Weighted modular estimates for a generalized maximal operator on spaces of homogeneous type

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Abstract

We study weighted modular inequalities for a generalized maximal operator associated to a Young function in the context of spaces of homogeneous type. We prove the equivalence between these inequalities and a Dini-type condition, which involves the function associated to the operator and the functions related to the modular estimates. Particularly we obtain a generalization of a result of C. Perez and R. Wheeden ([PW]). In addition we prove a characterization of the A_1-Muckenhoupt class, that extends and improves the corresponding results proved by H. Kita ([K3][K2]).

1 Introduction and Preliminaries

The Hardy-Littlewood maximal operator M of a locally integrable function f is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|Q|} \int_Q |f| \tag{1.1}$$

where the supremum is taken over all balls B containing x. A modular inequality for this operator involving the growth functions ϕ and ψ that has been widely studied is given by

$$\int_{\mathbb{R}^n} \phi(Mf(x))\, dx \leq C \int_{\mathbb{R}^n} \psi(C|f(x)|)\, dx.$$

For example, in [HSV] the authors prove that a Dini- type condition on the growth functions ϕ and ψ characterizes the modular boundedness of certain versions of the operator in (1.1) associated to an open bounded set Ω. Moreover, their results include norm estimates for these operators between Orlicz spaces related to the functions ϕ and ψ and extend those results contained in [K1].

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Related to weighted modular inequalities involving M, in [K3] the author proves that if w is a weight in the A_1 Muckenhoupt class and the function ϕ and ψ satisfy a Dini-type condition then

$$\int_{\mathbb{R}^n} \phi(Mf(x))w(x)\,dx \leq C \int_{\mathbb{R}^n} \psi(C|f(x)|)w(x)\,dx. \tag{1.2}$$

A reciprocal result is also obtained for weights w satisfying a RH_∞ condition, that is,

$$C \sup_Q w \leq \frac{1}{|Q|}\int_Q w.$$

Moreover, for weights in $A_1 \cap RH_\infty$ the author proves the equivalence between the Dini-type condition and the modular estimate in (1.2).

Let k be a positive integer and $M^k = \overbrace{M \circ \cdots \circ M}^{k}$. Let a and b be certain positive functions, $\phi(t) = \int_0^t a(s)\,ds$ and $\psi(t) = \int_0^t b(s)\,ds$, $t \geq 0$. In [K2] the author considers weighted modular inequalities involving the functions ϕ and ψ for M^k and A_1-weights. Concretely, he proves that if the functions ϕ and ψ satisfy the following Dini-type condition

$$\int_0^t \frac{a(s)}{s}\left(\log\frac{t}{s}\right)^{k-1} ds \leq C_1 b(C_2 t) \tag{1.3}$$

for every $t > 0$, then there exist positive constants C_3 and C_4 such that

$$\int_{\mathbb{R}^n} \phi(M^k f)w \leq C_3 \int_{\mathbb{R}^n} \psi(C_4|f|)w. \tag{1.4}$$

For the case $k = 1$, the author also gives a characterization of A_1 weights in terms of the Dini condition (1.3) and the modular inequality (1.4).

For a general Young function η, let

$$M_\eta f(x) = \sup_{Q \ni x} \|f\|_{\eta,Q}$$

where

$$\|f\|_{\eta,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q \eta(|f|/\lambda) \leq 1\}.$$

Particularly, when $\eta_k(t) = t\log^{k-1}(e+t)$, it is well known that $M^k \cong M_{\eta_k}$ (See, for example, [PW] or [BHP]). On the other hand, it is easy to check that $\eta_k'(t) \cong \eta_k(t)/t \cong (\log t)^{k-1}$, for $t > 1$. Thus, the Dini-type condition (1.3) can be written as

$$\int_0^t \frac{a(s)}{s}\eta_k'(t/s)\,ds \leq C_1 b(C_2 t), \tag{1.5}$$

and (1.4) gives a A_1-weighted modular estimate for M_{η_k}.

For a general Young function η, the continuity properties of M_η between L^p spaces have been studied in [P] in the euclidean setting and in [PW] in the context of spaces of homogeneous type with the additional hypotheses that every annuli in the space is not empty. Later, in [PS] the authors avoid this hypotheses and proved the result on spaces of homogeneous type with infinite measure. In all the cases the authors prove that the operator M_η is bounded from L^p into L^p if and only if η belongs to the B_p class, that is $\int_c^\infty \frac{\eta(t)}{t^p}\frac{dt}{t} < \infty$, which is the Dini-type condition (1.5) with $a(t) = b(t) = t^{p-1}$ and η_k replaced by η. Moreover, the B_p condition characterizes weighted estimates of M_η.

In this paper we give a characterization of weighted modular and norm estimates in Orlicz spaces of the generalized maximal function M_η via a Dini-type condition in the general setting of spaces of homogeneous type. Our results extends those contained in [P] in the euclidean context and in [PW] and [PS] on spaces of homogeneous type. Moreover, as a consequence of these results we obtain a new characterizations of A_1 weights in the spirit of the results contained in [K2].

The paper is organized as follows. In §2 we state the main results described above; in §3 we give some technical lemmas that allow us to prove in §4, the main results.

Before stating the main results of this article, we give some standard notation.

Let X be a set. A function $d : X\text{x}X \to [0, \infty)$ is called a quasi-distance on X if the following conditions are satisfied:

$i)$ for every x and y in X, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,

$ii)$ for every x and y in X, $d(x, y) = d(y, x)$,

$iii)$ there exists a constant K such that $d(x, y) \leq K(d(x, z) + d(z, y))$ for every x, y and z in X.

Let μ be a positive measure defined on the σ-algebra of subsets of X generated by the d-balls $B(x, r) = \{y : d(x, y) < r\}$, with $x \in X$ and $r > 0$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2Kr)) \leq A\mu(B(x, r)) < \infty \tag{1.6}$$

holds for every ball $B \subset X$. A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type and it was introduced for the first time in [CW] (for more details, see [MS1] and [MS2], for instance).

We say that (X, d, μ) is a space of homogeneous type regular in measure if μ is regular, that is for every measurable set E, given $\epsilon > 0$, there exists an open set G such that $E \subset G$ and $\mu(G - E) < \epsilon$. In what follows we always assume that the space (X, d, μ) is regular in measure.

A non negative function w defined on X will be called a weight if it is a locally integrable function. Given E a measurable set we denote $w(E) = \int_E w\, d\mu$.

A weight w is in the Muckenhoupt's class A_1 respect to μ if there exists a positive constant C such that the inequality

$$\frac{w(B)}{\mu(B)} \leq Cw(x) \tag{1.7}$$

holds for almost every $x \in B$.

We summarize now a few facts about Orlicz spaces. For more information see, for instance, [RR]. Recall that a non negative increasing function φ, defined on $[0,\infty)$ is called a Young function if it is convex and satisfies $\varphi(0)=0$, $\lim_{s\to\infty}\varphi(s)=\infty$. It follows that $\varphi(t)/t$ is increasing. Each Young function φ has an associated complementary Young function $\tilde{\varphi}$ satisfying

$$t \le \varphi^{-1}(t)\tilde{\varphi}^{-1}(t) \le 2t, \tag{1.8}$$

for all $t>0$. We shall be also concerned with submultiplicative functions φ, which means that

$$\varphi(ts) \le \varphi(t)\varphi(s)$$

for every positive numbers t and s. It is immediate that if φ is a submultiplicative Young function, then it satisfies the Δ_2 condition, that is, $\varphi(2s) \le C\varphi(s)$ which, in particular, implies that $\varphi'(t) \approx \varphi(t)/t$.

If φ is a Young function, we define the weighted φ-average of a function f over a ball B as

$$\|f\|_{\varphi,B,w} = \inf\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \varphi(|f|/\lambda)w\,d\mu \le 1\}.$$

If $w=1$ we simply write $\|f\|_{\varphi,B}$. The following generalization of Hölder's inequality holds

$$\frac{1}{\mu(B)}\int_B |fg|\,d\mu \le \|f\|_{\varphi,B}\|g\|_{\tilde{\varphi},B}. \tag{1.9}$$

2 Main results

Before stating our main results, we include some basic definitions.

Let a and b be positive continuous functions defined on $[0,\infty)$ with $a(0)=b(0)=0$. We also suppose that b is non decreasing, $b(s)\to\infty$ as $s\to\infty$. We define

$$\phi(t) = \int_0^t a(s)\,ds \qquad \psi(t) = \int_0^t b(s)\,ds \tag{2.1}$$

for $t \ge 0$. Observe that ϕ is not necessary a Young function.
From definition (2.1) and the fact that b is non decreasing the following property follows

$$\frac{1}{2}b\left(\frac{t}{2}\right) \le \frac{\psi(t)}{t} \le b(t). \tag{2.2}$$

Finally, we will be working with normalized Young functions η, which means that $\eta(1)=1$.

With these definitions, we can introduce our first result.

(2.3) **Theorem:** *Let (X, d, μ) be a space of homogeneous type with $\mu(X) = \infty$ and such that X contains a point measuring zero. Let η be a doubling Young function and a, b, ϕ and ψ defined as in (2.1). Then the following statements are equivalent*

(2.4) *There exists a positive constant C such that the inequality*

$$\int_0^{4t} \frac{a(s)}{s}\,\eta'(t/s)\,ds \le Cb(Ct)$$

holds for every $t > 0$.

(2.5) *There exists a positive constant C such that the inequality*

$$\int_X \phi(M_\eta f(x))w(x)\,d\mu(x) \le C\int_X \psi(|f(x)|)Mw(x)\,d\mu(x) \tag{2.6}$$

holds for every positive function f and every weight w.

(2.7) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_{\phi,w} \le C\|f\|_{\psi,Mw},$$

holds for every positive function f and every weight w.

(2.8) *There exists a positive constant C such that the inequality*

$$\int_X \phi(M_\eta f(x))\,d\mu(x) \le C\int_X \psi(|f(x)|)\,d\mu(x)$$

holds for every positive function f.

(2.9) *There exists a positive constant C such that the inequality*

$$\int_X \phi\left(\frac{Mf(x)}{M_{\tilde\eta}(\psi^{-1}(u))}\right)w(x)\,d\mu(x) \le C\int_X \psi\left(\frac{|f(x)|}{\psi^{-1}(u)}\right)Mw(x)\,d\mu(x) \tag{2.10}$$

holds for every positive function f and all weights w and u.

(2.11) *There exists a positive constant C such that the inequality*

$$\|M_\eta f\|_\phi \le C\|f\|_\psi,$$

holds for every positive function f.

(2.12) **Remark:** For $a(t) = b(t) = pt^{p-1}$, the theorem above was proved in the euclidean context by C. Pérez in [P]. In the setting of spaces of homogeneous type and for the same functions a and b, it was obtained in [PW] but under the assumption that every annuli in the space is non empty, which implies, for instance, that the space has infinite measure and no atoms (that is, points with positive measure). In [PS], the authors remove the last assumption and prove that the result is valid in any space of homogeneous type with infinite measure.

(2.13) **Remark:** The implications (2.4) $\Rightarrow$ (2.5) $\Rightarrow$ (2.7) $\Rightarrow$ (2.8) $\Rightarrow$ (2.11) and (2.5) $\Rightarrow$ (2.9) do not require $\mu(X) = \infty$. Moreover, from the proof of the theorem it can be seen that the hypothesis about the existence of a point with zero measure is only needed to prove (2.11) $\Rightarrow$ (2.4). So, from this fact and since (2.4) does not depends on X, the theorem is true for every space with infinite measure in staments (2.5) to (2.11).

One version of the above theorem when the measure of the whole space is finite is the following result.

(2.14) **Theorem:** *Let (X, d, μ) be a space of homogeneous type with $\mu(X) < \infty$ and such that X contains a point measuring zero. Let η be a doubling Young function and a, b, ϕ and ψ defined as in (2.1). Then the statements (2.4) to (2.8) are equivalent.*

(2.15) **Remark:** When $\mu(X) < \infty$ and $\mu(\{x\}) \neq 0$ for every x in X, (2.4) does not follows from (2.8). In fact, let us assume $X = \{0, 1\}$ such that $\mu(\{0\}) = \mu(\{1\}) = 1$ and the euclidean metric. For $\eta(t) = \phi(t) = \psi(t) = t$ the modular inequality (2.8) clearly holds, but (ϕ, ψ) does not satisfy the Dini condition (2.4).

(2.16) **Remark:** In Orlicz spaces related to a bounded subset Ω of $\mathbb{R}^n$, Harboure, Salinas and Viviani proved for $\eta(t) = t$ in [HSV], that a strong inequality $\|Mf\|_\phi \leq C\|f\|_\psi$ is equivalent to a slightly different Dini type condition, that is, $\int_1^t a(s)/s ds \leq Cb(Ct)$ whenever $t > 1$ and a modular inequality that differs in a constant.

The next results involve one weight inequalities for the generalized maximal operator M_η on spaces of homogeneous type with infinite measure. The first one can be obtained as a simple corollary of theorem 2.3.

(2.17) **Theorem:** *Let η, a, b, ϕ and ψ be as in theorem 2.3. Then the following statements are equivalent*

(2.18) *There exists a positive constant C such that the inequality*

$$\int_0^{4t} \frac{a(s)}{s} \eta'(t/s) \, ds \leq Cb(Ct)$$

holds for every $t > 0$,

(2.19) *There exists a positive constant C such that the inequality*

$$\int_X \phi(M_\eta f(x)) w(x) \, d\mu(x) \leq C \int_X \psi(|f(x)|) w(x) \, d\mu(x)$$

holds for every positive function f and every weight w belonging to A_1.

(2.20) **Remark:** In the euclidean context theorem 2.17 was partially obtained by H. Kita in [K3] for the maximal function M, but in proving that (2.19) is a sufficient condition for (2.18) the author assumes that $w \in A_1 \cap RH_\infty$ weights respect to the Lebesgue measure.

A certain reciprocal of the theorem above is contained in the following result.

(2.21) **Theorem:** *Let η be a submultiplicative Young function. If condition (2.18) implies (2.19) for every positive, continuous functions a, b, ϕ and ψ as in theorem 2.17, then $w \in A_1$.*

(2.22) **Remark:** In the euclidean setting and for the Hardy-Littlewood maximal operator, the theorem above was proved in [K2].

(2.23) **Remark:** Note that, from theorem (2.17) and (2.21) we get a characterization for weights belonging to A_1.

3 Some technical lemmas

The following result is a classical covering lemma in spaces of homogeneous type. A proof can be found in [CW].

(3.1) **Lemma:** *Let E be a bounded subset in X. Let $\{B(x, r(x)) : x \in E\}$ be a covering of E by balls centered at each point of E. Then there exists a sequence of points $\{x_i\}_{i\in N} \subset E$ such that*
(i) $B(x_i, r(x_i)) \bigcap B(x_j, r(x_j)) = \emptyset$ if $i \neq j$,
(ii) $E \subset \bigcup_{i=1}^{\infty} B(x_i, 4Kr(x_i))$, where K is a constant of the space

(3.2) **Lemma:** *Let η be a Young function and let w be a weight. Then, the following estimate holds*

$$w(\{x \in X : M_\eta f(x) > \lambda\}) \leq C \int_{1/4}^{\infty} Mw(\{x \in X : |f(x)|/\lambda > s\})\eta'(s)\, ds.$$

Proof: By using lemma (3.1) and standard techniques it is not difficult to prove that the following endpoint modular inequality holds

$$w(\{x \in X : M_\eta f(x) > \lambda\}) \leq C \int_X \eta(|f(x)|/\lambda) Mw(x)\, d\mu(x). \tag{3.3}$$

Now, let $f = f_1 + f_2$, where $f_1 = f\chi_{\{|f|\leq\lambda\}}$. Then, by (3.3) we have

$$\begin{aligned} w(\{x \in X : M_\eta f(x) > 2\lambda\}) &\leq Cw(\{x \in X : M_\eta f_2(x) > \lambda\}) \\ &\leq C \int_{\{|f|>\lambda\}} \eta(|f(x)/\lambda|) Mw(x)\, d\mu(x). \end{aligned}$$

Then, to conclude the proof, it is enough to prove that

(3.4)

$$\int_{\{|f|>\lambda\}} \eta(|f(x)/\lambda|)Mw(x)\,d\mu(x) \leq C\int_{1/2}^{\infty} Mw(\{x\in X : |f(x)|/\lambda > s\})\eta'(s)\,ds.$$

Let us observe that

$$\begin{aligned}\int_{\{|f|>\lambda\}} \eta(|f(x)/\lambda|)Mw(x)\,d\mu(x) &\leq \int_0^{\infty} Mw(\{x\in X : |f(x)|/\lambda > \max\{1,s\}\})\eta'(s)\,ds \\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_0^1 Mw(\{x\in X : |f(x)|/\lambda > \max\{1,s\}\})\eta'(s)\,ds$$

and

$$I_2 = \int_1^{\infty} Mw(\{x\in X : |f(x)|/\lambda > \max\{1,s\}\})\eta'(s)\,ds.$$

For I_1 we have

$$\begin{aligned} I_1 &\leq CMw(\{x\in X : |f(x)| > \lambda\}) \\ &\leq C\int_{1/2}^{1} Mw(\{x\in X : |f(x)| > \lambda s\})\eta'(s)\,ds. \\ &\leq C\int_{1/2}^{\infty} Mw(\{x\in X : |f(x)| > \lambda s\})\eta'(s)\,ds. \end{aligned}$$

On the other hand, it is clear that

$$I_2 \leq \int_{1/2}^{\infty} Mw(\{x\in X : |f(x)| > \lambda s\})\eta'(s)\,ds.$$

Then, from the estimates for I_1 and I_2 we obtain (3.4). □

The next lemma gives us a Calderón-Zygmund decomposition related to Orlicz norms and it can be proved applying a similar reasoning to the one used by H. Aimar in [A] for the case $\phi(t) = t$.

(3.5) **Lemma:** *Let f be a nonnegative function belonging to $L(X)$. Then, given $\sigma > 1$, for each $\lambda \geq \|f\|_{\eta,X}$ there exists a sequence $\{B_i\}$ of pairwise disjoint balls such that, if $\tilde{B}_i$ is the dilation of B_i by σ, the following statements hold*

(3.6) $\|f\|_{\eta,\tilde{B}_i} \leq \lambda < \|f\|_{\eta,B_i}$.

(3.7) *For every $x \in X - \cup_i \tilde{B}_i$, we get $\|f\|_{\eta,B} \leq \lambda$ for all ball B containing x.*

(3.8) **Remark:** Observe that, if $\mu(X)=\infty$, the lemma holds for every positive number λ.

The following result give us a reverse modular weak type inequality.

(3.9) **Lemma:** *Let f be a non negative locally integrable function. Then there exists a positive constant C such that the inequality*

$$\mu(\{x\in X: M_\eta f(x)>\lambda\})\geq C\int_{\{x\in X:\eta(f/\lambda)>1\}}\eta(f/\lambda)\,d\mu$$

holds for every $\lambda>\|f\|_{\eta,X}$.

Proof: From lemma 3.5, given $\sigma>0$ and $\lambda>\|f\|_{\eta,X}$ there exists a sequence $\{B_i\}$ of pairwise disjoint balls which satisfies (3.6) and (3.7). Moreover, since μ is a regular measure, it is easy to check that $\{x\in X:\eta(|f|/\lambda)>1\}\subset\cup_i\tilde{B}_i$. From these considerations we have

$$\begin{aligned}
\mu\left(x\in X: M_\eta f(x)>\lambda\right) &\geq \sum_i \mu(B_i)\\
&\geq C\sum_i \mu(\tilde{B}_i)\\
&\geq C\sum_i \int_{\tilde{B}_i}\eta(|f|/\lambda)\,d\mu\\
&\geq C\int_{\cup_i\tilde{B}_i}\eta(|f|/\lambda)\,d\mu\\
&\geq C\int_{\{x\in X:\eta(|f|/\lambda)>1\}}\eta(|f|/\lambda)\,d\mu
\end{aligned}$$

which proves the lemma. □

In order to state the next result we define $\delta: X\times X\to\mathbb{R}^+\cup\{0\}$ as

$$\delta(x,y)=\begin{cases}\mu(B(x,d(x,y))) & \text{if } x\neq y,\\ 0 & \text{if } x=y,\end{cases}$$

It can be seen that the function δ satisfies

(i) $\delta(x,y)\geq 0$ and $\delta(x,y)=0$ is and only if $x=y$,

(ii) $\delta(x,y)\leq A\delta(y,x)$ and

(iii) $A^2(\delta(x,z)+\delta(y,z))$ for every x, y and z in X, where A is the constant in (1.6).

We observe that $\delta(x,y)$ does not necessarily satisfy a symmetric condition as d. The function δ is called the non-necessarily symmetric quasi-distance associated to (X,d,μ) and it was introduced in [MST]. We denote by $B_\delta(x,r)$ the set $\{y:\delta(x,y)<r\}$. The conditions above on δ imply the existence of a constant D such that

$$0<\mu(B_\delta(x,2Kr))\leq D\mu(B_\delta(x,r))<\infty.$$

The next lemma related to the δ-balls was proved in [PS], (see [BS], too).

(3.10) **Lemma:** *Assuming $\mu(X)=\infty$, there exist two constants C_0 and C_1, depending only on the constants of the space (X,d,μ), such that*

$$\mu(B_\delta(z,C_0R))-\mu(B_\delta(z,R))\geq C_1R$$

for every z in X and every $R>\mu(\{z\})/2A^2$, where A is the constant in (1.6).

4 Proof of the main results

In this section we prove theorems 2.3, 2.14 and 2.21.

Proof of theorem 2.3:

We are going to do the proof in the following way:

i) We prove the chain of implications (2.4) $\Rightarrow$ (2.5) $\Rightarrow$ (2.7) $\Rightarrow$ (2.8) $\Rightarrow$ (2.4).

ii) We proceed with the proof of (2.5) $\Rightarrow$ (2.9) $\Rightarrow$ (2.4).

iii) We finally prove that (2.8) $\Rightarrow$ (2.11) $\Rightarrow$ (2.4).

i) Let us prove that (2.4) $\Rightarrow$ (2.5). Let $f\in L^{\psi}_{Mw}(X)$. From lemma 3.2 and the hypothesis we obtain

$$\begin{aligned}\int_X\phi(M_\eta f(x))w(x)\,d\mu(x) &= \int_0^\infty a(\lambda)w(\{x\in X: M_\eta f(x)>\lambda\})\,d\lambda\\ &\leq C\int_0^\infty a(\lambda)\int_{1/4}^\infty Mw(\{x\in X:|f(x)|/\lambda>s\})\eta'(s)\,ds\,d\lambda\\ &\leq C\int_0^\infty Mw(\{x\in X:|f(x)|>s\})\left(\int_0^{4s}\frac{a(\lambda)}{\lambda}\eta'(s/\lambda)\,d\lambda\right)ds\\ &\leq C\int_0^\infty b(Cs)Mw(\{x\in X:|f(x)|>s\})\,ds\\ &\leq C\int_X\psi(C|f(x)|)Mw(x)\,d\mu(x),\end{aligned}$$

which is (2.5).

From the fact that ψ is a convex function, (2.7) follows easily from (2.5) by taking $f/C\|f\|_{\psi,Mw}$ instead of f. Let us prove (2.7) $\Rightarrow$ (2.8). Let f be a positive function, and let α be a constant such that $\alpha=1/(\int_X\psi(C|f(x)|)d\mu(x))$. Clearly

$$\int_X\psi(C|f(x)|)\alpha d\mu(x)=1,$$

and then $\|f\|_{\psi,\alpha} = 1/C$, where $w \equiv \alpha$. Since the constant in (2.7) is independent of the weight and the function f, we can rewrite for that weight the inequality of the hypothesis as

$$\int_X \phi(M_\eta f(x))\alpha d\mu(x) \leq 1.$$

Then we have

$$\int_X \phi(M_\eta f(x))d\mu(x) \leq C\int_X \psi(C|f(x)|)d\mu(x),$$

which is (2.8).

Finally to complete (i), let us prove (2.8) $\Rightarrow$ (2.4). By hypothesis we obtain

$$\begin{aligned} C\int_X \psi(C|f(x)|)d\mu(x) &\geq \int_X \phi(M_\eta(f(x)))d\mu(x) \\ &= \int_0^\infty a(\lambda)\mu\{x\in X : M_\eta f(x) > \lambda\}d\lambda. \end{aligned}$$

From the weak type reverse inequality for M_η stated in lemma 3.9 it follows that

$$C\int_X \psi(C|f(x)|)d\mu(x) \geq \int_0^\infty a(\lambda)\int_{\{x\in:\eta(|f(x)|/\lambda)>1\}} \eta(|f(x)|/\lambda)d\mu(x)d\lambda.$$

Let $\lambda_o > 0$ and a fixed ball B_o. Let $f = \lambda_o\chi_{B_o}$. Replacing f in the previous inequality we get that

$$\begin{aligned} C\psi(C\lambda_o)\mu(B_o) &\geq \int_0^{\lambda_o} a(\lambda)\eta\left(\frac{\lambda_o}{\lambda}\right)\mu(B_o)d\lambda \\ &\geq C\int_0^{\lambda_o} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{\lambda_o}{\lambda}\right)\lambda_o\mu(B_o)d\lambda, \end{aligned}$$

having applied latter that $\eta'(t) \approx \eta(t)/t$. Finally as $\psi(t)/t \leq b(t)$ we have

$$\begin{aligned} Cb(C\lambda_o) &\geq C\frac{\psi(\lambda_o)}{\lambda_o} \\ &\geq C\int_0^{\lambda_o} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{\lambda_o}{\lambda}\right)d\lambda \end{aligned}$$

for all $\lambda_o > 0$ from which it follows (2.4).

ii) We use similar arguments to those in theorem 5.1 of [PW] to prove that (2.5) $\Rightarrow$ (2.9). Since (2.10) is equivalent to

$$\int_X \phi\left(\frac{M(fg)(x)}{M_{\tilde\eta}(g)}\right)w(x)\,d\mu(x) \leq C\int_X \psi\left(|f(x)|\right)Mw(x)\,d\mu(x)$$

for all non negative functions f, g and w, (2.9) follows from (2.5) after an application of the following inequality

$$M(fg)(x) \leq M_{\eta}(f)(x) M_{\tilde{\eta}}(g)(x) \qquad x \in X,$$

which is a consequence of the generalized Hölder's inequality (1.9).

Note that if $\eta(t) = t$, then $\tilde{\eta}(t) = 0$ if $t \leq 1$ and $\tilde{\eta}(t) = \infty$ if $t > 1$, which implies that $M_{\tilde{\eta}}(g)(x) = \|g\|_{\infty}$. In this case the results can be obtained by following similar arguments with the obvious changes.

The proof of (2.9) $\Rightarrow$ (2.4) follows similar arguments as those in the proof of theorem 1.4 in [PS]. For a sake of completeness we include it.

Let us take $w = 1$ in the hypotheses to obtain

$$\int_X \phi\left(\frac{Mf(x)}{M_{\tilde{\eta}}(\psi^{-1}(u))}\right) d\mu(x) \leq C \int_X \psi\left(\frac{|f(x)|}{\psi^{-1}(u)}\right) d\mu(x). \tag{4.1}$$

For $t > 0$, $z \in X$ and $R > 0$, let us take $f = t\chi_{B_0}$ and $\psi^{-1}(u) = \chi_{B_0}$, where $B_0 = B(z, R)$. Let $x \in X$ such that $d(x, z) > \theta R$, $\theta > 1$ large enough and let $\alpha > 1$ such that $\mu(B(z, \alpha\theta R)) > \mu(B(z, \theta R))$. Let Ω be the set defined by $\Omega = \{x : \mu(B(z, d(z, x))) \geq \mu(B(z, \alpha\theta R))\}$. It is clear that $\Omega \subset \{x : d(z, x) > \theta R\}$.

On the other hand, it is easy to check that there exists a positive constant C such that

$$M(t\chi_{B_0})(x) \cong \frac{t}{\mu(B(x, d(x, z)))}$$

and

$$M_{\tilde{\eta}}(\chi_{B_0})(x) \cong \frac{1}{\tilde{\eta}^{-1}(C\mu(B(x, d(x, z))))}.$$

Thus, proceeding as in the proof of theorem 1.4 in [PS], from (4.1) we obtain

$$C\psi(t) \geq \int_{\Omega} \phi\left(\frac{t\tilde{\eta}^{-1}(C\mu(B(z, d(x, z))))}{\mu(B(z, d(x, z)))}\right) d\mu.$$

Then, taking $R_0 = \mu(B(z, \alpha\theta R))$, from (1.8) applied to η and lemma (3.10) we get

$$\begin{aligned}
C\psi(t) &\geq \sum_{j=0}^{\infty} \int_{C_0^j R_0 \leq \mu(B(z,d(x,z))) < C_0^{j+1} R_0} \phi\left(\frac{t\tilde{\eta}^{-1}(C\mu(B(z, d(x, z))))}{\mu(B(z, d(x, z)))}\right) d\mu \\
&\geq \sum_{j=0}^{\infty} \phi\left(\frac{t\tilde{\eta}^{-1}(CC_0^j R_0)}{C_0^{j+1} R_0}\right) \mu\left(B_\delta(z, C_0^{j+1} R_0) - B_\delta(z, C_0^j R_0)\right) \\
&\geq C_1 \sum_{j=0}^{\infty} \phi\left(\frac{Ct}{C_0 \eta^{-1}(CC_0^j R_0)}\right) C_0^j R_0 \\
&\geq C_1 \sum_{j=0}^{\infty} \int_{C_0^j R_0}^{C_0^{j+1} R_0} s\phi\left(\frac{Ct}{\eta^{-1}(s)}\right) \frac{ds}{s} \\
&\geq C_1 \int_C^{\infty} s\phi\left(\frac{Ct}{\eta^{-1}(s)}\right) \frac{ds}{s}.
\end{aligned}$$

By changing variables in the expression above we obtain

$$
\begin{aligned}
C\frac{\psi(t)}{t} &\geq \int_0^{Ct} \frac{\phi(s)}{s}\eta'(t/s)\frac{ds}{s} \\
&\geq \int_0^{Ct} \frac{a(s)}{s}\eta'(t/s)\, ds.
\end{aligned}
$$

From the fact that $b(t) \geq \psi(t)/t$ (see (2.2)), we obtain (2.4).

It is important to note that if $\eta(t) = t$ then $M_{\tilde{\eta}}(\psi^{-1}(u)) = 1$ and the same proof works by making a slight modification.

iii) If we set $w = 1$ in (2.5) we get (2.8). On the other hand, by taking $w = 1$ and proceeding as in the proof of (2.5) $\Rightarrow$ (2.7), we obtain that (2.8) implies (2.11).

Let us now see that (2.11) $\Rightarrow$ (2.4). For $x_0 \in X$, such that $\mu(\{x_o\}) = 0$ and $0 < r < 1$ we set $B_r = B(x_0, r)$ and $f_r = \frac{1}{\mu(B_r)}\chi_{B_r}(x)$. If $0 < \lambda < 1/\mu(B_r)$, it is easy to prove that $B_r \subset \{x \in X : \eta(f_r/\lambda) > 1\}$. Since $\mu(X) = \infty$, we have that $\|f_r\|_{\eta,X} = 0$, thus, by applying lemma 3.9 we obtain

$$
\begin{aligned}
\mu(\{x \in X : M_\eta f_r(x) > \lambda\}) &\geq C\int_{\{x\in X:\eta(f_r/\lambda)>1\}} \eta(f_r/\lambda)\, d\mu \\
&\geq C\int_{B_r} \eta\left(\frac{1}{\mu(B_r)\lambda}\right) d\mu \\
&= C\mu(B_r)\, \eta\left(\frac{1}{\mu(B_r)\lambda}\right).
\end{aligned}
$$

On the other hand, we have that $\|f_r\|_\psi = \frac{1}{\mu(B_r)\psi^{-1}(1/\mu(B_r))}$. From (2.11) and the properties of η we get

$$
\begin{aligned}
1 &\geq \int_X \phi\left(\frac{M_\eta f_r}{C\|f_r\|_\psi}\right) d\mu \\
&= \int_0^\infty a(\lambda)\mu(\{M_\eta f_r > C\lambda\|f_r\|_\psi\})\, d\lambda \\
&= \int_0^\infty a\left(\frac{\lambda}{C\|f_r\|_\psi}\right)\mu(\{M_\eta f_r > \lambda\})\, \frac{d\lambda}{C\|f_r\|_\psi} \\
&\geq C\int_0^{1/\mu(B_r)} \frac{a(\lambda/C\|f_r\|_\psi)}{\lambda} 4\lambda\mu(B_r)\, \eta\left(\frac{1}{4\lambda\mu(B_r)}\right) \frac{d\lambda}{C\|f_r\|_\psi} \\
&\geq C\int_0^{1/\mu(B_r)} \frac{a(\lambda/C\|f_r\|_\psi)}{\lambda}\, \eta'\left(\frac{1}{4\lambda\mu(B_r)}\right) \frac{d\lambda}{C\|f_r\|_\psi} \\
&= C\int_0^{1/C\mu(B_r)\|f_r\|_\psi} \frac{a(\lambda)}{\lambda}\, \eta'\left(\frac{1}{4C\lambda\mu(B_r)\|f_r\|_\psi}\right) \frac{d\lambda}{C\|f_r\|_\psi}.
\end{aligned}
$$

If we set $t = \frac{1}{4C\mu(B_r)\|f_r\|_\psi}$ from the estimates above we obtain that

$$\begin{aligned}\int_0^{4t} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{t}{\lambda}\right)d\lambda &\leq C\|f_r\|_\psi \\ &= C\frac{\psi(Ct)}{Ct} \\ &\leq Cb(Ct),\end{aligned}$$

which proves (2.4). $(\square)$

Proof of theorem 2.14: Keeping in mind remark 2.13, in order to complete the proof we have to see that (2.4) follows from (2.8). Let $\lambda_o > 0$, $M > 0$, $x_o \in X$ such that $\mu(\{x_o\}) = 0)$ and the ball $B_o = B(x_o, M)$. Let $f = \lambda_o\chi_{B_o}$,then by the norm definition it follows that $\|f\|_{\eta,X} = \lambda_o/\eta^{-1}(\mu(X)/\mu(B_o))$. From the hypothesis

$$\begin{aligned}C\int_X \psi(C|f(x)|)d\mu(x) &\geq \int_X \phi(M_\eta(f(x)))d\mu(x) \\ &= \int_0^\infty a(\lambda)\mu\{x \in X : M_\eta f(x) > \lambda\}d\lambda.\end{aligned}$$

Applying the weak type reverse inequality (3.9) for M_η with $\lambda \geq \|f\|_{\eta,X}$ we have that

$$C\int_X \psi(C|f(x)|)d\mu(x) \geq \int_{\|f\|_{\eta,X}}^{\lambda_o} a(\lambda)\int_{\{x\in:\eta(|f(x)|/\lambda)>1\}} \eta(|f(x)|/\lambda)d\mu(x)d\lambda$$

From the definition of f we get

$$\begin{aligned}C\psi(C\lambda_o)\mu(B_o) &\geq \int_{\frac{\lambda_o}{\eta^{-1}(\mu(X)/\mu(B_o))}}^{\lambda_o} a(\lambda)\eta\left(\frac{\lambda_o}{\lambda}\right)\mu(B_o)d\lambda \\ &\geq \bar{C}\int_{\frac{\lambda_o}{\eta^{-1}(\mu(X)/\mu(B_o))}}^{\lambda_o} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{\lambda_o}{\lambda}\right)\lambda_o\mu(B_o)d\lambda.\end{aligned}$$

In the latter it has been applied that $\eta'(t) \approx \eta(t)/t$.
Now, applying that $\psi(t)/t \leq b(t)$, we obtain

$$\begin{aligned}Cb(C\lambda_o) &\geq C\frac{\psi(C\lambda_o)}{\lambda_o} \\ &\geq \int_{\frac{\lambda_o}{\eta^{-1}(\mu(X)/\mu(B_o))}}^{\lambda_o} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{\lambda_o}{\lambda}\right)d\lambda.\end{aligned}$$

Since $\frac{\lambda_o}{\eta^{-1}(\mu(X)/\mu(B_o))}$ tends to zero when the radius of B_o tends to zero and C is independent of it, we obtain

$$Cb(C\lambda_o) \geq \int_0^{\lambda_o} \frac{a(\lambda)}{\lambda}\eta'\left(\frac{\lambda_o}{\lambda}\right)d\lambda$$

for all $\lambda_o > 0$, which is (2.4) and the theorem is proved. $(\Box)$

Before starting with the proof of theorem 2.21, we introduce another class of weigths and then a characterization result.

Let ψ and η be Young functions. By B_ψ^η we denote the class of weights w for which there exists a positive constant C such that the inequality

$$w(\{x \in X : M_\eta f(x) > \lambda\}) \leq \int_X \psi \circ \eta \left(C \frac{|f(x)|}{\lambda} \right) w(x)\, d\mu(x) \tag{4.2}$$

holds for every $\lambda > 0$ and for all $f \in L_w^{\phi\circ\eta}(X)$. When $\eta(t) = t$ we simply write B_ψ. In the euclidean setting the class B_ψ was widely studied in [B].

It is not difficult to see that, if $w \in B_\psi^\eta$ then w satisfies a doubling condition. In fact, let $B = B(x_B, r)$ and $\tilde{B} = B(x_B, 2Kr)$ where K is the constant associated to the quasi-distance d. If $f = \chi_B$ then it is easy to check that $M_\eta f(x) \geq 1/\eta^{-1}(A)$ for $x \in \tilde{B}$, where A is the constant in (1.6). Then, from (4.2) we have

$$w(\tilde{B}) \leq w(\{x \in X : M_\eta f(x) > 1/\eta^{-1}(A)\}) \leq \psi(CA) w(B)$$

which proves the desired result.

The next theorem gives a characterization of the weights in the class B_ψ^η on spaces of homogeneous type. This result proves that both classes B_ψ^η and B_ψ coincide.

(4.3) **Theorem:** *Let w be a weight and let ψ and η be a Young functions such that η is submultiplicative. The following statements are equivalent*
(4.4) $w \in B_\psi$.
(4.5) *w satisfies a doubling condition and there exists a positive constant C such that the inequality*

$$\|1/w\|_{\tilde{\psi},B,w} \leq C\mu(B)/w(B) \tag{4.6}$$

holds for every ball $B \subset X$.
(4.7) $w \in B_\psi^\eta$.

(4.8) **Remark:** It is easy to see that (4.6) is equivalent to the existence of a positive number ϵ such that the inequality

$$\int_B \tilde{\psi} \left(\frac{\epsilon w(B)}{\mu(B) w} \right) w \, d\mu \leq w(B)$$

holds for every ball B in X.

Proof of theorem 4.3: To prove that (4.4) $\Rightarrow$ (4.5) we followed similar arguments to those applied by Bagby in $\mathbb{R}^n$ with obvious changes and we omit the details, (see theorem 3.3 in [B]). Let us first suppose that (4.5) holds. If B is any ball such that

$\int_B \eta(|f|/\lambda)\, d\mu > \mu(B)$ then, by proceeding as in the proof of theorem 3.3 in [B] and from the hypotheses on w we obtain that

$$w(B) \leq \int_B \psi(C\eta(|f|/\lambda))w\, d\mu. \tag{4.9}$$

Let E_λ be any subset of $\{x : M_\eta f(x) > \lambda\}$. Let $\tilde{M}_\eta$ be the centered version of M_η, because of there exists $C > 0$ such that the inequality $M_\eta \leq C\tilde{M}_\eta$ holds, we have

$$E_\lambda \subset \{x \in X : \tilde{M}_\eta f(x) > \lambda/C\}.$$

Then, if $x \in E_\lambda$ there exists a ball $B(x, r(x))$ such that $\|f\|_{\eta,B(x,r(x))} > \lambda/C$. From lemma (3.1) there exists a sequence of points $\{x_i\}_{i\in N} \subset E_\lambda$ such that the balls $B(x_i, r(x_i)) \bigcap B(x_j, r(x_j)) = \emptyset\, if\, i \neq j$ and

$$E_\lambda \subset \bigcup_{i=1}^{\infty} B(x_i, 4Kr(x_i)).$$

Since w satisfies a doubling condition, from (4.9) we obtain

$$\begin{aligned} w(E_\lambda) &\leq \sum_i w(B(x_i, 4Kr(x_i))) \\ &\leq \sum_i Cw(B(x_i, r(x_i))) \\ &\leq \sum_i \int_{B_i} C\psi \circ \eta \left(C\frac{|f(x)|}{\lambda}\right))(x)d\mu(x) \\ &\leq \int_X \psi \circ \eta \left(C\frac{|f(x)|}{\lambda}\right))(x)d\mu(x), \end{aligned}$$

where the last inequality follows from the fact that $\phi \circ \eta$ is a convex function. Then, by a standard approximation argument we obtain (4.7).

Let us now prove that (4.7) $\Rightarrow$ (4.4). Let $\widetilde{M_\eta}$ the maximal function defined by

$$\widetilde{M_\eta} f(x) = \sup_{B \ni x} \eta^{-1}\left(\frac{1}{\mu(B)}\int_B \eta(|f|)\, d\mu\right).$$

From the fact that η is a submultiplicative function it is easy to see that $\widetilde{M_\eta} f(x) \leq M_\eta f(x)$. Thus, since $w \in B_\psi$ we have

$$\begin{aligned} w(\{x \in X : M(\eta(|f|)) > \eta(\lambda)\}) &= w(\{x \in X : \widetilde{M_\eta}(|f|) > \lambda\}) \\ &\leq w(\{x \in X : M_\eta(|f|) > \lambda\}) \\ &\leq \int_X \psi(C\eta(|f|/\lambda))w\, d\mu. \end{aligned}$$

Particularly, if $\lambda = 1$ and $g = t\,\eta(|f|)$, $t > 0$, we obtain

$$w(\{x \in X : M(g) > t\}) \leq \int_X \psi(Cg/t)w\, d\mu,$$

and then $w \in B_\psi$ and (4.4) is true. $\square$

Proof of theorem 2.21: Let us first suppose that $w \notin A_1$. Proceeding as in the proof of theorem 2.3 in [K2] we can find a sequence of balls $\{B_n\}_{n\geq 1}$ such that, if

$$G_n = \{x \in B_n : w(B_n)/\mu(B_n) > 2^{2n} w(x)\},$$

then $\mu(G_n) > 0$ and consequently $w(G_n) > 0$. From this sequence we can construct a suitable positive continue function b defined in $[0, \infty)$ such that $\psi(t) = \int_0^t b(s)\, ds$ results a Young function with the property that $\tilde{\psi}(2^n) \geq w(B_n)/\mu(B_n)$, $n \geq 1$.
We define $a(s) = (b \circ \eta)'(s)/\eta(1/s)$. Since η is submultiplicative, we obtain that η is a doubling function and thus $\eta'(t) \approx \eta(t)/t$. Then we have

$$\begin{aligned} \int_0^t \frac{a(s)}{s} \eta'(t/s)\, ds &\leq C\eta'(t) \int_0^t a(s)\eta(1/s)\, ds \\ &\leq C\eta'(t) \int_0^t (b \circ \eta)'(s)\, ds \\ &\leq Cb \circ \eta(t)\ \eta'(t) \\ &\leq C(\psi \circ \eta)'(t). \end{aligned} \tag{4.10}$$

Let us now consider the functions $\phi(t) = \int_0^t a(s) ds$ and $\overline{\psi}(t) = \psi \circ \eta(t)$. By (4.10) and the hypothesis we obtain that the pair $(\phi, \overline{\psi})$ satisfies

$$\begin{aligned} w(\{x \in X : M_\eta f > \lambda\}) &\leq \int_X \phi\left(M_\eta f/\lambda\right) w\, d\mu \\ &\leq \int_X \psi\left(\eta(|f|/\lambda)\right) w\, d\mu \end{aligned}$$

and thus $w \in B_\psi^\eta$. From theorem 4.3 and remark 4.8 there exists $\epsilon > 0$ such that the inequality

$$\int_B \tilde{\psi}\left(\frac{\epsilon w(B)}{\mu(B) w}\right) w\, d\mu \leq w(B) \tag{4.11}$$

holds for every ball B.

Let $n \in \mathbb{N}$ big enough such that $1/2^n < \epsilon$, then from (4.11) we get

$$\begin{aligned} w(B_n) &\geq \int_{G_n} \tilde{\psi}\left(\frac{2^{2n}\mu(B_n) w}{2^n \mu(B) w}\right) w\, d\mu \\ &\geq \tilde{\psi}(2^n) w(G_n) \\ &> w(B_n), \end{aligned}$$

which is a contradiction. Then $w \in A_1$ and the proof is done. $\square$

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