ a projection and $q \in \mathcal{U}(p)$. Then*

1. $\|p - q\| < 1$ *if and only if* $d_r(p, q) < \pi/2$.

   In this case:

2. *there exists a geodesic in $\mathcal{U}(p)$ joining p with q whose length is minimal and therefore equals $d_r(p,q)$.*

3. $d_r(p,q) = arcsin(\|p-q\|)$.

Proof. We use a result of [16] which says that if $\|p-q\| < 1$ then there exists a geodesic curve γ joining p and q with $\ell(\gamma) = d_r(p,q) \leq \pi/2$. Let x be the velocity vector of this geodesic. That is, $\gamma(t) = e^{tx}pe^{-tx}$. In matrix form (in terms of p):

$$x = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$$

and $\ell(\gamma) = \|x\| = \|a\| \leq \pi/2$.

On the other hand $\|p-q\| = \|e^x p - pe^x\|$. Easy calculations show that

$$e^x = \begin{pmatrix} \cos(|a|) & -a^* f(|a^*|) \\ af(|a|) & \cos(|a^*|) \end{pmatrix}$$

where $f(t) = \frac{\sin(t)}{t}$, defined for $t \geq 0$. It is easy to see that $\|af(|a|)\| = \|\sin(|a|)\|$ and the same for a^*. Since $\|a\| \leq \pi/2$, we can deduce that $\|\sin(|a|)\| = \sin(\|a\|)$. Therefore

$$\|p-q\| = \|pe^x - e^x p\| = \max\{\|af(|a|)\|, \|a^* f(|a^*|)\|\} = \sin(\|a\|).$$

This shows one implication in 1 via the formula 3. Now, if $d_r(p,q) < \pi/2$, the argument consists on taking a short curve ρ joining p and q and a partition such that each pair of contiguous projections have chordal distance less than one. By the previous result a polygonal γ of geodesics shorter than ρ can be constructed. Therefore the sum of its lengths $(= \ell(\gamma))$ is less than $\pi/2$. Finally we use the following result (Lemma 3 of [2]): Given three projections r, s, w and nonnegative numbers t_1, t_2 such that $t_1 + t_2 < \pi/2$, $\|r-s\| = \sin t_1$ and $\|s-w\| = \sin t_2$, then $\|r-w\| \leq \sin(t_1+t_2)$. In our case, it can be easily deduced that $\|p-q\| \leq \sin(\ell(\gamma)) < 1$. This concludes the proof. □

Remark 3.6 The contents of the theorem above are essentially known. Items 2 and a part of 3 of (3.5) were proved in [16]. Phillips [15] proved equality 3 and the existence of curves of minimal length. The proof of 3 presented here uses the original ideas of Porta and Recht [16]. It should be noted that in [16] the results are stated in terms of selfadjoint symmetries (i.e. elements $\varepsilon \in A$ with $\varepsilon^2 = 1$ and $\varepsilon^* = \varepsilon$). One passes from projections to symmetries by $p \to \varepsilon = 2p - 1$, and therefore the metric in the space of symmetries carries a factor 2 with respect to the metric in the space of projections.

In [16] it was shown that if $\|p-q\| < 1/2$ then p and q are joined by a **unique** geodesic of the linear connection. The following proposition says that this fact still holds if $\|p-q\| < 1$. Note that if $\|p-q\| < 1$ there are many curves joining p and q with minimal length. What the following statement says is that only one of them is a geodesic.

Proposition 3.7 *Let $D = \{z \in A : z^* = -z\,,\, pz = z(1-p)$ and $\|z\| < \pi/2\}$. Then*

$$\exp : D \to \{q \in \mathcal{E}_p : \|p - q\| < 1\}, \quad \exp(z) = e^z p e^{-z}$$

is a C^∞ diffeomorphism.

Proof. Let $x \in A$ with $x^* = -x$ and $px = x(1-p)$. As in the previous result

$$x = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$$

and

$$e^x = \begin{pmatrix} \cos(|a|) & -a^* f(|a^*|) \\ a f(|a|) & \cos(|a^*|) \end{pmatrix}$$

with $f(t)$ as in 3.5. Put $\varepsilon = 2p - 1$. Then condition $pz = z(1-p)$ becomes $\varepsilon z = -z\varepsilon$, and therefore $e^z \varepsilon = \varepsilon e^{-z}$.

Clearly 3.5 implies that the mapping exp is surjective between the domains considered. Let us prove that it is also one to one. Suppose that $z_1, z_2 \in D$ satisfy $e^{z_1}\varepsilon e^{-z_1} = e^{z_2}\varepsilon e^{-z_2}$. Then $e^{2z_1}\varepsilon = e^{2z_2}\varepsilon$, and since ε is invertible, this implies $e^{2z_1} = e^{2z_2}$. Both exponentials have matrix forms as above,

$$e^{2z_i} = \begin{pmatrix} \cos(|a_i|) & -a_i^* f(|a_i^*|) \\ a_i f(|a_i|) & \cos(|a_i^*|) \end{pmatrix}$$

with $\|a_i\| < \pi$, $i = 1, 2$. The function cos is a diffeomorphism on the set of positive elements of A with norm strictly less than π. Therefore $\cos(|a_1|) = \cos(|a_2|)$ implies $|a_1| = |a_2|$. Another functional calculus argument shows that both $f(|a_i|)$, $i = 1, 2$ are invertible elements of A, and therefore $a_1 = a_2$. Moreover, this same sort of argument shows that exp is a diffeomorphism, since its inverse can be explicitely computed. □

Remark 3.8 Equality 3 of (3.5) implies that the metric d_r on $\mathcal{P}(p)$ agrees with the spherical distance defined in [19] as the arcsin of the chordal distance, under the hypothesis that the chordal distance between the pair of points is less than one. Moreover, if the diameter of $\mathcal{U}(p)$ (using the metric d_r) is $\pi/2$, then the spherical distance equals d_r in each unitary orbit of $\mathcal{P}(p)$. Indeed, easy computations show that if $q, r \in \mathcal{U}(p)$ and $d_r(q, r) = \pi/2$ then $\|q - r\| = 1$. In [15] it is shown that a large class of C*-algebras satisfy this diameter condition.

4 Projectivities and finite points of $\mathcal{P}(p)$.

Let H be a Hilbert space, $A \subseteq \mathcal{B}(H)$ a C*-algebra and $p \in A$ a projection.

Definition 4.1 Let $g \in G_A$. We denote by $T_g : \mathcal{P}(p) \to \mathcal{P}(p)$ the map

$$T_g([a]) = [ga] \quad , \quad [a] \in \mathcal{P}(p).$$

It is clearly well defined and is a diffeomorphism. Following [19], we call these maps the **projectivities** of $\mathcal{P}(p)$.

4.2 If we identify $[u] \in \mathcal{P}(p)$ for $u \in \mathcal{K}_p$ with $q = uu^* \in \mathcal{E}_p$, we can describe the projection in $\mathcal{E}_p$ which corresponds to $T_g([u])$. Note that if $a \in \mathcal{L}_p$ and $u \in \mathcal{K}_p$ is the partial isometry appearing in the polar decomposition of a in A, then $[a] = [u] \in \mathcal{P}(p)$. Since $a(H) = u(H)$ by construction, we deduce that $q = uu^* = P_{a(H)}$. The same happens for $ga \in \mathcal{L}_p$. Therefore $T_g([a]) = [ga]$ can be identified with the projection $T_g(q) = P_{ga(H)}$. Note also that $gqg^{-1}(H) = ga(H)$. Therefore $T_g(q)$ is the projection onto the image of the idempotent gqg^{-1}. Therefore (see [10]),

$$\begin{aligned} T_g(q) = P_{gqg^{-1}(H)} &= gqg^{-1}(gqg^{-1})^*(1 + (gqg^{-1} - (gqg^{-1})^*)^2)^{-1} \\ &= gqg^{-1}(gqg^{-1} - (gqg^{-1})^*)^{-1}. \end{aligned}$$

Definition 4.3 We denote by $\mathcal{P}_f(p) = \{[a] \in \mathcal{P}(p) : pap \in G_{pAp}\}$. The elements of $\mathcal{P}_f(p)$ are called the "finite points" of $\mathcal{P}(p)$. In proposition 4.6 below we shall see that they are the analogue in this setting of the finite points in the usual projective spaces and in the matrix projective spaces of Schwartz and Zaks.

Lemma 4.4 $\mathcal{P}_f(p) \subseteq \mathcal{P}_0(p) = \{[up] : u \in \mathcal{U}_A\}$. *Moreover, if* $v \in \mathcal{K}_p$ *and* $[v] \in \mathcal{P}_f(p)$, *then* $v \in \mathcal{U}_A \cdot p$.

Proof. Let $v \in \mathcal{K}_p$ such that $pvp = pv \in G_{pAp}$, which means that $[v] \in \mathcal{P}_f(p)$. We must prove that $v \in \mathcal{U}_A\ p$. In matrix form, in terms of p, we can write $v = \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix}$. Then $v_1 \in G_{pAp}$. Let $x = v_2 v_1^{-1} \in (1-p)Ap$. Then

$$v \sim_1 \begin{pmatrix} p & 0 \\ x & 0 \end{pmatrix} = p + x$$

The curve $\gamma(t) = [p + tx]$ joins $[p]$ with $[v]$. Then $\varrho_p(v) = vv^* \in (\mathcal{E}_p)_p$, the connected component of p in $\mathcal{E}_p$. Since $(\mathcal{E}_p)_p \subseteq \mathcal{U}(p)$ by (2.14), there exists $u \in \mathcal{U}_A$ such that $vv^* = upu^*$, i.e. $[v] = [up]$. Let $w \in U(pAp)$ such that $v = upw$. Then

$$v = upw = u \begin{pmatrix} w & 0 \\ 0 & 1-p \end{pmatrix} p \in \mathcal{U}_A p.$$

□

Example 4.5 Suppose that $A = B^{2\times 2}$ for a C*-algebra B and $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We can embed B into $\mathcal{P}(p)$ via

$$B \ni b \mapsto \left[\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}\right] \in \mathcal{P}(p).$$

Then

$$[a] = \left[\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}\right] \in \mathcal{P}_f(p) \quad \text{iff} \quad a_{11} \in G_B.$$

It is also easy to see that the map $k : B \to \mathcal{P}_f(p)$ given by

$$k(b) = \left[\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}\right] \quad , \quad b \in B,$$

is a homeomorphism (see (4.6) below).

Let $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G_A$. Denote by

$$D(g) = \{ b \in B \ : \ T_g(k(b)) \in \mathcal{P}_f(p) \} \subseteq B. \tag{3}$$

Consider the map $M_g : D(g) \to B$ defined by

$$M_g(b) = k^{-1}(\ T_g\ (\ k(b)\ )\ ) \quad , \quad b \in D(g)$$

(the Möebius map on B defined by g). Easy calculations show that, for $b \in D(g)$,

$$M_g(b) = k^{-1}\left(\left[\begin{pmatrix} x+yb & 0 \\ z+wb & 0 \end{pmatrix} \right] \right) = (z+wb)(x+yb)^{-1},$$

a picture that justifies the name Möebius map. Note that

$$D(g) = \{ \ b \in B \ : \ x + yb \in G_B \ \}.$$

This set is not easy to characterize, and could well be empty. Another way to regard this domain is given in the following:

Proposition 4.6 *Let A be a C^*-algebra and $p \in A$ a projection. Then*

i) The linear manifold $H_p = (1-p)Ap$ is C^∞-diffeomorphic to $\mathcal{P}_f(p)$ via the map

$$k : H_p \to \mathcal{P}_f(p) \quad \textit{given by} \quad k(x) = \left[\begin{pmatrix} p & 0 \\ x & 0 \end{pmatrix} \right] = [p + x] \quad , \quad x \in H_p.$$

ii) The diffeomorphism $\varrho_p : \mathcal{P}(p) \to \mathcal{E}_p$ given by $\varrho_p([v]) = vv^*$ for $v \in \mathcal{K}_p$ maps finite points onto projections q such that $\|p - q\| < 1$. That is

$$\begin{aligned} \varrho_p(\mathcal{P}_f(p)) &= \{ \ q \in U(p) \ : \ \|p-q\| < 1 \ \} \\ &= \{ \ q \in \mathcal{E}_p \ : \ d_r(p,q) < \pi/2 \ \}. \end{aligned}$$

Proof. ii) Let $v \in \mathcal{K}_p$ such that $\|vv^* - p\| < 1$. If $v = \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix}$, we have that $\|vv^* - p\| < 1 \Rightarrow \|v_1 v_1^* - 1\|_{pAp} < 1 \Rightarrow v_1 v_1^* \in G_{pAp}$.

On the other hand, $\|vv^* - p\| < 1$ implies that $\|v_2^* v_2\| = \|v_2 v_2^*\| < 1$. Because $v \in \mathcal{K}_p$ we know that $v_2^* v_2 + v_1^* v_1 = p$. Hence $\|v_1^* v_1 - 1\|_{pAp} = \|v_2^* v_2\| < 1$ and also $v_1^* v_1 \in G_{pAp}$. Then $v_1 \in G_{pAp}$ and one inclusion is proved.

Conversely, suppose that $v \in \mathcal{K}_p$, $v = \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix}$ and $v_1 \in G_{pAp}$. Let $x = v_2 v_1^{-1}$ and $a = \begin{pmatrix} p & 0 \\ x & 0 \end{pmatrix} = p + x \sim_1 v$. Let $a = w|a|$ be the polar decomposition of a. Note that $v \sim_1 a \sim_1 w$, and

$$|a| = (a^* a)^{1/2} = \begin{pmatrix} (p + x^* x)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad w = \begin{pmatrix} (p + x^* x)^{-1/2} & 0 \\ z & 0 \end{pmatrix}.$$

By Lemma 4.4, we know that $w \in \mathcal{U}_A \cdot p$. Let $u \in \mathcal{U}_A$ such that $w = up$. Then

$$u = \begin{pmatrix} (p + x^*x)^{-1/2} & y \\ z & t \end{pmatrix}.$$

Now,

$$\|w^*w - p\| = \|upu^* - p\| = \|up - pu\| = \| \begin{pmatrix} 0 & -y \\ z & 0 \end{pmatrix} \|.$$

Since $u \in \mathcal{U}_A$, $(p + x^*x)^{-1} + yy^* = (p + x^*x)^{-1} + z^*z = p$. Then

$$\|yy^*\| = \|z^*z\| = \|p - (p + x^*x)^{-1}\|_{pAp}.$$

Claim: $\|p - (p + x^*x)^{-1}\|_{pAp} < 1$.
Indeed, $p + x^*x \geq p \Rightarrow \sigma_{pAp}((p + x^*x)^{-1}) \subseteq (0, 1]$. Then $\sigma_{pAp}((p + x^*x)^{-1} - 1) \subseteq (-1, 0] \Rightarrow \|(p + x^*x)^{-1} - 1\|_{pAp} < 1$, because it is selfadjoint. Therefore $\|vv^* - p\| = \|ww^* - p\| = \max(\|y\|, \|z\|) < 1$, and the proof of (ii) is finished. i) It is easy to see that the map k is bijective (note that $[p + x] = [p + y]$ in $\mathcal{P}(p) \Rightarrow x = y$). k is C^∞ because is the composition of the C^∞ maps $x \mapsto x + p$ and $x + p \mapsto [x + p]$. Moreover, if $V_p = \{q \in U(p) : \|p - q\| < 1\}$, there exists a C^∞ map (see [10])

$$s_p : V_p \to \mathcal{U}_A \quad \text{such that} \quad s_p(q)\; p\; s_p(q)^* = q \quad , \quad q \in V_p.$$

By ii) we know that $\varrho_p(\mathcal{P}_f(p)) = V_p$. Then the map

$$\begin{array}{ccccccc} & \varrho_p & & s_p & & & \\ ([p+x]) & \mapsto & q = \varrho_p([p+x]) & \mapsto & s_p(q) & \mapsto & (1-p)s_p(q)p\,[p\; s_p(q)\; p]^{-1} = x \\ & & \Cap & & \Cap & & \Cap \\ & & V_p & & \mathcal{U}_A & & H_p \end{array}$$

for $[p + x] \in \mathcal{P}_f(p)$, is the inverse of k and is C^∞. Note that $ps_p(q)p \in G_{pAp}$, since $[s_p(q)p] = [p + x] \in \mathcal{P}_f(p)$. □

Remark 4.7 Suppose that $A \subseteq \mathcal{B}(H)$ for a Hilbert space H. Via the identifications

$$k : H_p \to \mathcal{P}_f(p) \quad \text{and} \quad \varrho_p : \mathcal{P}_f(p) \to \{q \in \mathcal{E}_p : \|p - q\| < 1\},$$

of (4.6), we can deduce that, for $g \in G_A$, the domain of the Möebius map M_g induced by the projectivity T_g is

$$D(g) = \{q \in \mathcal{E}_p : \|p - q\| < 1 \;\text{ and }\; \|P_{g(p(H))} - p\| < 1\}.$$

4.8 In (4.1) we defined the projectivities T_g for $g \in G_A$. Denote by

$$\mathcal{T}(\mathcal{P}(p)) \;=\; \{\; T_g \;:\; g \in G_A \;\},$$

the group of all projectivities on $\mathcal{P}(p)$. In order to characterize the group $\mathcal{T}(\mathcal{P}(p))$, we have to describe the isotropy group

$$\mathcal{N}(\mathcal{P}(p)) \; = \; \{ \; g \in G_A \; : \; T_g = Id_{\mathcal{P}(p)} \; \}.$$

In [19] it is shown that, for $A = M_{2n}(\mathbb{C})$, $\mathcal{N}(\mathcal{P}(p)) = G_{\mathbb{C}.I}$, the invertible scalar matrices. For a general C*-algebra A, denote by

$$Z(A) \; = \; \{ \; a \in A \; : \; ab = ba \; \text{ for all } \; b \in A \; \}$$

the center of A.

Proposition 4.9 *Let A be a C*-algebra and $p \in A$ a projection. Then*

1. $\mathcal{N}(\mathcal{P}(p)) \; = \; \{ \; g \in G_A \; : \; g(q(H)) = q(H) \;$ *for all* $q \in \mathcal{E}_p \; \}$.
2. $G_{Z(A)} \subseteq \{ g \in G_A : gq = qg \quad$ *for all* $\quad q \in \mathcal{E}_p \; \} \subseteq \mathcal{N}(\mathcal{P}(p))$.
3. *If $A = \mathcal{B}(H)$ for a separable Hilbert space H, then $\mathcal{N}(\mathcal{P}(p)) = G_{\mathbb{C}.I}$.*
4. *If A is a von Neumann factor of type III (on a separable Hilbert space), then $\mathcal{N}(\mathcal{P}(p)) = G_{\mathbb{C}.I}$.*

Proof. Item 1 is apparent from the definitions. Since $gq(H) = gqg^{-1}(H)$ for all $g \in G_A$ and $q \in \mathcal{E}_p$, item 2 follows from item 1.

To prove item 3 consider first the case when $\dim p(H) = \infty$. In this case, for all $x \in H$ the subspace $< x >^{\perp}$ is the image of some $q \in \mathcal{E}_p(\mathcal{B}(H)) = \{$ projections $q \in \mathcal{B}(H) \; : \; \dim q(H) = \infty \}$, since two projections are equivalent iff their images have the same dimension in $\mathcal{B}(H)$. Therefore, if $g \in \mathcal{N}(\mathcal{P}(p))$ then x is an eigenvector for g^* for all $x \in H$. Hence $g \in \mathbb{C}I$. If $\dim p(H) = n < \infty$ then any $g \in \mathcal{N}(\mathcal{P}(p))$ should have all subspaces of dimension n as invariant spaces. It is easy to see that such operators must be scalar multiples of the identity.

In order to prove item 4 recall that all non zero projections a factor of type III are equivalent. If $q \in A$ is a projection such that $0 \neq q \neq 1$ then both $q \sim 1 - q \sim p$. Let $g \in \mathcal{N}(\mathcal{P}(p))$. Then by item 1 we have that

$$qgq = gq \quad \text{ and } \quad (1-q)g(1-q) = g(1-q).$$

Hence $gq = qg$ for all projections $q \in A$, and $g \in Z(A) = \mathbb{C}I$. □

Remark 4.10 We conjecture that for every C*-algebra A and $p \in A$,

$$\mathcal{N}(\mathcal{P}(p)) = \{ g \in G_A : gq = qg \quad \text{for all} \quad q \in \mathcal{E}_p \; \}$$

Under reasonable hypothesis, this implies that $\mathcal{N}(\mathcal{P}(p)) = G_{Z(A)}$. A slight improvement of the argument used to show (4.9), item 4 can be used to show that this conjecture is valid if A is a type III von Neumann algebra with separable predual and p has central carrier 1.

5 The holomorphic structure of $\mathcal{P}(p)$ and $\mathcal{E}_p$.

As a homogeneous space of the complex analytic Lie group G_A, $\mathcal{P}(p)$ inherits a natural complex structure. It will be shown that the projectivities T_g are biholomorphic. These facts can be shown in an explicit way, making use of the local charts $g \times \mathcal{P}_f(p)$, $g \in G_A$.

In [23] Wilkins introduced a complex structure for the grassmannians. Under the identification $\mathcal{E}_p \simeq \mathcal{P}(p)$, both structures coincide. Let us denote by $\varrho_q : \mathcal{P}(q) \to \mathcal{E}_q = \mathcal{E}_p$ the homeomorphisms (and isometries) of equation (2), for all $q \in \mathcal{E}_p$.

Let A be a C*-algebra and p, q two equivalent projections in A. We consider the isometry

$$\psi_{p,q} : \mathcal{P}(q) \to \mathcal{P}(p) \quad \text{given by} \quad \psi_{p,q} = \varrho_p^{-1} \circ \varrho_q. \tag{4}$$

Remark 5.1 By proposition 4.6, for each $q \in \mathcal{E}_p$ we have that,

$$\psi_{p,q}(\mathcal{P}_f(q)) = \{b \in \mathcal{P}(p) : d_c(b, \varrho_p^{-1}(q)) < 1\} := B_{\mathcal{P}(p)}(\varrho_p^{-1}(q), 1), \tag{5}$$

because both sets are mapped onto $\{r \in \mathcal{E}_p : \|r - q\| < 1\}$ by ϱ_q and ϱ_p, respectively. We denote by $k_q : H_q = (1-q)Aq \to \mathcal{P}_f(q)$ the homeomorphisms of proposition 4.6, for each $q \in \mathcal{E}_p$. We can define now the homeomorphisms

$$k'_q : H_q \to B_{\mathcal{P}(p)}(\varrho_p^{-1}(q), 1) \quad \text{given by} \quad k'_q = \psi_{p,q} \circ k_q. \tag{6}$$

The maps k'_q, for $q \in \mathcal{E}_p$ are almost the local charts for $\mathcal{P}(p)$. It just remains to uniformize the different Banach spaces $(1-q)Aq = H_q$, for different projections q. Note that the different connected components of $\mathcal{P}(p)$ lie at chordal distance greater than 1. Therefore in order to study the differential structure of $\mathcal{P}(p)$ we can work in each component. For simplicity, we shall define the complex structure only for the space $\mathcal{P}_0(p)$ which is the union of several connected components of $\mathcal{P}(p)$. Note that $\varrho_p(\mathcal{P}_0(p)) = \{upu^* : u \in \mathcal{U}_A\} = \mathcal{U}(p)$, the unitary orbit of p. If $q \in \mathcal{U}(p)$ and $w \in \mathcal{U}_A$ such that $wpw^* = q$, then

$$Ad_w(H_p) = wH_pw^* = w(1-p)Apw^* = (1-q)Aq = H_q, \tag{7}$$

where Ad_w is the inner automorphism of A defined by w.

Let $a \in \mathcal{P}_0(p)$, $q = \varrho_p(a) \in \mathcal{U}(p)$ and $w \in \mathcal{U}_A$ such that $wpw^* = q$. Using equations (5), (6) and (7) we define the homeomorphism

$$\phi_a : H_p \to B_{\mathcal{P}(p)}(a, 1) \quad \text{given by} \quad \phi_a = k'_q \circ Ad_w = \psi_{p,q} \circ k_q \circ Ad_w. \tag{8}$$

Theorem 5.2 *The family of local charts $(\phi_a)_{a \in \mathcal{P}_0(p)}$ (choosing one appropiate w for each a) defines a complex holomorphic structure for the space $\mathcal{P}_0(p)$.*

Proof. We already know that all maps $\phi_a : H_p \to B_{\mathcal{P}(p)}(a, 1)$ are homeomorphisms. So it remains to check taht these maps are compatibible with the analytic structure of H_p. In other words, if $q, r \in \mathcal{U}(p)$ and there exist $s \in \mathcal{U}(p)$ such that $\|q - s\| < 1$ and $\|r - s\| < 1$, we

must show that $(k'_q)^{-1} \circ k'_r$ is analytic. This will suffice because the maps Ad_w are analytic for all $w \in \mathcal{U}_A$.
Case 1: Suppose that $\|q - r\| < 1$. In this case, easy computations show the following formula: if $x \in H_r$ and $k'_r(x) \in B_{\mathcal{P}(p)}(\varrho_p^{-1}(q), 1) = k'_q(H_q)$, then

$$(k'_q)^{-1} \circ k'_r(x) = (1-q)(r+x)q.(q(r+x)q)^{-1}, \tag{9}$$

where the inverse of $q(r+x)q$ is taken in qAq. It is clear that the formula (9) defines an analytic map of the variable x.
Case 2: Suppose that $q, r \in \mathcal{U}(p)$ and there exist $s \in \mathcal{U}(p)$ such that $\|q - s\| < 1$ and $\|r - s\| < 1$. Then, in the adequate domain,

$$(k'_q)^{-1} \circ k'_r = [(k'_q)^{-1} \circ k'_s] \circ [(k'_s)^{-1} \circ k'_r].$$

Since both maps on the right hand side are analytic by Case 1, the proof is complete. □

Remark 5.3 The analytic structure can be extended to the whole $\mathcal{P}(p)$ since, modulo the maps $\psi_{p,q}$, each connected component of $\mathcal{P}(p)$ is included in $\mathcal{P}_0(q)$ for some $q \in \mathcal{E}_p$. Then the analytic manifold structure can be defined around q in the same way as in Theorem 5.2.

Remark 5.4 The following properties of $\mathcal{P}(p)$ are now easy to see:

1. Each projectivity T_g, for $g \in G_A$, is biholomorphic.

2. The action of G_A over $\mathcal{P}_0(p)$ given by the map $\pi_p : G_A \to \mathcal{P}_0(p)$ defined by $\pi_p(g) = T_g([p])$, $g \in G_A$, defines an analytic homogeneous space. The structure group is the isotropy group

$$I_p = \{g \in G_A : T_g([p]) = [p]\} = \{g \in G_A : (1-p)gp = 0 \text{ and } pgp \in G_{pAp}\},$$

which is an union of connected components of the group of invertible elements of the subalgebra

$$T_p(A) = \{a \in A : (1-p)ap = 0\} \subseteq A$$

of p-upper triangular elements of A. This algebra is the tangent space at the identity of the group I_p. It is also the kernel of the differential $T(\pi_p)_1$ of π_p at 1, since $T(\pi_p)_1(a) = (1-p)ap$, for all $a \in A$.

3. The homogeneous space given by $\pi_p : G_A \to \mathcal{P}_0(p)$ admits a reductive structure given by the horizontal space $H_p = (1-p)Ap$ which can be tranported homogeneously to all elements of G_A. Note that this horizontal space is precisely the domain of our local charts and can also be naturally identified with the tangent space $T(\mathcal{P}(p))_{[p]}$ of $\mathcal{P}(p)$ at $[p]$.

Remark 5.5 A complex structure can be defined in the Grassmannian $\mathcal{U}(p)$ via the map ϱ_p, i.e. pulling back the complex structure of $\mathcal{P}_0(p)$. This structure is compatible with the real structure, since ϱ_p is a C^∞ diffeomorphism by remark 2.15. It also allows us to define the analytic homogeneous reductive structure of $\mathcal{U}(p)$ given by the new action of G_A over $\mathcal{U}(p)$:

$$\pi_p : G_A \to \mathcal{U}(p) \quad \text{given by} \quad \pi_p(g) = P_{g(Im(p))}.$$

Note that this action was described in (4.2) as $\pi_p(g) = T_g(p)$. A remarkable fact is that the formula for $T_g(p)$ given in (4.2) becomes analytic in the variable p, although the involution is involved in its description. It can also be remarked that this complex structure agrees with the complex structure defined in the Grassmannians by Wilkins in [23].

6 The non-Euclidean metrics on $\mathcal{P}(p)$.

Suppose that A is represented on a Hilbert space H. The projection p induces a Krein structure on H, by means of the selfadjoint symmetry $\varepsilon = 2p-1$. The set $\mathcal{U}_\varepsilon(A)$ of operators of A which are unitaries for this form is called the group of ε-unitaries. In this section we study a subset $\Delta^+(p) \subset \mathcal{P}(p)$, defined in (12), which is homogeneous under the action of $\mathcal{U}_\varepsilon(A)$. Moreover, it is shown that $\Delta^+(p)$ can be regarded as a copy of $\mathcal{U}_\varepsilon(A)^+ = \mathcal{U}_\varepsilon(A) \cap A^+$ inside $\mathcal{P}(p)$, where A^+ denotes the space of positive invertible elements of A. $\Delta^+(p)$ can be identified also with the space $N(A,p)$ of "normal" idempotents over the projection p, following the theory of [9] and [10], pp 60.

The space $\mathcal{U}_\varepsilon(A)^+$ is a totally geodesic submanifold of A^+, which is a hyperbolic space, that is, a (non riemannian) manifold of non positive curvature (in the sense of Gromov [12]). Therefore $\mathcal{U}_\varepsilon(A)^+$ is a hyperbolic space in itself. In particular, it has a rectifiable metric whose short curves can be explicitely computed. This metric can be translated to $\Delta^+(p)$. This translation, which has a natural intrinsic definition in terms of the already considered metrics of $\mathcal{P}(p)$, will be called the *non euclidean* metric E_n. Summarizing, $\Delta^+(p)$ will be shown to be a hyperbolic space inside $\mathcal{P}(p)$, with an isometric action of $\mathcal{U}_\varepsilon(A)$.

Definition 6.1

1. Let $p \in A \subseteq \mathcal{B}(H)$ a projection. Consider the symmetry

$$\varepsilon = 2p - 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Denote by $\mathcal{U}_\varepsilon(A)$ the space of ε-unitary elements of A, i.e. those $u \in A$ such that $< \varepsilon u(\xi), u(\eta) >=< \varepsilon(\xi), \eta >$, for all $\xi, \eta \in H$. Easy computations show that

$$\mathcal{U}_\varepsilon(A) = \{u \in A : u^*\varepsilon u = \varepsilon\} = \{u \in A : \varepsilon u^* \varepsilon = u^{-1}\}. \tag{10}$$

3. Denote by

$$\mathcal{U}_\varepsilon(A)^+ = \mathcal{U}_\varepsilon(A) \cap A^+, \tag{11}$$

the set of positive ε-unitary elements of A.

In the following remark we state several well known properties of the sets $\mathcal{U}_\varepsilon(A)$ and $\mathcal{U}_\varepsilon(A)^+$ (see [9] and [10]) :

Remark 6.2

1. $\mathcal{U}_\varepsilon(A)$ is a closed subgroup of G_A. Actually it is a real Banach-Lie group.
2. If $u \in \mathcal{U}_\varepsilon(A)$, then u^*, u^*u and $u^{-1} \in \mathcal{U}_\varepsilon(A)$.
3. $\lambda \in \mathcal{U}_\varepsilon(A)^+$ if and only if $\lambda\varepsilon = \varepsilon\lambda^{-1}$.
4. For all $\lambda \in \mathcal{U}_\varepsilon(A)^+$, $X = \log\lambda \in A$, verifies that $X = X^*$ and $\varepsilon X = -X\varepsilon$.
5. In matrix form, we have that $\lambda \in \mathcal{U}_\varepsilon(A)^+$ if and only if there exists $x \in H_p$ such that
$$\lambda = e^X \qquad \text{where} \qquad X = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}.$$
In this case, x is **unique** .
6. Using item 5, one deduces that
$$\lambda \in \mathcal{U}_\varepsilon(A)^+ \Rightarrow \lambda^t \in \mathcal{U}_\varepsilon(A)^+ \quad \text{for all} \quad t \in \mathbb{R} .$$
7. In particular, if $u \in \mathcal{U}_\varepsilon(A)$, then $|u| = (u^*u)^{1/2} \in \mathcal{U}_\varepsilon(A)$. In other words, the unitary and positive parts of each $u \in \mathcal{U}_\varepsilon(A)$ in its polar decomposition remain in $\mathcal{U}_\varepsilon(A)$. Note also that $\mathcal{U}_A \cap \mathcal{U}_\varepsilon(A) = \{u \in \mathcal{U}_A : u\varepsilon = \varepsilon u\} = \{u \in \mathcal{U}_A : up = pu\}$.
8. The metric of A^+ and its geodesics (see [8], [9] and [11]) can be restricted to $\mathcal{U}_\varepsilon(A)^+$. Indeed, if λ, $\mu \in A^+$, the unique geodesic of A^+ joining them is given by
$$\gamma_{\lambda\mu}(t) = \mu^{1/2}(\mu^{-1/2}\lambda\mu^{-1/2})^t\mu^{1/2}, \quad t \in [0,1].$$
Using items 1 and 6 one shows that if λ, $\mu \in \mathcal{U}_\varepsilon(A)^+$ then $\gamma_{\lambda\mu}(t) \in \mathcal{U}_\varepsilon(A)^+$.
9. The Finsler structure of A^+ (see [9]) induces a rectifiable metric on A^+ given by
$$d_+(\lambda,\mu) = \|\log(\mu^{-1/2}\lambda\mu^{-1/2})\| \quad \text{for} \quad \lambda,\ \mu \in A^+.$$
For this metric the geodesics $\gamma_{\lambda\mu}$ are of minimal length. Restricted to $\mathcal{U}_\varepsilon(A)^+$, this metric is also rectifiable by item 8, because the geodesic curves remain in $\mathcal{U}_\varepsilon(A)^+$ and are of course of minimal length.

Remark 6.3 Using item 5 of (6.2), easy computations, very similar to those of Theorem 3.5, show that for each $\lambda \in \mathcal{U}_\varepsilon(A)^+$ there exists a unique $x \in H_p$ such that
$$\lambda = \begin{pmatrix} \cosh(|x|) & x^*(\frac{\sinh t}{t})(|x^*|) \\ x(\frac{\sinh t}{t})(|x|) & \cosh(|x^*|) \end{pmatrix}.$$
In particular, using item 7 of (6.2), this implies that for all $u \in \mathcal{U}_\varepsilon(A)$,
$$\|(1-p)up\| = \|(1-p)u^{-1}p\| = \|(1-p)u^*p\| = \|\sin(|x|)\|,$$
where $x \in H_p$ verifies that $|u| = e^{x+x^*}$

6.4 Return now to the projective space $\mathcal{P}(p)$. Given $u \in A$, easy matrix computations using (10) show that if $u = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $u \in \mathcal{U}_\varepsilon(A)$ if and only if u is invertible and the following three conditions hold:

i) $a^*a - b^*b = p$,

ii) $d^*d - c^*c = 1 - p$ and

iii) $a^*c - b^*d = 0$.

Consider the set

$$\Delta^+(p) = \{[up] \ : \ u \in \mathcal{U}_\varepsilon(A)\} \subseteq \mathcal{P}(p). \tag{12}$$

Denote by

$$K_p'(A) = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in Ap : a \in G_{pAp} \ \text{ and } \ a^*a - b^*b = p\} \subseteq \mathcal{L}_p. \tag{13}$$

Proposition 6.5

$$\begin{aligned} \Delta^+(p) &= \{[u] : u \in K_p'(A)\} \\ &= \{[\lambda p] : \lambda \in \mathcal{U}_\varepsilon(A)^+\} \\ &= \{\left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right] : a \in G_{pAp} \ \textit{ and } \ a^*a - b^*b > 0 \ \textit{ in } \ pAp\} \subseteq \mathcal{P}_f(p), \end{aligned}$$

and the map $Y : \mathcal{U}_\varepsilon(A)^+ \to \Delta^+(p)$ *given by* $Y(\lambda) = [\lambda^{1/2}p]$ *is a homeomorphism.*

Proof. We have the following trivial inclusions:

$$\begin{aligned} \{[\lambda p] : \lambda \in \mathcal{U}_\varepsilon(A)^+\} &\subseteq \Delta^+(p) \\ &\subseteq \{[u] : u \in K_p'(A)\} \\ &\subseteq \left\{\left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right] : a \in G_{pAp} \ \text{ and } a^*a - b^*b > 0 \ \text{ in } \ pAp\right\}. \end{aligned}$$

So we have to show that $\{\left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right] : a \in G_{pAp} \ \text{ and } \ a^*a - b^*b > 0 \ \text{ in } \ pAp\} \subseteq \{[\lambda p] : \lambda \in \mathcal{U}_\varepsilon(A)^+\}$. Let $v = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ such that $x \in G_{pAp}$ and $d = x^*x - y^*y > 0$. Then $w = vd^{-1/2} \in K_p'(A)$, $[v] = [w]$ and $w = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ with $a^*a - b^*b = p$. Since $a \in G_{pAp}$, by taking the (right) polar decomposition of a in pAp we can also suppose that $a > 0$, that is $a = (p + b^*b)^{1/2}$. Consider

$$\lambda = \begin{pmatrix} a & b^* \\ b & (1-p+bb^*)^{1/2} \end{pmatrix} > \begin{pmatrix} |b| & b^* \\ b & |b^*| \end{pmatrix} \geq 0, \tag{14}$$

because $a = (p+b^*b)^{1/2} > (b^*b)^{1/2} = |b|$ in pAp and $(1-p+bb^*)^{1/2} > |b^*|$ in $(1-p)A(1-p)$. Then $\lambda \in A^+$. It is easy to see that λ verifies the three conditions of (6.4). So $\lambda \in \mathcal{U}_\varepsilon(A)^+$ and $[\lambda p] = [v]$.

The map $Y_0 : \mathcal{U}_\varepsilon(A)^+ \to \Delta^+(p)$ given by $Y_0(\lambda) = [\lambda p]$ is therefore continuous and surjective. To see that Y_0 is injective, suppose $\lambda, \mu \in \mathcal{U}_\varepsilon(A)^+$ with $[\lambda p] = [\mu p]$. Put $\lambda p = a + b$ and $\mu p = c + d$ with $a, c \in pAp^+$ and $b, d \in H_p$. Then

$$ba^{-1} = b(p + b^*b)^{-1/2} = d(p + d^*d)^{-1/2} = dc^{-1}.$$

Taking their polar decompositions in $\mathcal{B}(H)$, both elements have the same partial isometry, say u, and therefore

$$ba^{-1} = u|b|(p + |b|^2)^{-1/2} = u|d|(p + |d|^2)^{-1/2} = dc^{-1},$$

proving that u is also the partial isometry for b and d in their polar decompositions. This implies that $|b| = |d|$ since that map $f(t) = \frac{t}{(1+t^2)^{1/2}}$ has inverse $g(s) = \frac{s}{(1-s^2)^{1/2}}$. Then $b = d$ and $\lambda = \mu$ by (6.4) and equation (14). Note that we have already constructed the inverse of Y_0 by passing through H_p:

$$Y_0^{-1}\left(\left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right]\right) = \begin{pmatrix} (p + d^*d)^{1/2} & d^* \\ d & (1 - p + dd^*)^{1/2} \end{pmatrix} \in \mathcal{U}_\varepsilon(A)^+,$$

where $d = ba^{-1}(1 - |ba^{-1}|^2)^{-1/2}$. Clearly this map is also continuous. Finally, note that the map Y is the composition of the homeomorphism of $\mathcal{U}_\varepsilon(A)^+$ which consists of taking square roots, with the homeomorphism Y_0. Then the proof is complete. □

Remark 6.6

1. From the proof of (6.5), it follows quite easily that

$$K_p'(A) = \mathcal{U}_\varepsilon(A) \cdot p.$$

2. We have shown some characterizations of $\Delta^+(p)$ in terms of its representatives in A. Now we give other characterizations of $\Delta^+(p)$ in terms of the three natural metrics on $\mathcal{P}_f(p)$: the chordal and spherical metrics and the new metric d_k on $\mathcal{P}_f(p)$ given by the map $k_p^{-1} : \mathcal{P}_f(p) \to H_p$ of (4.6):

$$d_k(l, m) = \|k_p^{-1}(l) - k_p^{-1}(m)\| \quad \text{for} \quad l,\ m \in \mathcal{P}_f(p).$$

Corollary 6.7

$$\begin{aligned}\Delta^+(p) &= \{m \in \mathcal{P}_f(p) : d_k(m,[p]) = \|k_p^{-1}(m)\| < 1\} \\ &= \{m \in \mathcal{P}_f(p) : d_c(m,[p]) = \|\varrho_p(m) - p\| < \tfrac{\sqrt{2}}{2}\} \\ &= \{m \in \mathcal{P}_f(p) : d_r(m,[p]) < \pi/4\}.\end{aligned}$$

Proof. Let $m \in \Delta^+(p)$ and $\lambda \in \mathcal{U}_\varepsilon(A)^+$ such that $[\lambda p] = m$. If $\lambda p = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ then

$$\|k_p^{-1}(m) - k_p^{-1}([p])\| = \|ba^{-1}\| = \|b(p + b^*b)^{-1/2}\| < 1.$$

On the other hand, if $x \in H_p$ and $\|x\| < 1$ then $k_p(x) = [p + x]$ and $p - x^*x > 0$ in pAp. Then $k_p(x) \in \Delta^+(p)$.
In order to prove the other equalities, recall from the proof of (3.5), that if $m \in \mathcal{P}_f(p)$, then there exist $y \in H_p$ such that $d_r(m,[p]) = \|y\|$, and

$$m = \left[\begin{pmatrix} \cos(|y|) & 0 \\ y(\frac{\sin t}{t})(|y|) & 0 \end{pmatrix}\right] \quad \Rightarrow \quad k_p^{-1}(m) = y(\frac{\tan t}{t})(|y|).$$

Hence

$$d_k(m,[p]) = \|k_p^{-1}(m)\| = \|y(\frac{\tan t}{t})(|y|)\| = \|\tan(|y|)\| = \tan(\|y\|), \tag{15}$$

because the map $f(t) = \tan(t)$ is monotone increasing. Therefore $\|k_p^{-1}(m)\| < 1$ if and only if $d_r(m,[p]) = \|y\| = \arctan(d_k(m,[p]) < \pi/4$. Now the remainding equality becomes apparent using that, by (3.5), $d_c(m,[p]) = \sin(d_r(m,[p]))$. □

Remark 6.8 From (3.5) and the proof of (6.7), we can deduce the following facts: for all $m, n \in \mathcal{P}_f(p)$,

1. $d_k(m,[p]) = \tan(d_r(m,[p]))$.
2. $d_c(m,n) = \sin(d_r(m,n))$.

Note that the three metrics have clear geometrical senses: the chordal metric is the one associated to the norm in $\mathcal{U}(p)$ via the identification (which is in fact a bi-analytic map) ϱ_p. The metric d_r is the rectifiable metric generated by d_c taking the infima of the lengths of curves (and having the geodesics of the linear connection as minimal curves). On the other hand d_k is the metric induced on $\mathcal{P}_f(p)$ by the atlas of local charts of its complex manifold structure. Note also that they are related by the previous formulae and depend on the norm of some particular vectors in H_p, which is the tangent space at p of $\mathcal{P}_f(p)$. On $\Delta^+(p)$ we have a fourth metric, induced by the metric d_+ (see (7) of (6.2)) of $\mathcal{U}_\varepsilon(A)^+$ via the map $Y : \mathcal{U}_\varepsilon(A)^+ \to \mathcal{P}_f(p)$ of (6.5) (which is a C^∞ diffeomorphism). Let $m, n \in \Delta^+(p)$ and $\mu, \nu \in \mathcal{U}_\varepsilon(A)^+$ such that $m = [\mu^{1/2}p] = Y(\mu)$ and $n = [\nu^{1/2}p] = Y(\nu)$. Then

$$d_+(m,n) = d_+(\mu,\nu) = \|\log(\nu^{-1/2}\mu\nu^{-1/2})\|.$$

We define now the non-Euclidean metrics on $\Delta^+(p)$ following [19]:

6.9 Let $m, n \in \Delta^+(p)$ and $u, v \in \mathcal{U}_\varepsilon(A)$ such that $m = [up]$ and $n = [vp]$. We consider the following three functions:

1. $\rho(m,n) = \|(1-p)u^*\varepsilon vp\|$. This function is clearly well defined but is not a metric.

2. The "pseudo-chordal" metric:

$$d_{pc}(m,n) = \frac{\rho(m,n)}{(1+\rho(m,n)^2)^{1/2}}.$$

3. The non-Euyclidean metric:

$$E_n(m,n) = \frac{1}{2}\ \log\ \frac{1+d_{pc}(m,n)}{1-d_{pc}(m,n)}.$$

Remark 6.10 In order to relate the metrics just defined, we recall from item 5 of (6.2) the action of the group $\mathcal{U}_\varepsilon(A)$ over $\mathcal{U}_\varepsilon(A)^+$ given by $u \times \lambda = u\lambda u^*$, for $u \in \mathcal{U}_\varepsilon(A)$ and $\lambda \in \mathcal{U}_\varepsilon(A)^+$. By item 8 of (6.2), this action is isometric with respect to the Finsler metric of $\mathcal{U}_\varepsilon(A)^+$, since the same action is isometric at A^+ (see [9]). Therefore this action is also isometric for the geodesic metric d_+ defined in item 9 of (6.2). This fact yields

$$d_+(\mu,\nu) = d_+(\nu^{-1/2}\mu\nu^{-1/2}, 1) = \|\log(\nu^{-1/2}\mu\nu^{-1/2})\|.$$

We consider also the action of $\mathcal{U}_\varepsilon(A)$ over $\Delta^+(p)$ induced via the map Y of (6.5). More explicitely, for $u \in \mathcal{U}_\varepsilon(A)$ and $\lambda \in \mathcal{U}_\varepsilon(A)^+$,

$$u \times [\lambda^{1/2}p] = u \times Y(\lambda) := Y(u \times \lambda) = [(u\lambda u^*)^{1/2}p].$$

Proposition 6.11

1. *The pseudo-chordal and non-Euclidean metrics are symmetric and invariant under the action of $\mathcal{U}_\varepsilon(A)$ on $\Delta^+(p)$ defined in (6.10).*

2. *For $m, n \in \Delta^+(p)$, let $Y^{-1}(m) = \mu \in \mathcal{U}_\varepsilon(A)^+$ and $Y^{-1}(n) = \nu \in \mathcal{U}_\varepsilon(A)^+$. Then*

$$d_{pc}(m,n) = \|k_p^{-1}([\nu^{-1/2}\mu^{1/2}p])\| = \|k_p^{-1}([\nu^{1/2}\mu^{-1/2}p])\| \quad \textit{and, in particular}$$

$$d_{pc}(m,[p]) = d_k(m,[p]).$$

Proof. 1) First note that, using (6.3),

$$\rho(m,n) = \|(1-p)\mu\varepsilon\nu p\| = \|(1-p)\mu\nu^{-1}p\| = \|(1-p)\nu\mu^{-1}p\| = \rho(n,m),$$

and ρ is symmetric. On the other hand, if $u \in \mathcal{U}_\varepsilon(A)$, its left polar decomposition is given by $u = (uu^*)^{1/2}w$, where $w \in \mathcal{U}_A \cap \mathcal{U}_\varepsilon(A)$ commutes with p by item 7 of (6.2). Hence, for all $u \in \mathcal{U}_\varepsilon(A)$ we have that

$$[up] = [(uu^*)^{1/2}wp] = [(uu^*)^{1/2}p(pwp)] = [(uu^*)^{1/2}p] = [|u^*|p]. \tag{16}$$

Now we can describe more clearly the action of $\mathcal{U}_\varepsilon(A)$ over $\Delta^+(p)$ of (6.10): let $u \in \mathcal{U}_\varepsilon(A)$ and $\lambda \in \mathcal{U}_\varepsilon(A)^+$, then, by (6.12),

$$u \times [\lambda^{1/2}p] = [(u\lambda u^*)^{1/2}p] = [u\lambda^{1/2}p].$$

Then, for $u \in \mathcal{U}_\varepsilon(A)$,

$$\begin{aligned}
\rho(u \times [\mu^{1/2}p], u \times [\nu^{1/2}p]) &= \rho([u\mu^{1/2}p], [u\nu^{1/2}p]) \\
&= \|(1-p)\mu^{1/2}u^*\varepsilon u\nu^{1/2}p\| \\
&= \|(1-p)\mu^{1/2}\varepsilon\nu^{1/2}p\| \\
&= \rho([\mu^{1/2}p], [\nu^{1/2}p]).
\end{aligned}$$

Therefore ρ is symmetric and invariant under the action of $\mathcal{U}_\varepsilon(A)$ on $\Delta^+(p)$. It is clear that the same happens for d_{pc} and E_n, since they are defined in terms of ρ.
2) Using 1) we have that

$$\begin{aligned}
d_{pc}(m,n) &= d_{pc}([\mu^{1/2}p], [\nu^{1/2}p]) \\
&= d_{pc}([\nu^{-1/2}\mu^{1/2}p], [p]) \\
&= d_{pc}([(\nu^{-1/2}\mu\nu^{-1/2})^{1/2}p], [p]).
\end{aligned}$$

Then, using again (6.10), we can suppose that $n = [p]$, since the action of $\nu^{-1/2}$ transforms $n = [\nu^{1/2}p]$ to $[p]$ in $\Delta^+(p)$ and ν to 1 in $\mathcal{U}_\varepsilon(A)^+$. Now, let $\mu^{1/2}p = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. Then $x = k_p^{-1}(m) = ba^{-1} = b(p + b^*b)^{-1/2}$ and

$$d_k(m,[p])^2 = \|x^*x\| = \|b^*b(p+b^*b)^{-1}\| = \frac{\|b\|^2}{(1+\|b\|^2)}.$$

Note that $\|b\| = \|(1-p)\mu^{1/2}p\| = \rho([\mu^{1/2}p],[p]) = \rho(m,[p])$. Therefore $d_{pc}(m,[p]) = d_k(m,[p])$. □

Theorem 6.12

1. *If $m, n \in \Delta^+(p)$, let $Y^{-1}(m) = \mu \in \mathcal{U}_\varepsilon(A)^+$ and $Y^{-1}(n) = \nu \in \mathcal{U}_\varepsilon(A)^+$. Then*

$$2\ E_n(m,n) = d_+(m,n) = d_+(\mu,\nu) = \|\log(\nu^{-1/2}\mu\nu^{-1/2})\|,$$

where d_k is the metric on $\mathcal{P}_f(p)$ defined in (6.6) and d_+ is the metric on $\Delta^+(p)$ defined in (6.8).

2. *The map $Y : \mathcal{U}_\varepsilon(A)^+ \to \Delta^+(p)$ allows the translation of the C^∞ homogeneous reductive structure and Finsler metric of $\mathcal{U}_\varepsilon(A)^+$ to $\Delta^+(p)$. In this sense, the geodesics $\gamma_{\mu,\nu}$ defined in item 8 of (6.2) yield minimal length geodesics in $\Delta^+(p)$ via Y:*

$$E_n(m,n) = \frac{1}{2}\, d_+(\mu,\nu) = \frac{1}{2}\, \ell_{\mathcal{U}_\varepsilon(A)^+}(\gamma_{\mu,\nu}) = \ell_{\Delta^+(p)}(\gamma_{m,n}),$$

where

$$\begin{aligned} \gamma_{m,n}(t) &= Y \circ \gamma_{\mu,\nu}(t) \\ &= Y(\mu^{1/2}(\mu^{-1/2}\nu\mu^{-1/2})^t\mu^{1/2}) \\ &= [\mu^{1/2}(\mu^{-1/2}\nu^{1/2}\mu^{-1/2})^{t/2}p]\ , \quad t \in \mathrm{R}\,. \end{aligned}$$

Proof. Using (6.3), there exists $z \in H_p$ such that, if $Z = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$, then

$$\mu^{1/2} = e^Z = \begin{pmatrix} \cosh(|z|) & z^*(\frac{\sinh t}{t})(|z^*|) \\ z(\frac{\sinh t}{t})(|z|) & \cosh(|z^*|) \end{pmatrix}$$

and, by (6.2),

$$d_+(m,[p]) = d_+(\mu,1) = \|\log(\mu)\| = 2\|\log(\mu^{1/2})\| = 2\|\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}\| = 2\|z\|.$$

Easy computations similar as those of (15), show that

$$d_{pc}(m,[p]) = d_k(m,[p]) = \|z(\frac{\tanh(t)}{t})(|z|)\| = \|\tanh(|z|)\| = \tanh(\|z\|).$$

Therefore $d_+(m,[p]) = 2\ \mathrm{argtanh}(d_{pc}(m,[p]))$. Elementary computations show that, for all $t \in (-1,1)$, $\mathrm{argtanh}(t) = \frac{1}{2}\log\frac{1+t^2}{1-t^2}$. Therefore we have proved that the metrics $2E_n$ and d_+ coincide.
2) It follows because the geodesics $\gamma_{\mu,\nu}$ are minimal for d_+ in $\mathcal{U}_\varepsilon(A)^+$ and we have translated the metric and Finsler structure from $\mathcal{U}_\varepsilon(A)^+$ to $\Delta^+(p)$ via Y. □

References

[1] E. Andruchow, G. Corach and D. Stojanoff, Geometry of the sphere of a Hilbert module, Math. Proc. Cambridge Philos. Soc. (to appear).

[2] L. G. Brown, The rectifiable metric on the set of closed subspaces of Hilbert space, Trans. Amer. Math. Soc. 337, (1993), 279-289.

[3] L. G. Brown and G. K. Pedersen, On the geometry of the unit ball of a C^*-algebra, J. Reine Angew. Math. 469, 113-147 (1995).

[4] L. G. Brown and G. K. Pedersen, Approximation and convex decomposition by extremals in a C^*-algebra, Math. Scand. 81, No.1, 69-85 (1997)

[5] G. Corach, H. Porta and L. Recht, Differential geometry of spaces of relatively regular operators, Integral Equations and Oper. Th. 13 (1990), 771-794.

[6] G. Corach, H. Porta and L. Recht, Differential geometry of systems of projections in Banach algebras, Pacific J. Math. 141, (1990), 209-228.

[7] G. Corach, H. Porta and L. Recht, Splitting of the positive set of a C^*-algebra, Indag. Mathem. N.S. 2(4) (1991), 461-468.

[8] G. Corach, H. Porta and L. Recht, A geometric interpretation of the inequality $\|e^{X+Y}\| \leq \|e^{X/2}e^{Y}e^{X/2}\|$, Proc. Amer. Math. Soc. 115 (1992), 229-231.

[9] G. Corach, H. Porta and L. Recht, The geometry of spaces of selfadjoint invertible elements of a C^*-algebra, Integral Equations and Oper. Th. 16 (1993), 333-359.

[10] G. Corach, H. Porta and L. Recht, The geometry of the spaces of projections in C^*-algebras, Adv. Math. 101 (1993), 59-77.

[11] G. Corach, H. Porta and L. Recht, Geodesics and operator means in the space of positive operators, International Journal of Math. 4 (1993), 193-202.

[12] M. Gromov, Structures métriques pour les variétés riemannienes, CEDIC/Fernand Nathan, Paris, 1981.

[13] M. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Publ., New York-London-Sydney, 1969.

[14] G. K. Pedersen, Factorization in C^*-algebras, Expo. Math. 16, No.2, 145-156 (1998).

[15] N. C. Phillips, The rectifiable metric on the space of projections in a C*-algebra, Intern. J. Math. 3, (1992), 679-698.

[16] H. Porta and L. Recht, Minimality of geodesics in Grassman manifolds, Proc. Amer. Math. Soc. 100, (1987), 464-466.

[17] F. Riesz and Sz.-Nagy, Functional Analysis, Frederick Ungar Publishing Co., New York, 1966.

[18] B. Schwarz and A. Zaks, Matrix Möebius transformations, Communications in Algebra 9(19) (1981), 1913-1968.

[19] B. Schwarz and A. Zaks, Geometries of the projective matrix space, J. Algebra 95 (1985), 263-307.

[20] B. Schwarz and A. Zaks, On the embeddings of projective spaces in lines, Linear and Multilinear Algebra 18 (1985), 319-336.

[21] B. Schwarz and A. Zaks, Higher dimensional euclidean and hyperbolic matrix spaces, J. Analyse Math. 46 (1986), 271-282.

[22] B. Schwarz and A. Zaks, On geometries of the projective matrix space, J. Algebra 124 (1989), 334-336.

[23] D. R. Wilkins, The Grassmann manifold of a C*-algebra, Proc. Royal Irish Acad. 90A (1990), 99-116.

[24] S. Zhang; Exponential rank and exponential length of operators on Hilbert C*-modules, Ann. of Math. 137 (1993), 281-306.

Esteban Andruchow
Instituto de Ciencias, UNGS, San Miguel, Argentina
e-mail : eandruch@ungs.edu.ar

Gustavo Corach
Depto. de Matemática, FCEN-UBA, Buenos Aires, Argentina and
Instituto Argentino de Matemática, Buenos Aires, Argentina
e-mail: gcorach@mate.dm.uba.ar
and
Demetrio Stojanoff
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and
Instituto Argentino de Matemática, Buenos Aires, Argentina
e-mail: demetrio@mate.dm.uba.ar

Mathematical Subject Classification 1991: Primary 46L05, 58B20.

Submitted: June 3, 1999
Revised: October 14, 1999