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An explicit expression for singular integral operators with non-necessarily doubling measures

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Abstract We study singular integral operators with Hilbert-valued kernels in the setting of R^n with non-necessarily doubling measures. We obtain an explicit formula for these operators following a similar approach as in [MST]. By using this formula and a result due to Krein we get a $T1$-theorem in this context. Finally, we develop a theory for antisymmetric kernels and we apply the results to the Oscillation Operators related to the Riesz Transform.

Keywords Singular integrals, non-doubling measures, vector-valued functions, oscillation operators.

Mathematics Subject Classification (2000) 42B20

1 Introduction

The purpose of this work is to study singular integral operators with kernels taking values on a Hilbert space, in the setting of measures on euclidean spaces satisfying a growth condition.

The classical theory of singular integral operators starts with the celebrated work of Calderón-Zygmund in [CZ]. Since then, many authors have been studying these operators in different contexts. The necessary and sufficient conditions for L^2-boundedness ($T1$ Theorem) were developed by David and Journé in [DJ].

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The Calderón-Zygmund theory for operators with vector-valued kernels is linked to the names of Benedek, Calderón and Panzone (see [BCP]) and the works included in [RFRT] and [RFT]. The versions of $T1$ Theorem for operators with vector-valued kernels have been recently treated in [HW] and [H1]. Also, $T1$ Theorems in the case of non-necessarily doubling measures were considered, using different approaches and techniques, by Nazarov, Treil and Volberg in [NTV], Tolsa in [T1], [T2] and [T3], Melnikov and Verdera in [MV], Verdera in [V] and Hytönen in [H2].

One of our motivations to consider vector-valued singular integral operators is their applications to the study of the behavior of quadratic functions like

$$\mathcal{O}(\tau)f(x) = \left(\sum_{\ell=1}^{\infty} \left| \tau_{\epsilon_i} f(x) - \tau_{\epsilon_{i+1}} f(x) \right|^2 \right)^{1/2},$$

related to the scalar family of operators $\{\tau_\epsilon\}_{\epsilon>0}$ such that $\lim_{\epsilon\to 0} \tau_\epsilon f(x) = \tau f(x)$.

These kind of quadratic functions (Oscillation Operators) were earlier considered in connection with ergodic theory in [B1], [J], [JOR] and [JKRW]. Our approach allows us to derive L^2 boundedness results for these operators without using Fourier transform and Plancherel Theorem.

We start with an operator T defined on scalar-valued Lipschitz functions, with μ a Radón measure, "formally" associated to a vector-valued kernel $k(x,y)$ in the following sense:

$$Tf(x) = \int k(x,y) f(y) d\mu(y),$$

whenever x is not in the support of f.

First, we prove several results by following the approach of Macías, Segovia and Torrea in [MST], in order to obtain an explicit formula for $Tf(x)$ for μ-almost every x (see Theorem 2.4). This formula is new in the scalar case, in the setting of non-necessarily doubling measures, and provide us an explicit expression for the Cauchy integral on Lipschitz graphs.

Secondly, we apply the previous results to get Lipschitz bounds, when $T1 = 0$. As a consequence we obtain $T1$-Theorems in the setting of non-necessarily doubling measures with the aid of a result by Krein (see Theorems 2.6 and 2.9). The application of the Krein's Theorem to derive L^2-boundedness for singular integral operators appears for the first time in the work of Wittman ([W]). (See also [B2]).

Since Theorem 2.9 is obtained for vector-valued antisymmetric kernels satisfying integral conditions (24) and (25), we are allowed to apply it to obtain boundedness for the Oscillation operator of the Riesz transforms (see [CJRW1] and [CJRW2] for works in this direction). This result is interesting by itself, although it is only considered in the case of the Lebesgue measure.

The paper is organized as follows. In the next section the main results are listed. In Section 3 we give definitions, notation and technical results. In Sections 4, 5 and 6 we provide the proofs of the main results. Finally in Section 7 we study the Oscillation operator of the Riesz transform.

2 Main results

First, we provide the definitions and notation in order to state the main theorems of this paper. The technical details appear in Section 3.

We consider the setting of n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the quasidistance

$$d(x,y) = |x-y|_\infty^n, \quad \text{where} \quad |a|_\infty = \max_{i=1,\dots,n} |a_i|, \; a=(a_1,\dots,a_n). \tag{1}$$

The properties of d are shown in (12) and (13). The **balls** for the metric d with center x and radius $s>0$ will be denoted by $B_s(x)$

We will consider a Radón measure μ defined on $\mathbb{R}^n$, absolutely continuous with respect to the Lebesgue measure. We assume in the sequel that μ satisfies the next growth condition: There is a number ν, $0<\nu\le 1$, such that for all $s>0$ and $x\in\mathbb{R}^n$

$$\mu(B_s(x)) \le c_0 s^\nu, \tag{2}$$

where $c_0>0$ is a constant non-depending on s and x. In what follows we assume that $\mu(\{x\})=0$ for every $x\in\mathbb{R}^n$ (μ is non-athomic).

The classical notation $\mathbb{N},\mathbb{Z},\mathbb{R},\mathbb{C}$ will stand for the usual set of numbers, and $\mathbb{H}$ will denote an arbitrary Hilbert space. In addition, we will use several geometrical constants, for example: $\alpha=1/n$, A and ρ (see Subsections 3.1 to 3.3).

The involved function spaces related to the measure μ will be the scalar and vector valued versions of the Lebesgue spaces L^p_μ, the bounded mean oscillation spaces BMO_ρ (where $\rho\ge 1$) and several classes of Lipschitz spaces, such that Λ_0^γ (Lipschitz functions with bounded support), Λ_b^γ (bounded Lipschitz functions), etc. The Subsection 3.2 provides all the neccessary definitions.

As we say in the introduction our investigation will be focused on singular integral operators T associated to a vector valued kernel $k(x,y)$ which satisfy certain "size" and "smoothness" conditions(see (24) and (25)). Also, we study the adjoint operator T^* associated to the kernel $k^*(x,y)=k(y,x)$. We refer to Subsection 3.4 for these and other important definitions.

Hypothesis 2.1. *We say that a linear operator T* **satisfies the main hypothesis** *if the following conditions are fulfilled:*

(i) For all $0<\gamma\le\alpha$, the operator $T:\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{C})\to(\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{H}))'$ is continuous.
(ii) T is associated to a kernel $k(x,y)$ (with values in $\mathbb{H}$).
(iii) T satisfies the W.B.P. for some η, $0<\eta\le\alpha$ (see Definition 3.10).
(iv) For some $1\le r\le\infty$, both kernels k and k^* satisfy condition (24), and condition (25) with exponent $\eta+\epsilon$, $\epsilon>0$,
(v) T satisfies the Meyer Commutation Property (see Definition 3.11).

In the sequel we will put on play a list of vector constants in $\mathbb{H}$, denoted as $\mathtt{C}_{B,\hat{B}}[g]$, $\bar{\mathtt{C}}_{B,\hat{B}}$, and $\bar{\mathtt{C}}^{(*)}_{B,\hat{B}}$ (see Definition 3.18), explicitly depending on some fixed ball B and its associated doubling ball $\hat{B}$ (see Definition 3.2). In the case of $\mathtt{C}_{B,\hat{B}}[g]$, the expression also depends on g, a given function of the space $BMO_\rho(\mathbb{R}^n,\mathbb{H})$.

Furthermore, for each ball B with center z and functions $\phi \in L^\infty_\mu(\mathbb{R}^n,\mathbb{C})$, $\overrightarrow{\psi} \in L^\infty_\mu(\mathbb{R}^n,\mathbb{H})$, we define

$$I_B\phi(x) := \int (k(x,y)-k(z,y))\,\phi(y)\,(1-h_B(y))\,d\mu(y) \tag{3}$$

$$I_B^{(*)}\overrightarrow{\psi}(x) := \int \left(k^*(x,y)-k^*(z,y),\overrightarrow{\psi}(y)\right)_{\mathbb{H}}(1-h_B(y))\,d\mu(y), \tag{4}$$

$$J_B\phi(x) \;=\; \int \left(k^*(x,y)-k^*(z,y)\right)(1-h_B(y))\,\phi(y)\,d\mu(y). \tag{5}$$

Definition 2.2. *. Given a function $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n,\mathbb{H})$ supported in a ball $B=B_s(z)$ we define*

$$\left\langle T1,\overrightarrow{\psi}\right\rangle := \left\langle Th_B,\overrightarrow{\psi}\right\rangle + \left\langle I_B1,\overrightarrow{\psi}\right\rangle,$$

where h_B is the cut function defined in subsection 3.3. In a similar way, given a vector $\overrightarrow{w} \in \mathbb{H}$ and a function $\phi \in \Lambda_0^\gamma(\mathbb{R}^n,\mathbb{C})$ supported in a ball $B=B_s(z)$ we define

$$\left\langle T^*(1\overrightarrow{w}),\phi\right\rangle := \left\langle T^*(h_B\overrightarrow{w}),\phi\right\rangle + \left\langle I_B^{(*)}(1\overrightarrow{w}),\phi\right\rangle.$$

Definition 2.3. *Let us assume that $\mathbb{B}_1,\mathbb{B}_2$ are Banach spaces. Given $0<\gamma\le\alpha$, we say that T is a* ***bounded Lipschitz operator of order*** *γ if T satisfies*

$$\|Tf\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{B}_2)} \le C\|f\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{B}_1)} \tag{6}$$

and

$$\|Tf\|_{L^\infty_\mu(\mathbb{R}^n,\mathbb{B}_2)} \le C\,s_0^\gamma\|f\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{B}_1)}, \tag{7}$$

for any function $f\in\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{B}_1)$ supported in a ball with radius s_0.

If E is a set with positive μ-measure and f is a μ-integrable function over E, we write

$$m_E := \frac{1}{\mu(E)}\int_E f\,d\mu.$$

We are in position to state the main theorems.

Theorem 2.4. *Let T be an operator satisfying Hypothesis 2.1 and $T1=g$ with $g\in BMO_\rho(\mathbb{R}^n,\mathbb{H})$.*

(i) If $\phi\in\Lambda_0^\eta(\mathbb{R}^n,\mathbb{C})$ then, for every ball B containing the support of ϕ and for μ-almost every $x\in\mathbb{R}^n$, we have

$$\begin{aligned} T\phi(x) &= (g(x)-m_{A\hat{B}}g)\phi(x) + \mathtt{C}_{B,\hat{B}}[g]\phi(x) - I_B1(x)\phi(x) \\ &\quad + \int[\phi(y)-\phi(x)]k(x,y)h_B(y)\,d\mu(y), \end{aligned} \tag{8}$$

where

$$\sup_{B\ a\ ball}\left|\mathtt{C}_{B,\hat{B}}[g]\right|_{\mathbb{H}} \le C_g, \tag{9}$$

with C_g only depending on $\|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})}$.

(ii) In addition, if $T1=0_{\mathbb{H}}$, then T is a bounded-Lipschitz operator of order η for any $0<\eta\le\alpha$ with $\mathbb{B}_1=\mathbb{C}$ and $\mathbb{B}_2=\mathbb{H}$.

Remark. We observe that the Meyer Commutation Property is not assumed in the setting of spaces of homogeneous type, because it is true in such context. Further, in [MST, Prop. 2.29] it is possible to obtain a reciprocal of Theorem 2.4(ii).

Theorem 2.5. *Let T be an operator satisfying Hypothesis 2.1. In addition, assume that, for fixed $\overrightarrow{w}_1, \dots, \overrightarrow{w}_N \in \mathbb{H}$, we have $T^*(1\overrightarrow{w}_i) = f_{\overrightarrow{w}_i}$, with $f_{\overrightarrow{w}_i} \in BMO_\rho(\mathbb{R}^n, \mathbb{C})$, $i = 1, \dots, N$. If $\overrightarrow{\psi}(x) = \sum_{i=1}^N \psi_i(x)\overrightarrow{w}_i$, where $\psi_i \in \Lambda_0^\eta(\mathbb{R}^n, \mathbb{C})$, then for every ball B containing the support of $\overrightarrow{\psi}$ and μ-almost every $x \in \mathbb{R}^n$, we have*

$$T^*(\overrightarrow{\psi})(x) = \mathcal{F}_B[\overrightarrow{\psi}](x) + \big(\bar{\mathtt{C}}^{(*)}_{B,\hat{B}}, \overrightarrow{\psi}(x)\big)_{\mathbb{H}} - \big(J_B 1(x), \overrightarrow{\psi}(x)\big)_{\mathbb{H}} \qquad (10)$$
$$+ \int \big(k^*(x,y)h_B(y), \overrightarrow{\psi}(y) - \overrightarrow{\psi}(x)\big)_{\mathbb{H}} d\mu(y),$$

where

$$\mathcal{F}_B[\overrightarrow{\psi}](x) = \sum_{i=1}^N \Big[(f_{\overrightarrow{w}_i}(x) - m_{A\hat{B}} f_{\overrightarrow{w}_i})\psi_i(x) - \big\langle (f_{\overrightarrow{w}_i}(x) - m_{A\hat{B}} f_{\overrightarrow{w}_i}), l_{\hat{B}} \big\rangle\Big].$$

Suppose that $T^*(1\overrightarrow{w}_i) = 0$ for $i = 1, \dots, N$. In this case we have $\mathcal{F}_B[\overrightarrow{\psi}](x) = 0$ in (10)*. Also, if $\mathbb{A}_N = span\{\overrightarrow{w}_1, \dots, \overrightarrow{w}_N\}$, then T^* is a bounded-Lipschitz operator of order η for all $0 < \eta \le \alpha$ with $\mathbb{B}_1 = \mathbb{A}_N$ and $\mathbb{B}_2 = \mathbb{C}$.*

Moreover, the constant C appearing in (6) *and* (7) *is independent of the vectors $\overrightarrow{w}_1, \dots, \overrightarrow{w}_N$.*

From the above result, the Krein's Theorem (see Section 5) plays a key role in order to get L^2_μ boundedness of T and T^*.

Theorem 2.6. *Let T be an operator satisfying Hypothesis 2.1 with $\mathbb{H}$ a separable Hilbert space. In addition, let us suppose that $T1 = 0_{\mathbb{H}}$ and, for all vector $\overrightarrow{e_j}$ in the canonical base of $\mathbb{H}$, $T^*(1\overrightarrow{e_j}) = 0$. Then T and T^* can be extended to linear operators $T : L^2_\mu(\mathbb{R}^n, \mathbb{C}) \to L^2_\mu(\mathbb{R}^n, \mathbb{H})$ and $T^* : L^2_\mu(\mathbb{R}^n, \mathbb{H}) \to L^2_\mu(\mathbb{R}^n, \mathbb{C})$ respectively, in such manner that*

$$\|T\phi\|_{L^2_\mu(\mathbb{R}^n,\mathbb{H})} \le C\|\phi\|_{L^2_\mu(\mathbb{R}^n,\mathbb{C})} \quad \text{and} \quad \|T^*\overrightarrow{\psi}\|_{L^2_\mu(\mathbb{R}^n,\mathbb{C})} \le C\|\overrightarrow{\psi}\|_{L^2_\mu(\mathbb{R}^n,\mathbb{H})}.$$

Corollary 2.7. *Under the same Hypothesis as Theorem 2.6, the operator T^* satisfies equation* (10) *with $\mathcal{F}_B[\overrightarrow{\psi}] = 0$, for all $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$.*

The next results deal with antisymmetric kernels. The proofs are in Section 6.

Theorem 2.8. *If T is a linear operator associated to an* **antisymmetric** *kernel k satisfying (24) and (25), then T satisfies Hypothesis 2.1.*

In addition, if $T1 = g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$, then Theorem 2.4 is fulfilled. Also, for any $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$ with B a ball containing the support of $\overrightarrow{\psi}$ and μ-almost every $x \in B$, we have

$$T^*\overrightarrow{\psi}(x) = -\big((g(x) - m_{A\hat{B}} g), \overrightarrow{\psi}(x)\big)_{\mathbb{H}} + \big\langle (g - m_{A\hat{B}}(g)), l_{\hat{B}}\ \overrightarrow{\psi}(x)\big\rangle \qquad (11)$$
$$- \big(\bar{\mathtt{C}}_{B,\hat{B}}, \overrightarrow{\psi}(x)\big)_{\mathbb{H}} + \big(I_B 1(x), \overrightarrow{\psi}(x)\big)_{\mathbb{H}}$$
$$- \int \big(k(x,y), (\overrightarrow{\psi}(y) - \overrightarrow{\psi}(x))h_B(x)\big)_{\mathbb{H}}\, d\mu(y).$$

In particular, if $T1 = 0_{\mathbb{H}}$, then $T^*(1\overrightarrow{w}) = 0$ for all $\overrightarrow{w} \in \mathbb{H}$; also T^* is a bounded-Lipschitz operator of order η for all $0 < \eta \le \alpha$ with $\mathbb{B}_1 = \mathbb{H}$ and $\mathbb{B}_2 = \mathbb{C}$. .

Again, the Krein's Theorem leads us to proof L^2 boundedness by knowing the Lipschitz boundedness of the operators T and T^*.

Theorem 2.9. *Let T be as in Theorem 2.8. If $T1 = 0_{\mathbb{H}}$, then T and T^* are bounded linear operators from $L^2_\mu(\mathbb{R}^n, \mathbb{C})$ to $L^2_\mu(\mathbb{R}^n, \mathbb{H})$.*

Remark. The last two results have no restriction on the Hilbert space $\mathbb{H}$.

3 Technical details and previous results

3.1 Non-necessarily doubling measures on the Euclidean Space

The function d defined in (1) satisfies

$$d(x,z) \le c_n(d(x,y) + d(y,z)), \qquad \text{(we denote } c_n = 2^{n-1}\text{)}, \tag{12}$$

and

$$\left|d(x,y) - d(x',y)\right| \le n s^{1-\alpha} d(x,x')^\alpha, \qquad \text{(we denote } \alpha = 1/n\text{)}. \tag{13}$$

for every $x, x', y \in \mathbb{R}^n$, whenever $d(x,y) < s$ and $d(x',y) < s$.

Also, any constant A greater than c_n works on (12). We fix the value of A in all the paper.

The **ball centered in** $x \in \mathbb{R}^n$ **and radius** $s > 0$ is the set $B_s(x) = \{y | d(x,y) < s\}$. We denote aB the ball concentric with B and radius a times the radius of B. Also, for $z \in \mathbb{R}^n$ and $0 < R_1 < R_2$ we define the **annulus** $E(z, R_1, R_2) = B_{R_2}(x) \setminus B_{R_1}(x)$.

If m_n is the Lebesgue measure on $\mathbb{R}^n$, we have $m_n(B_s(x)) = 2^n s$.

Given a ball B and numbers $a > 1, b > 0$, we say that B is (a,b)**-doubling** if the following relation holds

$$\mu(aB) \le b\mu(B).$$

Lemma 3.1. *Let z be a point in $\operatorname{supp}(\mu)$ and σ be a positive number. If $b > a^\nu$ then there exists an (a,b)-doubling ball $\tilde{B}$ centered in z with radius$(\tilde{B}) \ge \sigma$.*

The proof is easy. See for example [T1].

If we consider a family of concentrinc balls $\{B_{r_\iota}(x_0)\}_{\iota \in I}$, with infimum radius $s > 0$, their intersection satisfies the inclusions

$$B_s(x_0) \subset \bigcap_{\iota \in I} B_{r_\iota}(x_0) \subset \{y : d(x_0, y) \le s\}.$$

This kind of intersection may be an "open" or a "closed" ball.

Definition 3.2. *We say that a set B is a* ***doubling ball*** *if there are $x_0 \in \mathbb{R}^n$ and $s > 0$, such that B satisfies the inclusions*

$$B_s(x_0) \subset B \subset \{y : d(x_0, y) \le s\}$$

and also B fulfills the condition

$$\mu(\mathfrak{a}B) \le \mathfrak{b}\mu(B), \tag{14}$$

with

$$\mathfrak{a} = 8A^4\rho, \qquad and \qquad \mathfrak{b} = 2\mathfrak{a},$$

where $\rho \geq 1$ *is a constant appearing in* (17).

Consequently, we will say that B *has radius* s *and is centered in* x_0.

Remark: The definition of $\mathfrak{a}B$ *is similar to which we have in the case that* B *is an "open" ball.*

Remark. In particular, since $0 < \nu \leq 1$ and $\mathfrak{a} > 1$, we have $\mathfrak{b} \geq 2\mathfrak{a}^\nu$. By using Lemma 3.1 it is not hard to see the following useful property

Proposition 3.3. *Given a ball* B *with radius* $s > 0$ *and center in the support of* μ, *there is a doubling ball containing* B *and concentric with* B, *having minimal radius.*

In the sequel we will denote by $\hat{B}$ the doubling ball associated to B given by Proposition 3.3.

The following Proposition states that there exist doubling balls of radius as small as we want, centered in every relevant point.

Proposition 3.4. *For* μ*-almost every point* $x \in \operatorname{supp}(\mu)$ *and any* $\delta_0 > 0$, *there exists a doubling ball* B *centered in* x *with radius* $\delta \leq \delta_0$.

See [T1] for a proof.

Lemma 3.5. ***(Differentiation)***. *Let* φ *be a (scalar or vector-valued) function that is locally* μ*-integrable. Then, for* μ*-almost every point* $x \in \mathbb{R}^n$ *there exists a sequence of doubling balls* $B_j = B_{s_j}(x)$ *such that* $s_j \to 0$ *and*

$$\lim_{j\to\infty} \frac{1}{\mu(B_j)} \int_{B_j} \varphi(y)d\mu(y) = \varphi(x).$$

Proof. It can be derived by standard arguments. □

3.2 Vector-valued measures and spaces of functions

For definitions and general theory of Bochner-measure, see for example [DU].

Let (X, μ) be some σ-finite measure space. Take two Banach spaces $\mathbb{A}, \mathbb{B}$, with norms $|\cdot|_{\mathbb{A}}$ and $|\cdot|_{\mathbb{B}}$, respectively. It is known that if f is a $\mathbb{B}$-valued μ-measurable function then f is Bochner-μ-integrable if and only if the scalar-valued function $|f|_{\mathbb{B}}$ is μ-integrable, in the usual sense.

We will often use the next Theorem without explicit mention.

Theorem 3.6. *Let* ℓ *be a bounded linear functional over the Banach space* $\mathbb{A}$. *If* F *is an* $\mathbb{A}$*-valued Bochner-*μ*-integrable function, then the following equality holds*

$$\ell\Big(\int F(x)\,d\mu(x)\Big) = \int \ell(F(x))\,d\mu(x).$$

We shall understand that $\mathbb{H}$ is a Hilbert space. For $\mathbb{H}$-valued functions, we consider the spaces of functions $L^p_\mu(\mathbb{R}^n,\mathbb{H})$, $L^\infty_\mu(\mathbb{R}^n,\mathbb{H})$ and $BMO_\rho(\mathbb{R}^n,\mathbb{H})$, endowed with the following norms, respectively

$$\|f\|_{L^p_\mu(\mathbb{R}^n,\mathbb{H})} := \Big(\int |f(x)|^p_{\mathbb{H}}\, d\mu(x)\Big)^{1/p}, \quad 1\le p<\infty, \tag{15}$$

$$\|f\|_{L^\infty_\mu(\mathbb{R}^n,\mathbb{H})} := \text{ess sup}_\mu\{|f(x)|_{\mathbb{H}} : x\in\mathbb{R}^n\}, \tag{16}$$

$$\|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})} := \sup_B \mu(\rho B)^{-1}\int_B |g(x)-m_B g|_{\mathbb{H}} d\mu(x), \tag{17}$$

where $m_B g = \mu(B)^{-1}\int_B g\, d\mu$ and $\rho\ge 1$ is fixed in the paper.

We denote by $\Lambda^\gamma(\mathbb{R}^n,\mathbb{H})$ the space of $\mathbb{H}$-valued functions $\overrightarrow{\psi}$ such that the next quantity is finite

$$\|\overrightarrow{\psi}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{H})} = \sup_{x,y\in \text{supp}(\mu), x\neq y} \frac{|\overrightarrow{\psi}(x)-\overrightarrow{\psi}(y)|_{\mathbb{H}}}{d(x,y)^\gamma}.$$

Also, we define

$$\Lambda^\gamma(B,\mathbb{H}) = \{\overrightarrow{\psi}\in\Lambda^\gamma(\mathbb{R}^n,\mathbb{H}) : \text{supp}(\overrightarrow{\psi})\subset B\} \qquad \text{with } B \text{ a ball}, \tag{18}$$

$$\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H}) = \bigcup_B \Lambda^\gamma(B,\mathbb{H}), B \text{ ranging on all balls of } \mathbb{R}^n, \tag{19}$$

$$\{\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H})\}_0 = \{\overrightarrow{\psi}\in\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H}) : \int \overrightarrow{\psi}\, d\mu = 0_{\mathbb{H}}\}, \tag{20}$$

$$\Lambda^\gamma_b(\mathbb{R}^n,\mathbb{H}) = \Lambda^\gamma(\mathbb{R}^n,\mathbb{H})\cap L^\infty_\mu(\mathbb{R}^n,\mathbb{H}) \tag{21}$$

and

$$\big(\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H})\big)' \text{ the space of all continuous linear functions on } \Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H}). \tag{22}$$

It is easy to prove that if $\overrightarrow{\psi}$ belongs to the space $\Lambda^\gamma(B,\mathbb{H})$, for some ball B with radius s, then the following inequality holds

$$\|\overrightarrow{\psi}\|_{L^\infty_\mu(\mathbb{R}^n,\mathbb{H})} \le C\|\overrightarrow{\psi}\|_{\Lambda^\gamma(B,\mathbb{H})} s^\gamma. \tag{23}$$

In $\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H})$ we define the topology which is the inductive limit of the spaces $\Lambda^\gamma(B,\mathbb{H})$. Also, it is not hard to see that the space $\Lambda^\gamma_0(\mathbb{R}^n,\mathbb{H})$ is dense in $L^p_\mu(\mathbb{R}^n,\mathbb{H})$ for each p, $1\le p<\infty$.

Notation. We will use the symbol $(\cdot,\cdot)_{\mathbb{H}}$ for the inner product in $\mathbb{H}$ and the symbol $\Big\langle F, G\Big\rangle$ for the distributional action. In particular, if G is an $\mathbb{H}$-valued locally sumable function and F has compact support, we write $\Big\langle F,G\Big\rangle = \int \Big(F(x),G(x)\Big)_{\mathbb{H}} d\mu$.

3.3 Auxiliary functions

We set certain auxiliary scalar-valued functions needed in the paper. We start with an infinitely differentiable function $h : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $h(t) = 1$ if $0 \leq t \leq 1$, $h(t) = 0$ if $t \geq A$ and $0 \leq h(t) \leq 1$ for all $t \geq 0$. For each ball B with center z and radius $s > 0$, we build functions h_B, h'_B, l_B, as follows

$$h_B(y) = h\left(\frac{d(y,z)}{7A^3 s}\right), h'_B(y) = h\left(\frac{d(y,z)}{s}\right) \text{ and } l_B(y) = \frac{1}{\int h'_B \, d\mu} h'_B(y).$$

Remark. We note that $\int l_B(y)\, d\mu(y) = 1$.

Proposition 3.7. *Given a ball $B = B_s(z)$, we have*

$$\|h_B\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})}, \|h'_B\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \leq Cs^{-\gamma}; \quad \|l_B\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \leq C\mu(B_s(z))^{-1} s^{-\gamma}.$$

Definition 3.8. *In order to get a more compact notation, we define:*

$$\mathfrak{c}^1 := 2A,\ \mathfrak{c}^2 := 5A^2,\ \mathfrak{c}^3 := 7A^3,\ \mathfrak{c}^4 := 8A^4,\ \mathfrak{c}^5 := 21A^5,\ \mathfrak{c}^6 := 32A^6.$$

3.4 Singular Integral Operators

Definition 3.9. *Let $1 \leq r \leq \infty$ and $r' = r/(r-1)$. We say that a function $k : (\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta) \to \mathbb{H}$ satisfies a L^r**-size condition**, if for any $R > 0$ the following condition holds*

$$\left[\int_{E(x,R,AR)} |k(x,y)|^r_{\mathbb{H}} \, d\mu(y)\right]^{1/r} \leq CR^{-\nu/r'}. \tag{24}$$

*We say that k satisfies a L^r**-smoothness condition**, if there exists η, $0 < \eta \leq \alpha$, such that*

$$\left[\int_{E(y,R,AR)} |k(y,x) - k(z,x)|^r_{\mathbb{H}} \, d\mu(x)\right]^{1/r} \leq CR^{-\nu/r'} \left(\frac{d(y,z)}{R}\right)^\eta, \tag{25}$$

whenever $Ad(y,z) \leq R$.

Remark. We denote $\Delta = \{(x,y) : x = y\}$.

Consider a linear continuous operator $T : \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C}) \to (\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H}))'$ for some γ, $0 < \gamma \leq \alpha$, associated to a kernel $k : (\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta) \to \mathbb{H}$ in the sense that for any function $\phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ and $x \notin \operatorname{supp}(\phi)$,

$$T\phi(x) = \int k(x,y)\phi(y)\, d\mu(y), \tag{26}$$

holds. The adjoint operator T^* of T is defined as the linear operator from $\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$ to $(\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C}))'$, by means of

$$\langle T^* \overrightarrow{\psi}, \phi\rangle = \langle T\phi, \overrightarrow{\psi}\rangle, \qquad (\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H}), \phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})). \tag{27}$$

This operator has the associated kernel $k^*(x,y) = k(y,x)$, in the sense that for any function $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$ and any point $x \notin \operatorname{supp}(\overrightarrow{\psi})$

$$T^* \overrightarrow{\psi}(x) = \int \left(k^*(x,y), \overrightarrow{\psi}(y)\right)_{\mathbb{H}} d\mu(y). \tag{28}$$

Definition 3.10. (W.B.P.) *We say that* T *has the* Weak Boundedness Property*, or* T *is* weakly bounded*, of order* γ*,* $0 < \gamma \leq \alpha$*, if for each* $\phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ *and* $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$ *such that* $\phi, \overrightarrow{\psi}$ *have their supports contained in* $B = B_s(x_0)$*, we have*

$$\left|\langle T\phi, \overrightarrow{\psi}\rangle\right| \leq C\mu(\rho B)\|\phi\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})}\|\overrightarrow{\psi}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{H})}s^{2\gamma}.$$

Definition 3.11. *We say that* T *satisfies the Meyer Commutation Property if, for all* $\phi, \varphi \in \Lambda_0^{\gamma'}(\mathbb{R}^n, \mathbb{C})$*,* $\overrightarrow{\psi} \in \Lambda_0^{\gamma'}(\mathbb{R}^n, \mathbb{H})$ *with* $\gamma' > \gamma$*, the following equality holds*

$$\langle T(\phi\varphi), \overrightarrow{\psi}\rangle = \langle T\phi, \overrightarrow{\psi}\varphi\rangle + \iint \left[\varphi(y) - \varphi(x)\right]\left(\phi(y)\, k(x,y), \overrightarrow{\psi}(x)\right)_{\mathbb{H}} d\mu(x)d\mu(y).$$

Remark. If T satisfies the Meyer Commutation Property, then T^* does it and we have

$$\langle T^*(\overrightarrow{\psi}\varphi), \phi\rangle = \langle T^*\overrightarrow{\psi}, \phi\varphi\rangle + \iint \left[\varphi(y) - \varphi(x)\right]\left(\overrightarrow{\psi}(y), k^*(x,y)\right)_{\mathbb{H}}\phi(x)\, d\mu(y)d\mu(x).$$

The proof is easily derived after applying the Meyer Commutation Property of T and the change of variables $(x, y) \to (y, x)$.

3.5 Previous Lemmas

The proof of several results of the present subsection follow the same lines as their analogues in Section 2 of [MST]. The differences are a consequence of the non-necessarily doubling context and the presence of vector-valued functions. Hence, we only develop those details that are actually different.

In this subsection we will suppose that T satisfies Hypothesis 2.1.

The following two Lemmas, using size and smoothness conditions of the kernels, are basic.

Lemma 3.12. *Suppose that* k *and* k^* *satisfy condition* (24)*.*

If $0 < \eta \leq \alpha$*, then for all* $s > 0$ *and* $x \in \mathbb{R}^n$

$$\int_{B_s(x)} d(x,y)^\eta |k(x,y)|_{\mathbb{H}} d\mu(y) \leq Cs^\eta. \tag{29}$$

$$\int_{E(x,A^{-2}(A-1)s,2A^2s)} |k(x,y)|_{\mathbb{H}} d\mu(y) \leq C. \tag{30}$$

Replacing $k(x,y)$ *by* $k^*(x,y)$ *the same inequalities are still valid.*

Proof. For (29) proceed as in Lemma 2.7 of [MST], using (2) and (24). For (30) use Hölder inequality, (2) and (24), to get

$$\begin{aligned}
&\int_{E(x,A^{-2}(A-1)s,2A^2s)} |k(x,y)|_{\mathbb{H}} d\mu(y) \\
&\leq C\Big(\int_{E(x,A^{-2}s,2A^2s)} |k(x,y)|_{\mathbb{H}}^r d\mu(y)\Big)^{1/r}\left(\mu(B_{2A^2s}(x))\right)^{1/r'} \leq C.
\end{aligned}$$

□

Lemma 3.13. *Suppose that k and k^* satisfy condition (25). Given a ball $B = B_s(z)$, if $x_1, x_2 \in B_{\mathfrak{c}^1 s}(z)$ then*

$$\Big|\int (k(x_1,y) - k(x_2,y))(1 - h_B(y))d\mu(y)\Big|_{\mathbb{H}} \leq C\Big[\frac{d(x_1,x_2)}{s}\Big]^{\eta} \leq C. \tag{31}$$

In addition, if k satisfies (25) with exponent $\eta + \epsilon$, $\epsilon > 0$, we have

$$\int_{Ad(x_1,x_2)<d(x_1,y)} d(x_2,y)^{\eta}|k(x_1,y) - k(x_2,y)|_{\mathbb{H}}d\mu(y) \leq C\, d(x_1,x_2)^{\eta}. \tag{32}$$

Analogous estimates are true for $k^*(x,y)$.

Proof. To obtain (31), we can proceed as in Lemma 2.13 of [MST].

For (32), applying Hölder inequality and using (25) and (2) we have

$$\begin{aligned}
&\int_{Ad(x_1,x_2)<d(x_1,y)} d(x_2,y)^{\eta}|k(x_1,y) - k(x_2,y)|_{\mathbb{H}}d\mu(y)\\
&\leq \sum_{j=0}^{\infty} \Big[\int_{E(x_1,A^jAd(x_1,x_2),A^{j+1}Ad(x_1,x_2))} |k(x_1,y) - k(x_2,y)|_{\mathbb{H}}^{r}d\mu(y)\Big]^{\frac{1}{r}} \cdot\\
&\qquad \cdot\Big[\int_{E(x_1,A^jAd(x_1,x_2),A^{j+1}Ad(x_1,x_2))} d(x_2,y)^{\eta r'}d\mu(y)\Big]^{\frac{1}{r'}}\\
&\leq C\sum_{j=0}^{\infty}(A^{j+1}d(x_1,x_2))^{-\frac{\nu}{r'}}(A^{-(j+1)})^{\eta+\epsilon}\,(A^{j}d(x_1,x_2))^{\eta+\frac{\nu}{r'}}\\
&\leq Cd(x_1,x_2)^{\eta}.
\end{aligned}$$

□

The next Lemma enables to extend the definition of T and T^* to bounded Lipschitz functions.

Lemma 3.14. *Suppose that k and k^* satisfy condition (25) and let B be a ball in $\mathbb{R}^n$. If $\phi \in L^{\infty}_{\mu}(\mathbb{R}^n,\mathbb{C})$ and $\overrightarrow{\psi} \in L^{\infty}_{\mu}(\mathbb{R}^n,\mathbb{H})$ then $I_B\phi(x)$, $I_B^{(*)}\overrightarrow{\psi}(x)$ and $J_B\phi(x)$ are well defined for μ-almost every $x \in \mathfrak{c}^1B$.*

Further, if $\phi \in \Lambda_b^{\gamma}(\mathbb{R}^n,\mathbb{C})$ and $\overrightarrow{\psi} \in \Lambda_b^{\gamma}(\mathbb{R}^n,\mathbb{H})$, $0 < \gamma \leq \alpha$, then $I_B\phi \in (\Lambda^{\gamma}(B,\mathbb{H}))'$, $I_B^{(*)}\overrightarrow{\psi} \in (\Lambda^{\gamma}(B,\mathbb{C}))'$, $J_B\phi \in (\Lambda^{\gamma}(B,\mathbb{H}))'$ and

$$\Big|\int \big(I_B\phi(x), \overrightarrow{\psi}(x)\big)_{\mathbb{H}}d\mu(x)\Big| \leq C\mu(\mathfrak{c}^1B)\|\phi\|_{L^{\infty}_{\mu}(\mathbb{R}^n,\mathbb{C})}\|\overrightarrow{\psi}\|_{L^{\infty}_{\mu}(\mathbb{R}^n,\mathbb{H})},$$

with estimates completely analogous for $I_B^{(*)}$ and J_B.

Proof. Apply Cauchy-Schwartz inequality to write

$$\Big|\int \Big(I_B\phi(x), \overrightarrow{\psi}(x)\Big)_{\mathbb{H}}d\mu(x)\Big| \leq \int |I_B\phi(x)|_{\mathbb{H}}\,|\overrightarrow{\psi}(x)|_{\mathbb{H}}d\mu(x),$$

and proceed as in [MST, Lemma 2.14]. Do the same for $I_B^{(*)}$ and J_B. □

Definition 3.15. *Given a ball B, we define $T_B : \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{C}) \to (\{\Lambda^\gamma(B, \mathbb{H})\}_0)'$ and $T_B^{(*)} : \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{H}) \to (\{\Lambda^\gamma(B, \mathbb{C})\}_0)'$ as follows*

$$T_B\phi = T(\phi h_B) + I_B\phi, \qquad \phi \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{C}),$$
$$T_B^{(*)}\overrightarrow{\psi} = T^*(\overrightarrow{\psi} h_B) + I_B^{(*)}\overrightarrow{\psi}, \quad \overrightarrow{\psi} \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{H}).$$

Lemma 3.16. *For each pair of balls $B_1 = B_{r_1}(z_1) \subset B_2 = B_{r_2}(z_2)$, for all $\phi \in \Lambda_b^\gamma(\mathbb{R}^n)$ and all $\overrightarrow{\psi} \in \{\Lambda^\gamma(B_1, \mathbb{H})\}_0$, the following equality holds*

$$\langle T_{B_1}\phi, \overrightarrow{\psi}\rangle = \langle T_{B_2}\phi, \overrightarrow{\psi}\rangle.$$

In similar way, for all $\phi \in \{\Lambda^\gamma(B_1, \mathbb{C})\}_0$ and all $\overrightarrow{\psi} \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{H})$ we have

$$\langle T_{B_1}^{(*)}\overrightarrow{\psi}, \phi\rangle = \langle T_{B_2}^{(*)}\overrightarrow{\psi}, \phi\rangle.$$

Proof. Proceeding as in [MST, Lemma 2.16], we can obtain that

$$-T(\phi(h_{B_2} - h_{B_1}))(z_1) = \int k(z_1, y)\phi(y)\big(h_{B_1}(y) - h_{B_2}(y)\big)\, d\mu(y)$$

and

$$-I_{B_2}\phi(z_1) = \int \big(k(z_2, y) - k(z_1, y)\big)(1 - h_{B_2}(y))\, \phi(y)\, d\mu(y).$$

So, since $\int \overrightarrow{\psi}\, d\mu = 0_{\mathbb{H}}$, we have

$$\begin{aligned}\langle T_{B_2}\phi, \overrightarrow{\psi}\rangle - \langle T_{B_1}\phi, \overrightarrow{\psi}\rangle &= \langle T_{B_2}\phi, \overrightarrow{\psi}\rangle - \langle T(\phi h_{B_1}), \overrightarrow{\psi}\rangle - \langle I_{B_1}\phi, \overrightarrow{\psi}\rangle \\ &= \int \bigg(\int \big(k(x, y) - k(z_1, y)\big)\phi(y)[1 - h_{B_1}(y)]d\mu(y), \overrightarrow{\psi}(x)\bigg)_{\mathbb{H}} d\mu(x) \\ &\quad - \langle I_{B_1}\phi, \overrightarrow{\psi}\rangle = 0.\end{aligned}$$

The proof that $\langle T_{B_1}^{(*)}\overrightarrow{\psi}, \phi\rangle = \langle T_{B_2}^{(*)}\overrightarrow{\psi}, \phi\rangle$ is very similar, using in this case $I_B^{(*)}$ in place of I_B and the identities

$$T^*(\overrightarrow{\psi}(h_{B_2} - h_{B_1}))(z_1) = \int \big(k^*(z_1, y), \overrightarrow{\psi}(y)\big)_{\mathbb{H}}\big(h_{B_2}(y) - h_{B_1}(y)\big)\, d\mu(y)$$

and

$$-I_{B_2}^{(*)}\overrightarrow{\psi}(z_1) = \int \big(\big(k^*(z_2, y) - k^*(z_1, y)\big)(1 - h_{B_2}(y)), \overrightarrow{\psi}(y)\big)_{\mathbb{H}} d\mu(y).$$

□

It is clear that $\langle T_B\phi, \overrightarrow{\psi}\rangle = \langle T\phi, \overrightarrow{\psi}\rangle$, whenever $\operatorname{supp}(\phi) \subset B$, and $\langle T_B^{(*)}\overrightarrow{\psi}, \phi\rangle = \langle T^*\overrightarrow{\psi}, \phi\rangle$, if $\operatorname{supp}(\overrightarrow{\psi}) \subset B$. So, the preceding Lemma enables to introduce the following extensions of T and T^*.

Definition 3.17. *For every $\phi \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{C})$ we define*

$$\langle T\phi, \overrightarrow{\psi}\rangle = \langle T_B\phi, \overrightarrow{\psi}\rangle, \text{ for all } \overrightarrow{\psi} \in \{\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})\}_0 \text{ and } \operatorname{supp}(\overrightarrow{\psi}) \subset B.$$

In the same way, for every $\overrightarrow{\psi} \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{H})$ we define

$$\langle T^*\overrightarrow{\psi}, \phi\rangle = \langle T_B^{(*)}\overrightarrow{\psi}, \phi\rangle \quad \text{for all } \phi \in \{\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})\}_0 \text{ and } \operatorname{supp}(\phi) \subset B.$$

Definition 3.18. *For a ball B and each function $g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$, by using the Riesz Representation Theorem, we define the vector $\mathtt{C}_{B,\hat{B}}[g] \in \mathbb{H}$ as*

$$(\mathtt{C}_{B,\hat{B}}[g], \overrightarrow{w})_{\mathbb{H}} = \left\langle Th_B + I_B 1 - (g - m_{A\hat{B}}(g)), l_{\hat{B}}\, \overrightarrow{w} \right\rangle \quad \textit{for all } \overrightarrow{w} \in \mathbb{H}.$$

On the other hand, for each function f in $BMO_\rho(\mathbb{R}^n, \mathbb{C})$, a fixed vector $\overrightarrow{w} \in \mathbb{H}$, and a ball B, we define the number $\mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}]$ by

$$\mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}] = \langle T^*(h_B\, \overrightarrow{w}) + I^{(*)}_B(1\overrightarrow{w}) - (f - m_{A\hat{B}}(f)), l_{\hat{B}} \rangle.$$

The quantity $\mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}]$ cannot be a vector of $\mathbb{H}$ since is not linear in $\overrightarrow{w}$.

In a similar way, we also define the vectors $\bar{\mathtt{C}}_{B,\hat{B}}$ and $\bar{\mathtt{C}}^{(*)}_{B,\hat{B}}$ by

$$(\bar{\mathtt{C}}_{B,\hat{B}}, \overrightarrow{w})_{\mathbb{H}} = \langle Th_B + I_B 1, l_{\hat{B}}\, \overrightarrow{w} \rangle \qquad \textit{for all } \overrightarrow{w} \in \mathbb{H}$$

and

$$(\bar{\mathtt{C}}^{(*)}_{B,\hat{B}}, \overrightarrow{w})_{\mathbb{H}} = \langle T^*(h_B \overrightarrow{w}) + I^{(*)}_B(1\overrightarrow{w}), l_{\hat{B}} \rangle \qquad \textit{for all } \overrightarrow{w} \in \mathbb{H}.$$

Remark. The quantities defined above, satisfy the next relations

$$\begin{aligned} (\mathtt{C}_{B,\hat{B}}[g], \overrightarrow{w})_{\mathbb{H}} &= (\bar{\mathtt{C}}_{B,\hat{B}}, \overrightarrow{w})_{\mathbb{H}} - \langle (g - m_{A\hat{B}}(g)), l_{\hat{B}}\, \overrightarrow{w} \rangle, \\ \mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}] &= (\bar{\mathtt{C}}^{(*)}_{B,\hat{B}}, \overrightarrow{w})_{\mathbb{H}} - \langle (f - m_{A\hat{B}}(f)), l_{\hat{B}} \rangle. \end{aligned}$$

The next Lemma follows easily in the context of doubling measures, however it is not straightforward in the case of non-necessarily doubling measures.

Lemma 3.19. *Given a ball B and $w \in \mathbb{H}$ we get*

$$\left| \left\langle Th_B, l_{\hat{B}} \overrightarrow{w} \right\rangle \right| \leq C |\overrightarrow{w}|_{\mathbb{H}}, \quad \left| \left\langle T^*(h_B\, \overrightarrow{w}), l_{\hat{B}} \right\rangle \right| \leq C |\overrightarrow{w}|_{\mathbb{H}}.$$

with constant C not depending on B and $\overrightarrow{w}$.

Proof. Without loss of generality we suppose that $|\overrightarrow{w}|_{\mathbb{H}} = 1$. Let $B = B_s(z)$ and denote $\hat{s}$ the radius of $\hat{B}$. Also, we denote $B^1 = B_{3A^2 s}(z)$.

In the following we assume that any ball is centered in z and we recall that:

The function h_B is 1 on $B_{\mathfrak{c}^3 s}$, null on $B^c_{7A^4 s}$, and supported on $B_{\mathfrak{c}^4 s}$.

The function $h'_{\hat{B}}$ is 1 on $B_{\hat{s}}$, null in $B^c_{A\hat{s}}$, and then supported on $B_{\mathfrak{c}^1 \hat{s}}$.

The function h_{B^1} is 1 on $B_{\mathfrak{c}^5 s}$ null on $B^c_{21A^6 s}$, and supported on $B_{\mathfrak{c}^6 s}$. Hence the function $1 - h_{B^1}$ is null on $B^c_{\mathfrak{c}^5 s}$.

1^{st} *case:* Suppose that $\mathfrak{c}^6 s \leq \hat{s}$. We write

$$\left\langle Th_B, l_{\hat{B}} \overrightarrow{w} \right\rangle = \left\langle Th_B, h_{B^1} l_{\hat{B}}\, \overrightarrow{w} \right\rangle + \left\langle Th_B, (1 - h_{B^1}) l_{\hat{B}}\, \overrightarrow{w} \right\rangle =: J_1 + J_2.$$

Observe that $B_{\mathfrak{c}^6 s} \subset \hat{B}$, and $\rho\hat{B} \subset \mathfrak{a}\hat{B}$. Also h_B and $h_{B^1} l_{\hat{B}}$ are both supported on $B_{\mathfrak{c}^6 s}$ and $h_{B^1} h'_{\hat{B}} = h_{B^1}$. Now, using the Weak Boundedness Property, Proposition 3.7, the assumption $|\overrightarrow{w}|_{\mathbb{H}} = 1$ and the doubling condition of $\hat{B}$, we get

$$\begin{aligned}
|J_1| &\leq \mu(\rho B_{\mathfrak{c}^6 s}) C s^{2\gamma} \|h_B\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \|h_{B^1} l_{\hat{B}}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \\
&\leq C\mu(\rho B_{\hat{s}}) s^\gamma \, \frac{1}{\int h_{\hat{B}}} \|h_{B^1} h'_{\hat{B}}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \\
&\leq C\mu(\mathfrak{a}\hat{B}) s^\gamma \, \frac{1}{\mu(\hat{B})} \|h_{B^1}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \\
&\leq C\mathfrak{b} s^\gamma \, (4A^2 \mathfrak{c}^3 s)^{-\gamma} \leq C\mathfrak{b}.
\end{aligned}$$

Since h_B and $(1 - h_{B^1}) l_{\hat{B}}$ have disjoint supports, we have

$$J_2 = \iint \Big(k(x,y), \overrightarrow{w}\Big)_{\mathbb{H}} \, h_B(y)(1 - h_{B^1}(x)) l_{\hat{B}}(x) \, d\mu(x) d\mu(y).$$

Then, applying Hölder inequality, condition (24) and the fact that $\mathfrak{c}^4 s \leq \hat{s}$, we have

$$\begin{aligned}
|J_2| \; &\leq \int_{\mathfrak{c}^4 B} \Big| \int_{x\in\hat{B}} \left(k(x,y), \overrightarrow{w}\right)_{\mathbb{H}} h_B(y)(1 - h_{B^1}(x)) l_{\hat{B}}(x) \, d\mu(x) \Big| d\mu(y) \\
&\leq \frac{1}{\int h'_{\hat{B}}} \int_{\mathfrak{c}^4 B} \int_{E(y,\mathfrak{c}^4 s, \mathfrak{c}^1 \hat{s})} |k(x,y)|_{\mathbb{H}} \; d\mu(x) d\mu(y) \\
&\leq \frac{\mu(\mathfrak{c}^1 \hat{B})^{\frac{1}{r'}}}{\mu(\hat{B})} \int_{\mathfrak{c}^4 B} \Big[\sum_{m=0}^{\infty} \int_{E(y,\mathfrak{c}^4 s A^m, \mathfrak{c}^4 s A^{m+1})} |k(x,y)|^r_{\mathbb{H}} \; d\mu(x) \Big]^{\frac{1}{r}} d\mu(y) \\
&\leq \frac{\mu(\mathfrak{c}^1 \hat{B})^{\frac{1}{r'}}}{\mu(\hat{B})} \mu(\mathfrak{c}^4 B) C \mu(\mathfrak{c}^4 B)^{-\frac{1}{r'}} \leq C\mathfrak{b} \left(\frac{\mu(\mathfrak{c}^4 B)}{\mu(\hat{B})} \right)^{\frac{1}{r}} \leq C.
\end{aligned}$$

2^{nd} *case.* We now analyze the opposite situation, having $s \leq \hat{s} < \mathfrak{c}^6 s$.

Since the functions h_B and $h'_{\hat{B}}$ have their supports contained in $\mathfrak{c}^4 \hat{B}$, applying the Weak Boundedness Property of T, Proposition 3.7, the doubling condition of $\hat{B}$ and the fact that $s \approx \hat{s}$, we obtain

$$\left| \left\langle T h_B, l_{\hat{B}} \overrightarrow{w} \right\rangle \right| = \frac{1}{\int h_{\hat{B}}} \left| \left\langle T h_B, h'_{\hat{B}} \overrightarrow{w} \right\rangle \right| \leq C \frac{1}{\mu(\hat{B})} \mu(\mathfrak{a}\hat{B}) \, \hat{s}^{2\gamma} s^{-\gamma} \hat{s}^{-\gamma} \leq \mathfrak{b} C.$$

The proof that $\left| \left\langle T^*(h_B \overrightarrow{w}), l_{\hat{B}} \right\rangle \right| \leq C |\overrightarrow{w}|_{\mathbb{H}}$ is very similar. □

Lemma 3.20. *Given $g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$ and a ball B, we get*

$$\left| \mathsf{C}_{B,\hat{B}}[g] \right|_{\mathbb{H}} \leq C_g \qquad \text{and} \qquad \left| \bar{\mathsf{C}}_{B,\hat{B}} \right|_{\mathbb{H}} \leq C, \tag{33}$$

where C_g only depends on $\|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})}$ and C does not depend on g and B. Further, for $f \in BMO_\rho(\mathbb{R}^n, \mathbb{C})$, $\overrightarrow{w} \in \mathbb{H}$ and a ball B, we get

$$\left| \mathsf{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}] \right|_{\mathbb{H}} \leq C |\overrightarrow{w}|_{\mathbb{H}} + C_f, \qquad \left| \bar{\mathsf{C}}^{(*)}_{B,\hat{B}} \right|_{\mathbb{H}} \leq C, \tag{34}$$

where C_f depends on $\|f\|_{BMO_\rho(\mathbb{R}^n,\mathbb{C})}$, but not on $\overrightarrow{w}$ or B.

Proof. By Lemma 3.14, inequality (23), Proposition 3.7, the chain of inclusions $\mathfrak{c}^1 B \subset \mathfrak{c}^1 \hat{B} \subset \mathfrak{a}\hat{B}$ and the fact that $\hat{B}$ is doubling, we have

$$\left| \left(\int I_B 1(x) l_{\hat{B}}(x) d\mu(x), \overrightarrow{w} \right)_{\mathbb{H}} \right| \leq C\mu(\mathfrak{c}^1 B)\hat{s}^\gamma \|l_{\hat{B}}\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} |\overrightarrow{w}|_{\mathbb{H}} \leq C|\overrightarrow{w}|_{\mathbb{H}}.$$

Analogously $\left| \int \left(I_B^{(*)}(1\overrightarrow{w})(x), l_{\hat{B}}(x) \right)_{\mathbb{H}} d\mu(x) \right| \leq C|\overrightarrow{w}|_{\mathbb{H}}$.

Let $g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$. Since $0 \leq l_{\hat{B}}(x) \leq 1/\mu(\hat{B})$, $\rho A\hat{B} \subset \mathfrak{a}\hat{B}$, and using the fact that $\hat{B}$ is doubling, we get

$$\begin{aligned} \left| \langle g - m_{A\hat{B}} g, l_{\hat{B}} \overrightarrow{w} \rangle \right| &\leq \frac{1}{\mu(\hat{B})} \int_{A\hat{B}} \left| g(x) - m_{A\hat{B}} g \right|_{\mathbb{H}} d\mu(x) |\overrightarrow{w}|_{\mathbb{H}} \\ &\leq C \frac{1}{\mu(\hat{B})} \mu(\rho A\hat{B}) \|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})} |\overrightarrow{w}|_{\mathbb{H}} \\ &\leq C\mathfrak{b} \|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})} |\overrightarrow{w}|_{\mathbb{H}}, \end{aligned}$$

In similar way, we can prove that if $f \in BMO_\rho(\mathbb{R}^n, \mathbb{C})$, then

$$\left| \left\langle (f - m_{A\hat{B}} f), l_{\hat{B}} \right\rangle \right| \leq C\|f\|_{BMO_\rho(\mathbb{R}^n,\mathbb{C})}.$$

From these estimates, Lemma 3.19 and the Riesz Representation Theorem, the proof of the Lemma is finished. □

Lemma 3.21. *If $T1 = g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$, for any ball B and $\overrightarrow{\psi} \in \Lambda^\gamma(B, \mathbb{H})$, we have*

$$\begin{aligned} \langle Th_B, \overrightarrow{\psi} \rangle = &\int \left(g(x) - m_{A\hat{B}}(g), \overrightarrow{\psi}(x) \right)_{\mathbb{H}} d\mu(x) \\ &+ \left(\mathtt{C}_{B,\hat{B}}[g], \int \overrightarrow{\psi}(x)\, d\mu(x) \right)_{\mathbb{H}} - \int \left(I_B 1(x), \overrightarrow{\psi}(x) \right)_{\mathbb{H}} d\mu(x), \end{aligned}$$

where $\sup_B \left| \mathtt{C}_{B,\hat{B}}[g] \right|_{\mathbb{H}} \leq C_g$.

Remark. If $\mathbb{H} = \mathbb{C}$ and B is $(\mathfrak{a}, \mathfrak{b})$-doubling (that is, $B = \hat{B}$), we recovered Lemma 2.18 of **[MST]**.

Proof. Proceeding as in the proof of Lemma 2.18 of **[MST]**, adding $\pm l_{\hat{B}} \int \overrightarrow{\psi}$ and $\pm \left(m_{A\hat{B}} g, \int \overrightarrow{\psi} d\mu \right)_{\mathbb{H}}$, we clearly have

$$\langle Th_B + I_B 1, \overrightarrow{\psi} \rangle = \int \left(g(x) - m_{A\hat{B}} g, \overrightarrow{\psi}(x) \right)_{\mathbb{H}} d\mu(x) + \left(\mathtt{C}_{B,\hat{B}}[g], \int \overrightarrow{\psi} d\mu \right)_{\mathbb{H}},$$

where $\mathtt{C}_{B,\hat{B}}[g]$ is the vector defined in 3.18. In view of Lemma 3.19 we get that $\sup_B \left| \mathtt{C}_{B,\hat{B}}[g] \right|_{\mathbb{H}} \leq C_g$, as we desired. □

Lemma 3.22. *If $\overrightarrow{w} \in \mathbb{H}$ and $T^*(1\overrightarrow{w}) = f \in BMO_\rho(\mathbb{R}^n, \mathbb{C})$ then for any ball B and $\phi \in \Lambda^\gamma(B, \mathbb{C})$, we have*

$$\begin{aligned} \left\langle T^*(h_B \overrightarrow{w}), \phi \right\rangle = &\int (f(x) - m_{A\hat{B}}(f))\phi(x) d\mu(x) + \mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}] \int \phi(x) d\mu(x) \\ &- \int I_B^{(*)}(1\overrightarrow{w})(x)\phi(x) d\mu(x), \end{aligned}$$

where $\sup_B \left| \mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}] \right| \leq C|\overrightarrow{w}|_{\mathbb{H}} + C_f$.

Proof. We write $\langle T^*(h_B\overrightarrow{w}) + I_B^{(*)}(1\overrightarrow{w}), \phi\rangle$ in similar way as in the preceding Lemma. Then we use (34). □

The next two Corollaries follow easily from Lemmas 3.21 and 3.22.

Corollary 3.23. *The function $g \in L^\infty_\mu(\mathbb{R}^n, \mathbb{H})$ if and only if $\left|\langle Th_B, \overrightarrow{\psi}\rangle\right| \le C\|\overrightarrow{\psi}\|_{L^1_\mu(B,\mathbb{H})}$ for all $\overrightarrow{\psi} \in \Lambda^\gamma(B, \mathbb{H})$, where C is a constant not depending on B.*

Corollary 3.24. *If $T^*(1\overrightarrow{w}) = 0$ for some $\overrightarrow{w} \in \mathbb{H}$, then $\left|\langle T^*(h_B\overrightarrow{w}), \phi\rangle\right| \le C|\overrightarrow{w}|_{\mathbb{H}}\|\phi\|_{L^1_\mu(B)}$ for all $\phi \in \Lambda^\gamma(B, \mathbb{C})$, where C neither depends on $\overrightarrow{w}$ nor B.*

Given a ball B, for $\phi \in \Lambda^\gamma(B, \mathbb{C})$ and $x \in B$, we define

$$T^B\phi(x) = (g(x) - m_{A\hat{B}}g)\phi(x) + \mathtt{C}_{B,\hat{B}}[g]\phi(x) - I_B 1(x)\phi(x) \qquad (35)$$
$$+ \int [\phi(y) - \phi(x)]k(x,y)h_B(y)\, d\mu(y).$$

Consider a pair of balls $B_1 = B_{r_1}(z_1) \subset B_2 = B_{r_2}(z_2)$. Then we can prove the equality

$$T^{B_2}\phi(x) = T^{B_1}\phi(x), \qquad \text{for } \mu\text{-almost } x \in B_1.$$

This can be done by writing $\left(\mathtt{C}_{B_2,\hat{B}_2}[g] - \mathtt{C}_{B_1,\hat{B}_1}[g], \overrightarrow{w}\right)_{\mathbb{H}}$, using Definition 3.18 and proceeding as in [MST, Lemma 2.21].

Given a ball B, for $\psi \in \Lambda^\gamma(B, \mathbb{C})$, $\overrightarrow{w} \in \mathbb{H}$ and $x \in B$, we define

$$S^B_{\overrightarrow{w}}\psi(x) = (f(x) - m_{A\hat{B}}f)\psi(x) + \mathtt{C}^{(*)}_{B,\hat{B}}[f, \overrightarrow{w}]\psi(x) - I_B^{(*)}(1\overrightarrow{w})(x)\psi(x) \qquad (36)$$
$$+ \int [\psi(y) - \psi(x)]\left(k^*(x,y), \overrightarrow{w}\right)_{\mathbb{H}} h_B(y) d\mu(y).$$

Again, we have that $S^{B_1}_{\overrightarrow{w}}\psi(x) = S^{B_2}_{\overrightarrow{w}}\psi(x)$ for all $x \in B_1$, with $B_1 \subset B_2$. is well defined.

The discusion above enables to define $\tilde{T}\phi$ as the function

$$\tilde{T}\phi(x) = T^B\phi(x) \qquad \text{for } \phi \in \Lambda^\gamma_0(\mathbb{R}^n, \mathbb{C}),$$

where B is a ball containing the support of ϕ and $x \in B$.

In the same way, we can define $\tilde{S}_{\overrightarrow{w}}\psi$ as the function

$$\tilde{S}_{\overrightarrow{w}}\psi(x) = S^B_{\overrightarrow{w}}\psi(x) \qquad (x \in B),$$

where B is a ball containing the support of ψ.

4 Proofs of main results: Lipschitz bounds and explicit formulas.

Lemma 4.1. *Let T be a linear operator satisfying Hypothesis 2.1. Given $x_1, x_2 \in \mathbb{R}^n$ consider the balls $B_1 = B_{d(x_1,x_2)}(x_1)$ and $B = B_s(x_1)$ such that $x_1, x_2 \in B$ with $Ad(x_1, x_2) < s$.*

If $T1 \in L^\infty_\mu(\mathbb{R}^n, \mathbb{H})$, then

$$\left|\int k(x_1,y)h_B(y)(1 - h_{B_1}(y))d\mu(y)\right|_{\mathbb{H}} \le C \qquad (37)$$

If $T^*(1\overrightarrow{w}) = 0$ for some $\overrightarrow{w} \in \mathbb{H}$, then

$$\left|\left(\int k^*(x_1,y)h_B(y)(1 - h_{B_1}(y))d\mu(y), w\right)_{\mathbb{H}}\right| \le C|w|_{\mathbb{H}} \qquad (38)$$

Remark. Actually, the inequalities (37) and (38) are true for almost every x_1. However, that is enough for the proof of Theorem 2.4.

Proof. Since $x_1 \notin \text{supp}(h_B(1-h_{B_1}))$ and $B_1 \subset B$, by definition of the associated kernel of T we get that the integral in (37) is equal to

$$|T(h_B - h_{B_1})(x_1)|_{\mathbb{H}} \leq \big(|Th_B(x_1)|_{\mathbb{H}} + |Th_{B_1}(x_1)|_{\mathbb{H}}\big).$$

Since $T1 \in L^\infty_\mu(\mathbb{R}^n, \mathbb{H})$, by Corollary 3.23 and duality, it follows (37).

For (38) we proceed in analogous way using now Corolary 3.24. □

Proof of Theorem 2.4. By using the Meyer Commutation Property and Lemma 3.21, it can be seen that $T\phi = \tilde{T}\phi$, and this proves (8).

Now, if $T1 = 0_{\mathbb{H}}$, using Lemma 3.13, Lemma 3.12(29) and inequalities (37) and (33), the proof that T is a bounded-Lipschitz operator follows the same lines of [MST, Theorem 2.32] □

Proof of Theorem 2.5. We apply the Meyer Commutation Property of T^* in order to prove that $\tilde{S}_{\overrightarrow{w}}\psi = T^*(\psi\, \overrightarrow{w})$, with ψ a scalar-valued function and $\overrightarrow{w} \in \mathbb{H}$. Now, we can easily deduce (10) for a finite linear combination $\overrightarrow{\psi} = \psi_1 \overrightarrow{w}_1 + \ldots + \psi_N \overrightarrow{w}_N$.

The rest of the Theorem again follows the same lines of [MST, Theorem 2.32], using this time Lemma 3.13, Lemma 3.12(30) and equations (38) and (34). We only note that the operator J_B is used after writing $I_B^{(*)}(1\overrightarrow{\psi}(x)) = \Big(J_B(1), \overrightarrow{\psi}(x)\Big)_{\mathbb{H}}$. Also, in some step of the proof (38) must be used with $\overrightarrow{w} = \overrightarrow{\psi}(x_2) - \overrightarrow{\psi}(x_1)$. □

Observation 4.2. *Let T be the operator $T\phi(x) = g(x)\phi(x)$. Suppose that $g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$ and T satisfies the Weak Boundedness Property of order γ. Then*

$$|g(x_0)|_{\mathbb{H}} \leq C, \tag{39}$$

for μ-almost every x_0 in the support of μ.

Proof. Take $x_0 \in \text{supp}(\mu)$ and let $\{\hat{B}_j = B_{\hat{s}_j}(x_0)\}_{j=1}^\infty$ be a sequence of doubling balls such that $\lim_{j\to} \hat{s}_j = 0$. Fix $\overrightarrow{w} \in \mathbb{H}$ with $|\overrightarrow{w}|_{\mathbb{H}} = 1$. Now, using that $\int l_{\hat{B}_j} d\mu = 1$ and the equalities

$$\begin{aligned}\int (g(x), \overrightarrow{w})_{\mathbb{H}} l_{\hat{B}_j}(x) d\mu(x) &= \int (h_{\hat{B}_j}(x) g(x), \overrightarrow{w})_{\mathbb{H}} l_{\hat{B}_j}(x) d\mu(x) \\ &= \langle Th_{\hat{B}_j}, l_{\hat{B}_j}\overrightarrow{w}\rangle,\end{aligned}$$

we obtain that

$$\begin{aligned}\Big|(m_{\hat{B}_j} g, \overrightarrow{w})_{\mathbb{H}}\Big| &= \Big|\int (m_{\hat{B}_j} g, \overrightarrow{w})_{\mathbb{H}}\, l_{\hat{B}_j}(x)\, d\mu(x)\Big| \\ &\leq \Big|\int (m_{\hat{B}_j} g - g(x), \overrightarrow{w})_{\mathbb{H}}\, l_{\hat{B}_j}(x)\, d\mu(x)\Big| + \Big|\langle Th_{\hat{B}_j}, l_{\hat{B}_j}\overrightarrow{w}\rangle\Big|.\end{aligned}$$

By Weak Boundedness Property and Proposition 3.7, we have

$$\Big|\langle Th_{\hat{B}_j}, l_{\hat{B}_j}\, \overrightarrow{w}\rangle\Big| \leq C.$$

On the other hand, since $\hat{B}_j$ is a doubling ball and $g \in BMO_\rho(\mathbb{R}^n, \mathbb{H})$, we get

$$\begin{aligned}\Big|\int (m_{\hat{B}_j} g - g(x), \overrightarrow{w})_{\mathbb{H}}\, l_{\hat{B}_j}(x)\, d\mu(x)\Big| &\le \Big|(m_{\hat{B}_j} g - m_{A\hat{B}_j} g, \overrightarrow{w})_{\mathbb{H}}\Big| \\ &\quad + \tfrac{1}{\mu(\hat{B}_j)} \int_{A\hat{B}_j} \big|(m_{A\hat{B}_j} g - g(x), \overrightarrow{w})_{\mathbb{H}}\big|\, d\mu(x) \\ &\le \tfrac{2}{\mu(\hat{B}_j)} \int_{A\hat{B}_j} \big|g(x) - m_{A\hat{B}_j} g\big|_{\mathbb{H}}\, d\mu(x) \\ &\le C\mathfrak{b}\|g\|_{BMO_\rho(\mathbb{R}^n,\mathbb{H})}.\end{aligned}$$

In consequence

$$\Big|\tfrac{1}{\mu(\hat{B}_j)} \Big(\int_{\hat{B}_j} g(x)\, d\mu(x), \overrightarrow{w}\Big)_{\mathbb{H}}\Big| \le C|w|_{\mathbb{H}}, \quad \text{for all } j = 1, 2, \ldots, \text{ and } \overrightarrow{w} \in \mathbb{H}.$$

By Lemma 3.5, inequality (39) follows. □

Corollary 4.3. *Let T be a linear operator as in Theorem 2.4. Then the kernel $k(x, y)$ is 0 if and only if $T\phi(x) = \beta(x)\phi(x)$ for some $\beta \in L^\infty_\mu(\mathbb{R}^n, \mathbb{H})$.*

Proof. If $k(x, y)$ is zero, then using (8) we obtain the desired expression of T with $\beta(x) = g(x) - m_{A\hat{B}} g + \mathsf{C}_{B,\hat{B}}[g]$. By Observation 4.2 it is clear that β belongs to $L^\infty_\mu(\mathbb{R}^n, \mathbb{H})$. □

5 Proof of main results: Krein's Theorem and L^2_μ-bounds of T.

In the next Lemma we denote by $\mathbb{A}$ some Hilbert space and by μ an arbitrary measure on $\mathbb{R}^n$.

Lemma 5.1. *Assume that $\mathbb{T}, \mathbb{S}$ are linear operators verifying*

$$\mathbb{T} : \Lambda^\gamma_0(\mathbb{R}^n, \mathbb{C}) \mapsto \Lambda^\gamma_b(\mathbb{R}^n, \mathbb{A}), \qquad \mathbb{S} : \Lambda^\gamma_0(\mathbb{R}^n, \mathbb{A}) \mapsto \Lambda^\gamma_b(\mathbb{R}^n, \mathbb{C})$$

and

$$\int f(x)\mathbb{S}\overrightarrow{g}(x)\, d\mu = \int \Big(\overrightarrow{g}(x), \mathbb{T}f(x)\Big)_{\mathbb{A}}\, d\mu$$

for all $f \in \Lambda^\gamma_0(\mathbb{R}^n, \mathbb{C})$ and $\overrightarrow{g} \in \Lambda^\gamma_0(\mathbb{R}^n, \mathbb{A})$. Also, suppose that $\mathbb{T}$ is a bounded-Lipschitz operator of order γ from $\Lambda^\gamma_0(\mathbb{R}^n, \mathbb{C})$ to $\Lambda^\gamma(\mathbb{R}^n, \mathbb{A})$ and $\mathbb{S}$ is a bounded-Lipschitz operator of order γ from $\Lambda^\gamma_0(\mathbb{R}^n, \mathbb{A})$ to $\Lambda^\gamma(\mathbb{R}^n, \mathbb{C})$.

Then, $\mathbb{T}$ is bounded from $L^2_\mu(\mathbb{R}^n, \mathbb{C})$ to $L^2_\mu(\mathbb{R}^n, \mathbb{A})$ and $\mathbb{S}$ is bounded from $L^2_\mu(\mathbb{R}^n, \mathbb{A})$ to $L^2_\mu(\mathbb{R}^n, \mathbb{C})$.

The proof is similar to that contained in [W, Lemma 2.4] and it invokes the Theorem of Krein ([GK]). In the present situation, a version of the Krein's Theorem for two different Hilbert spaces must be used.

Proof of Theorem 2.6 and Corollary 2.7 .

Let $\mathbb{A}_N = span\{e_1, \dots, e_N\}$ with $N \in \mathbb{N}$ and consider the projection operator $P_N : \mathbb{H} \to \mathbb{A}_N$ defined by $P_N(e_i) = e_i$ if $1 \le i \le N$ and $0_{\mathbb{H}}$ otherwise. Consider the operators T_N

$$\left(T_N\phi, \overrightarrow{\psi}\right) = \left(T\phi, P_N\overrightarrow{\psi}\right).$$

We clearly have that each T_N satisfy Hypothesis 2.1. On the other hand, since $T_N 1 = 0_{\mathbb{H}}$ and $T_N^*(1\overrightarrow{w}) = 0$ for all $\overrightarrow{w} \in \mathbb{H}$, the conclusion of Theorems 2.4 and 2.5 are true.

Consequently, we can apply Lemma 5.1 with $\mathbb{A} = \mathbb{A}_N$. In particular,

$$\left|\left(T\phi, \overrightarrow{\psi}\right)\right| \le C\|\phi\|_{L^2_\mu(\mathbb{R}^n,\mathbb{C})}\|\overrightarrow{\psi}\|_{L^2_\mu(\mathbb{R}^n,\mathbb{H})}$$

for all $\phi \in L^2_\mu(\mathbb{R}^n, \mathbb{C})$ and $\overrightarrow{\psi} \in L^2_\mu(\mathbb{R}^n, \mathbb{A}_N)$, with C not depending on N.

Since $\Omega = \bigcup_{N=1}^\infty L^2_\mu(\mathbb{R}^n, \mathbb{A}_N)$ is dense in $L^2_\mu(\mathbb{R}^n, \mathbb{H})$, the bilinear form $\langle T\phi, \overrightarrow{\psi}\rangle$ can be extended to a bounded operator on $L^2_\mu(\mathbb{R}^n, \mathbb{C}) \times L^2_\mu(\mathbb{R}^n, \mathbb{H})$. This proves Theorem 2.6.

Now we continue with Corollary 2.7. Clearly, given $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$, the sequence $\{\overrightarrow{\psi}_N(x)\}_N$ with $\overrightarrow{\psi}_N(x) = P_N\overrightarrow{\psi}(x)$, converges to $\overrightarrow{\psi}$ in $L^2_\mu(\mathbb{R}^n, \mathbb{H})$. Also, it can be shown that there is a subsequence $\{\overrightarrow{\psi}_{N_j}\}_j$ pointwisely convergent to $\overrightarrow{\psi}$ and such that $T^*\overrightarrow{\psi}_{N_j}(x) \to T^*\overrightarrow{\psi}(x)$ for μ-almost every x. Finally, applying (10) to each element of $\{\overrightarrow{\psi}_{N_j}\}_j$ and the Theorem of the Dominated Convergence, the proof is finished. □

6 Proof of main results: Antisymmetric kernels.

In this section we suppose that the operator T is associated to an antisymmetric kernel k, that is $k(x, y) = -k(y, x)$ for all $x, y \in \mathbb{R}^n$, satisfying (24) and (25). Further, we assume $\langle T\phi, \overrightarrow{\psi}\rangle := \lim_{\epsilon\to 0}\langle T_\epsilon\phi, \overrightarrow{\psi}\rangle$, where

$$\langle T_\epsilon\phi, \overrightarrow{\psi}\rangle := \iint_{d(x,y)\ge\epsilon} \left(k(x, y)\,\phi(y), \overrightarrow{\psi}(x)\right)_{\mathbb{H}} d\mu(x)d\mu(y). \tag{40}$$

We define the following expression that often appears when antisymmetric kernels are used.

Definition 6.1. *Given $h \in \Lambda_b^\gamma(\mathbb{R}^n, \mathbb{C})$ and F, G Lipschitz functions of order $0 < \gamma \le \alpha$ with compact support, such that F is scalar valued and G is $\mathbb{H}$-valued, or viceversa, we denote*

$$\mathfrak{C}^B_M(h, F, G) = \iint\limits_{M\cap(B\times B)} \left(k(x, y), (F(y) - F(x))\,h(y)G(x)\right)_{\mathbb{H}} d\mu(y)d\mu(x), \tag{41}$$

where M is some $\mu \times \mu$-measurable subset of $\mathbb{R}^n \times \mathbb{R}^n$ and B is either a ball or the full space $\mathbb{R}^n$. If $M = \mathbb{R}^n \times \mathbb{R}^n$, the subscript M is dropped, and the same is done for $B = \mathbb{R}^n$.

If h is supported on a ball B, the expression $\mathfrak{E}(h, F, G)$ is well defined, because the double integral in (41) is absolutely summable.

Proposition 6.2. *Let h be a function in $\Lambda_b^\gamma(\mathbb{R}^n, \mathbb{C})$ and suppose that F and G are Lipschitz functions of order γ with compact support such that F or G is $\mathbb{H}$-valued. Let $B = B_s(x_0)$ be a ball containing the support of F and G.*

If M is any $\mu \times \mu$-measurable set then the integral defining $\mathfrak{E}_M^B(h, F, G)$ is absolutely summable and, in addition, we have the following inequality

$$\left|\mathfrak{E}_M^B(h, F, G)\right| \le C\|h\|_{L_\mu^\infty(\mathbb{R}^n,\mathbb{C})} \|F\|_{\Lambda^\gamma} \|G\|_{\Lambda^\gamma}\, \mu(B) s^{2\gamma}. \tag{42}$$

with constant C independent of M, B, h, F and G.

Proof. Applying Lemma 3.12, we clearly have

$$\left|\mathfrak{E}_M^B(h, F, G)\right| \le C\|h\|_{L_\mu^\infty(\mathbb{R}^n,\mathbb{C})} \|F\|_{\Lambda^\gamma} \|G\|_{L_\mu^\infty} \mu(B) s^{\gamma}.$$

Inequality (23) ends the proof. □

We observe that, as a consequence of this Proposition, if M is some symmetric μ-measurable subset of $\mathbb{R}^n \times \mathbb{R}^n$, using the antisymmetric property of k we easily have that $\mathfrak{E}_M^B(h, F, G)$ **is equal to**

$$\frac{1}{2} \iint\limits_{M \cap (B \times B)} \Big(k(x, y), (F(y) - F(x))(h(y)G(x) + h(x)G(y))\Big)_{\mathbb{H}} d\mu(y) d\mu(x). \tag{43}$$

Proposition 6.3. *For every $0 < \gamma \le \alpha$ the operator T, given by (40), is well defined for all functions $\phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ and $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$. Also, the following equality holds*

$$\langle T\phi, \overrightarrow{\psi} \rangle = \frac{1}{2} \iint \big(k(x, y), \phi(y)\overrightarrow{\psi}(x) - \phi(x)\overrightarrow{\psi}(y)\big)_{\mathbb{H}}\, d\mu(x) d\mu(y), \tag{44}$$

and, in addition, the following Weak Boundedness Property is verified

$$\big|\langle T\phi, \overrightarrow{\psi} \rangle\big| \le C\mu(B) s^{2\gamma} \|\phi\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{C})} \|\psi\|_{\Lambda^\gamma(\mathbb{R}^n,\mathbb{H})}, \tag{45}$$

for functions ϕ and $\overrightarrow{\psi}$ supported in $B_s(x_0)$.

Moreover, for every $0 < \gamma \le \alpha$, the operator T satisfies the Meyer Commutation Property and is continuous from $\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ to $\big(\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})\big)'$.

Proof. By (24), the double integral (40) is absolutely summable and, in particular, is well defined.

We need to take $M_\epsilon = \{(x, y) | d(x, y) \ge \epsilon\}$ and to note that the support of $\phi(y)\overrightarrow{\psi}(x) - \phi(x)\overrightarrow{\psi}(y)$ is contained in $B \times B$. Now, we make the change of variable $(x, y) \mapsto (y, x)$ and apply the antisymmetric property of k to obtain

$$\langle T_\epsilon\phi, \overrightarrow{\psi} \rangle = \frac{1}{2} \iint_{M_\epsilon \cap (B \times B)} \big(k(x, y), \phi(y)\overrightarrow{\psi}(x) - \phi(x)\overrightarrow{\psi}(y)\big)_{\mathbb{H}} d\mu(x) d\mu(y). \tag{46}$$

Using equality (43) with $h = 1$, we get that

$$\langle T_\epsilon\phi, \overrightarrow{\psi} \rangle = \mathfrak{E}_{M_\epsilon}^B(1, \phi, \overrightarrow{\psi}) - \mathfrak{E}_{M_\epsilon}^B(1, \overrightarrow{\psi}, \phi).$$

Hence, noting that the support of $\phi(y)\overrightarrow{\psi}(x) - \phi(x)\overrightarrow{\psi}(y)$ is contained in $B \times B$, applying Proposition 6.2 and the Theorem of Dominated Convergence, we clearly have (44) after taking $\epsilon \to 0$.

To check the Meyer Commutation Property, let us take $h, \varphi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ and $\overrightarrow{\psi} \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$. Let B be a ball containing the support of the three functions. We have $\mathfrak{C}(h, \varphi, \overrightarrow{\psi}) = \mathfrak{C}^B(h, \varphi, \overrightarrow{\psi})$. Now, by (44) and (43) we get $\langle Th\varphi, \overrightarrow{\psi}\rangle - \langle Th, \varphi\overrightarrow{\psi}\rangle = \mathfrak{C}(h, \varphi, \overrightarrow{\psi})$.

To prove the continuity, we take into account the topology of the inductive limit on $\Lambda_0^\gamma(\mathbb{R}^n, \mathbb{H})$ (see for example [MS]) and we use (45). □

Observation 6.4. *We note that in equation* (43) *and Proposition 6.3, the antisymmetric property and* (24) *are the unique properties of k that were used, while the property* (25) *of k was not used.*

Proof of Theorem 2.8. Let $\phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ be a function supported in the ball B. So h_B, ϕ and $\overrightarrow{\psi}$ are all supported in $\mathfrak{c}^4 B$. Writing $\mathfrak{C}(h_B, \phi, \overrightarrow{\psi}) = \mathfrak{C}^{\mathfrak{c}^4 B}(h_B, \phi, \overrightarrow{\psi})$ and $\mathfrak{C}(h_B, \overrightarrow{\psi}, \phi) = \mathfrak{C}^{\mathfrak{c}^4 B}(h_B, \overrightarrow{\psi}, \phi)$, since $h_B(\cdot)\phi(\cdot) = \phi(\cdot)$ and $h_B(\cdot)\overrightarrow{\psi}(\cdot) = \overrightarrow{\psi}(\cdot)$, by equality (41) we get

$$\mathfrak{C}(h_B, \phi, \overrightarrow{\psi}) - \mathfrak{C}(h_B, \overrightarrow{\psi}, \phi) = 2\langle T\phi, \overrightarrow{\psi}\rangle = 2\langle T^*\overrightarrow{\psi}, \phi\rangle. \tag{47}$$

Denote by $Z^B\{\overrightarrow{\psi}, \phi\}$ to the following expression

$$\int \big((g(x) - m_{A\hat{B}}g), \overrightarrow{\psi}(x)\big)_{\mathbb{H}}\phi(x)d\mu(x) - \int \langle (g - m_{A\hat{B}}(g)), l_{\hat{B}}\, \overrightarrow{\psi}(x)\rangle\phi(x)d\mu(x)$$
$$+ \int \big(\bar{\mathfrak{C}}_{B,\hat{B}}, \overrightarrow{\psi}(x)\big)_{\mathbb{H}}\phi(x)d\mu(x) - \int \big(I_B 1(x), \overrightarrow{\psi}(x)\big)_{\mathbb{H}}\phi(x)d\mu(x).$$

Now, by using the formula (8) for $T\phi$ and (47), we have

$$\langle T^*\overrightarrow{\psi}, \phi\rangle = \langle T\phi, \psi\rangle = Z^B\{\overrightarrow{\psi}, \phi\} + 2\langle T^*\overrightarrow{\psi}, \phi\rangle + \mathfrak{C}(h_B, \overrightarrow{\psi}, \phi). \tag{48}$$

Therefore we obtain

$$\langle T^*\overrightarrow{\psi}, \phi\rangle = -Z^B\{\overrightarrow{\psi}, \phi\} - \mathfrak{C}(h_B, \overrightarrow{\psi}, \phi).$$

Then, for μ-almost every $x \in B$, the formula (11) holds.

Now, suppose that $T1 = 0_{\mathbb{H}}$. Let $\phi \in \Lambda_0^\gamma(\mathbb{R}^n, \mathbb{C})$ such that $\mathrm{supp}(\phi) \subset B = B_s(x_0)$ and $\int \phi\, d\mu = 0$. For $\overrightarrow{w} \in \mathbb{H}$, we know that

$$\Big\langle T^*(1\overrightarrow{w}), \phi\Big\rangle = \langle T^*(h_B\overrightarrow{w}), \phi\rangle + \langle I_B^{(*)}(1\overrightarrow{w}), \phi\rangle.$$

Using definition of T^* and (44), we get

$$\langle T^*(h_B\overrightarrow{w}), \phi\rangle = \Big\langle T\phi, h_B\overrightarrow{w}\Big\rangle = -\langle Th_B, \phi\overrightarrow{w}\rangle. \tag{49}$$

On the other hand, it is easy to check that

$$\langle I_B^{(*)}(1\overrightarrow{w}), \phi\rangle = -\langle I_B 1, \phi\overrightarrow{w}\rangle, \tag{50}$$

because k is antisymmetric.

Since $\phi\overrightarrow{w}$ is in $\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{H})$ and $\int(\phi\overrightarrow{w})d\mu = 0_{\mathbb{H}}$, by (49) and (50) we have $\left\langle T^*(1\overrightarrow{w}),\phi\right\rangle = -\left\langle Th_B,\phi\overrightarrow{w}\right\rangle - \left\langle I_B 1,\phi\overrightarrow{w}\right\rangle = -\left\langle T1,\phi\overrightarrow{w}\right\rangle = 0$.

Finally, for the proof that T^* is a bounded-Lipschitz operator, it can be used the same procedure as Theorem 2.5. □

Proof of Theorem 2.9. Since $T1 = 0$, we can apply Theorem 2.8 and then Lemma 5.1 with $\mathbb{T} = T$, $\mathbb{S} = T^*$ and $\mathbb{A} = \mathbb{H}$. This ends the proof. □

7 Oscillation operator

In this section, we fix $\mathbb{H} = \ell^2(\mathbb{C})$ and $m = m_n$, the normalized n-dimensional Lebesgue measure, with the exponent ν in (2) equal to 1. Also, we suppose $\kappa(x,y)$ is an antisymmetric and scalar-valued kernel associated to some singular integral operator τ, where τ is the limit of the family $\{\tau_\varepsilon\}_{\varepsilon>0}$ defined by

$$\left\langle \tau_\varepsilon f, g\right\rangle = \iint_{d(x,y)\geq\varepsilon} \kappa(x,y)f(y)g(x)dm(y)dm(x).$$

Next, we define the *Oscillation operator* $T = \mathcal{O}(\tau)$, by means of

$$\left\langle Tf,\overrightarrow{g}\right\rangle = \lim_{L\to\infty}\sum_{\ell=-L}^{L}\left\langle(\tau_{A^\ell} - \tau_{A^{\ell-1}})f, g\right\rangle, \tag{51}$$

for all $f \in \Lambda_0^\gamma(\mathbb{R}^n,\mathbb{C})$, $\overrightarrow{g} \in \Lambda_0^\gamma(\mathbb{R}^n,\mathbb{H})$, whenever the limit exists.

Observe that T has associated the vector-valued kernel

$$K(x,y) = \{\kappa(x,y)\chi_{[A^{\ell-1},A^\ell)}(d(x,y))\}_{\ell\in\mathbb{Z}}. \tag{52}$$

We shall study this operator and its L^2_m-bounds through $T1$-theorems.

Lemma 7.1. *If the kernel κ is antisymmetric and satisfies condition* (24) *then K satisfies both conditions too.*

The proof is easy by (52) and after observing that $|K(x,y)|_{\mathbb{H}} \leq |\kappa(x,y)|$.

Remark 7.2. *By Lemma 7.1 and Observation 6.4, the operator $\mathcal{O}(\tau)$ is continuous from $\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{C})$ to $(\Lambda_0^\gamma(\mathbb{R}^n,\mathbb{H}))'$ and satisfies the Weak Boundedness Property and the Meyer Commutation Property, for all $0 < \gamma \leq \alpha$.*

Now we study the Oscillation operators for the Riesz Transforms. The j-th Riesz Transform ($j = 1,\cdots,n$) for $f \in \Lambda_0^\gamma(\mathbb{R}^n,\mathbb{C})$ is defined by

$$\Re_j f(x) = \lim_{\varepsilon\to 0}\int_{d(x,y)\geq\varepsilon}\frac{x_j - y_j}{d(x,y)^{\frac{1}{n}(n+1)}}f(y)\,dm(y).$$

In what follows we only consider $j = 1$ and we shall denote $\Re = \Re_1$.

Obviously, the kernel $\kappa(x,y) = (x_1 - y_1)/d(x,y)^{\frac{1}{n}(n+1)}$ is antisymmetric.

It is well known that κ satisfies the following conditions

$$|\kappa(x,y)| \leq C d(x,y)^{-1}, \qquad \text{(for all } x,y,\ x\neq y),$$
$$\left|\kappa(x,y)-\kappa(x',y)\right| \leq C\frac{d(x,x')^{\alpha}}{d(x,y)^{1+\alpha}}, \quad \text{(for all } x,y,\ d(x,x')\leq d(x,y)/A).$$

We now define T and K by (51) and (52), respectively. We will prove that the kernel K satisfies a L^r-smoothness condition, at least for $r=2$.

Proposition 7.3. *For all $R>0$ and for all pair of points $x,x'\in\mathbb{R}^n$ such that $d(x,x')<R/A$, the following it is true*

$$\left(\int_{R\leq d(x,y)<AR}\left|K(x,y)-K(x',y)\right|_{\mathbb{H}}^{2}dm(y)\right)^{1/2} \leq CR^{-1/2}\left(\frac{d(x,x')}{R}\right)^{\alpha/2},$$

where $A>c_n$.

Proof. Let x,x' be a pair of points such that $d(x,x')<R/A$. Let $j\in\mathbb{Z}$ be the integer satisfiying $A^{j-1}\leq R<A^j$. We calculate

$$\begin{aligned}
&\int_{R<d(x,y)\leq AR}\left|K(x,y)-K(x',y)\right|^2 dm(y)\\
&\leq \int_{R<d(x,y)\leq AR}\left|\kappa(x,y)-\kappa(x',y)\right|^2\sum_{\ell\in\mathbb{Z}}\left|\chi_{[A^{\ell-1},A^{\ell})}(d(x,y))\right|^2 dm(y)\\
&\quad+\int_{R<d(x,y)\leq AR}\left|\kappa(x',y)\right|^2\sum_{\ell\in\mathbb{Z}}\left|\chi_{[A^{\ell-1},A^{\ell})}(d(x,y))-\chi_{[A^{\ell-1},A^{\ell})}(d(x',y))\right|^2 dm(y)\\
&=: J_1+J_2.
\end{aligned}$$

For J_1, since $R<d(x,y)\leq AR$ we get that $\chi_{[A^{\ell-1},A^{\ell})}(d(x,y))=0$ whenever $\ell\neq j,j+1$. Now, we can apply the L^2-smoothness condition of κ and the desired bound is obtained.

For J_2, we have that $\left|\kappa(x',y)\right|^2\leq C\,R^{-2}$. So, we pay attention to the series. Let us introduce the following notation:

$$\begin{aligned}
U_\ell(x,y) &:= \left|\chi_{[A^{\ell-1},A^{\ell})}(d(x,y))-\chi_{[A^{\ell-1},A^{\ell})}(d(x',y))\right|^2,\\
I_1^{(\ell)} &:= \int_{R<d(x,y)\leq A^j} U_\ell(x,y)dm(y),\\
I_2^{(\ell)} &:= \int_{A^j<d(x,y)\leq AR} U_\ell(x,y)dm(y).
\end{aligned}$$

Now, we can write

$$J_2\leq CR^{-2}\sum_{\ell\in\mathbb{Z}}\left(I_1^{(\ell)}+I_2^{(\ell)}\right).$$

We first analyze the terms $I_1^{(\ell)}$. Suppose that $d(x',y)<A^\ell\leq A^{j-1}$. We have

$$d(x,y)\leq c_n d(x',y)+c_n d(x,x')<c_nA^\ell+\frac{c_n}{A}R<A^{\ell+1}+\frac{c_n}{A}d(x,y).$$

From this we obtain $(1-c_n/A)d(x,y) < A^{\ell+1}$ and then, $(1-c_n/A)A^{j-1} < A^{\ell+1} \leq A^j$, which implies

$$(j-1) + \log_A(1-c_n/A) - 1 < \ell < j-1.$$

This means that the series above has at most $|\log_A(1-c_n/A) - 1|$ terms of the form $I_1^{(\ell)}$ not null with $\ell \leq j-1$. For these terms we have

$$0 \leq d(x,y) - d(x',y) \leq nd(x,x')^\alpha A^{j(1-\alpha)},$$

therefore

$$A^{j-1} \leq d(x,y) \leq A^\ell + nd(x,x')^\alpha A^{j(1-\alpha)} \leq A^{j-1} + Cd(x,x')^\alpha R^{1-\alpha}$$

and then

$$\begin{aligned} I_1^{(\ell)} &\leq \int_{A^{j-1}<d(x,y)\leq A^j} U_\ell(x,y)\, dm(y) \\ &\leq Cm\big(B(x, A^{j-1} + Cd(x,x')^\alpha R^{1-\alpha}) \setminus B(x, A^{j-1})\big) \\ &\leq Cd(x,x')^\alpha R^{1-\alpha}. \end{aligned}$$

On the another hand, we have $d(x',y) \leq c_n(d(x,x') + d(x,y)) \leq c_n(R/A + A^j) < A^{j+1}$. This implies that $U_\ell(x,y) = 0$ for $\ell \geq j+2$. It remains to study the terms $I_1^{(j)}$ and $I_1^{(j+1)}$.

For $I_1^{(j+1)}$, since $A^{j-1} < d(x,y) \leq A^j$, it is enough to consider $A^j < d(x',y) \leq A^{j+1}$. In this case we have

$$0 \leq d(x',y) - d(x,y) \leq nd(x',x)^\alpha A^{(j+1)(1-\alpha)} \leq C\, d(x,x')^\alpha R^{1-\alpha},$$

so, we obtain $A^j < d(x',y) \leq A^j + Cd(x,x')^\alpha R^{1-\alpha}$ and then

$$\big|I_1^{(j+1)}\big| \leq m(B(x', A^j + Cd(x,x')^\alpha R^{1-\alpha}) \setminus B(x', A^j)) \leq Cd(x,x')^\alpha R^{1-\alpha},$$

as desired.

Now, we focus on $I_1^{(j)}$. We have $A^{j-1} \leq d(x,y) < A^j \leq AR$. The expression $U_j(x,y)$ is not null only in the case that $d(x',y) < A^{j-1}$ or if $d(x',y) \geq A^j$.

In the first case we can write

$$A^{j-1} > d(x',y) = (d(x',y) - d(x,y)) + d(x,y),$$

but since $d(x,y) \geq A^{j-1}$, it happens that $d(x',y) - d(x,y) < 0$, so

$$A^{j-1} + \big|d(x',y) - d(x,y)\big| \geq d(x,y).$$

This implies

$$A^{j-1} \leq d(x,y) \leq A^{j-1} + nd(x,x')^\alpha (AR)^{1-\alpha}. \tag{53}$$

Suppose now that we are in the second case, that is $d(x',y) \geq A^j$. Since $d(x,y) < A^j$, we get that $d(x',y) - d(x,y) \geq 0$. We can write

$$Cd(x,x')^\alpha R^{1-\alpha} + d(x,y) \geq \big(d(x',y) - d(x,y)\big) + d(x,y) \geq A^j.$$

Hence

$$A^j \leq d(x',y) \leq A^j + Cd(x,x')^\alpha R^{1-\alpha}. \tag{54}$$

From (53) and (54) the function $U_j(x,y)$ is not null in the union of the two anulli

$$A_j^1 := B_{A^{j-1}+Cd(x,x')^\alpha R^{1-\alpha}}(x) \setminus B_{A^{j-1}}(x),$$
$$A_j^2 := B_{A^j+Cd(x,x')^\alpha R^{1-\alpha}}(x') \setminus B_{A^j}(x').$$

Since m is the measure of Lebesgue, we have

$$m(A_j^1) = Cd(x,x')^\alpha R^{1-\alpha}$$
$$m(A_j^2) = Cd(x,x')^\alpha R^{1-\alpha}.$$

This give us the desired bounds.

Proceeding in a similar way we can obtain the same bounds as above for the terms $I_2^{(\ell)}$, $\ell \in \mathbb{Z}$.

Summing up, we obtain the desired bound for J_2 and the proof is completed. □

Related to the condition $T1 = 0_{\mathbb{H}}$ we have the following Proposition, whose proof is easy and it is left to the reader.

Proposition 7.4. *If the kernel κ satisfies*

$$\int_{E(x,a,b)} \kappa(x,y)dm(y) = 0, \quad \text{all } x \in \mathbb{R}^n \text{ and any } 0 < a < b, \tag{55}$$

then $T(1) = \mathcal{O}(\tau)(1) = 0_{\mathbb{H}}$.

On view of Lemma 7.1, Remark 7.2, Proposition 7.3 and Proposition 7.4 we can state the following

Theorem 7.5. *The Oscillation operator associated to the j-th Riesz Transform ($j = 1,\ldots,n$), is bounded from $L^2_m(\mathbb{R}^n,\mathbb{C})$ to $L^2_m(\mathbb{R}^n,\mathbb{H})$.*

In particular, since the Hilbert transform is just the Riesz transform when the dimension is $n = 1$, we have that

Corollary 7.6. *The Oscillation operator of the Hilbert-transform can be extended to a bounded linear operator from $L^2_m(\mathbb{R}^n,\mathbb{C})$ to $L^2_m(\mathbb{R}^n,\mathbb{H})$.*

Remark. We note that Theorem 7.5 is still true for any singular integral operator τ associated to an antisymmetric kernel satisfying the standard conditions and (55).

References

[B1] J. Bourgain. Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes Études Sci. Publ. Math.*, 69(1989),5–45.

[B2] M. Bramanti. Singular integrals in nonhomogeneous spaces: L^2 and L^p continuity from Hölder estimates. *Rev. Mat. Iberoamericana*, 26(2010),347–366.

[BCP] A. Benedek, A.-P. Calderón, and R. Panzone. Convolution operators on Banach space valued functions. *Proc. Nat. Acad. Sci. U.S.A.*, 48(1962),356–365 .

[CJRW1] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl. Oscillation and variation for the Hilbert transform. *Duke Math. J.*, 105(2000),59–83.

[CJRW2] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl. Oscillation and variation for singular integrals in higher dimensions. *Trans. Amer. Math. Soc.*, 355(2003),2115–2137.

[CZ] A. P. Calderon and A. Zygmund. On the existence of certain singular integrals. *Acta Math.*, 88(1952),85–139.

[DJ] G. David and J.-L. Journé. A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. of Math. (2)*, 120(1984),371–397, .

[DJS] G. David, , J.-L. Journé, and S. Semmes. Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana*, 1(1985),1–56.

[DU] J. Diestel, J. J. Uhl, Jr. Vector Meaures. *Mathematical Suerveys Number 15, American Mathematical Society*, (1977).

[GK] I. Gohberg and N. Krupnik, Einführung in die Theorie der eindimensionalen singulären Integraloperatoren, *Birkhäuser* , (1979).

[H1] T. Hytönen. An Operator Valued Tb Theorem. *Journal of Functional Analysis*, 234(2006), 420-463 .

[H2] T. Hytönen. The vector-valued non-homogeneous Tb theorem. *Arxiv:0809.3097*.

[HW] T. Hytönen and L. Weis. A $T1$ theorem for integral transformation with operator-valued kernel. *Journal für die reine und angewandte Mathematik (Crelles's Journal)*, 599(2006),155-200.

[J] R. Jones. Ergodic theory and connections with analysis and probability *New York J. Math 3A.*, 3A(1997),31–67.

[JKRW] R. L. Jones, R. Kaufman, J. Rosenblatt, and M. Wierdl. Oscillation in ergodic theory. *Ergodic Theory Dynam. Systems*, 18(1998),889–935.

[JOR] R. Jones, I. Ostrovski, and J. Rosenblatt., Spaces functions in ergodic theory. *Ergodic Theory Dynamical Systems*, 16(1996),267–305

[MS] R.A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type *Advances in Math.*, 33(1976), 257-270.

[MST] R.A. Macías, C. Segovia, and J.L. Torrea. Singular integral operators with non-necessarily bounded kernels on spaces of homogeneous type. *Adv. Math.*, 93(1992),25–60.

[MV] M.S. Melnikov and J. Verdera. A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs. *Internat. Math. Res. Notices*, 7(1995),325-331.

[NTV] F. Nazarov, S. Treil, and A. Volberg. Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces. *Internat. Math. Res. Notices*, 15(1997),703–726.

[RFRT] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea. Calderón-Zygmund theory for operator-valued kernels. *Adv. in Math.*, 62(1986),7–48.

[RFT] J. L. Rubio de Francia and J. L. Torrea. Vector extensions of operators in L^p spaces. *Pacific J. Math.*, 105(1983),227–235.

[T1] X. Tolsa. Littlewood-Paley Theory and the $T(1)$ Theorem with Non-doubling Measures. *Advances in Math.*, 164(2001),57–116.

[T2] X. Tolsa. A $T(1)$ theorem for non-doubling measures with atoms. *Proc. London Math. Soc. (3)*, 82(2001),195–228.

[T3] X. Tolsa. L^2-boundedness of the Cauchy integral operator for continuous measures. *Duke Math. J.*, 98(1999),269–304.

[V] J. Verdera. On the $T(1)$-theorem for the Cauchy integral. *Trans. of the American Mathematical Society*, 299(1987),581–599.

[W] R. Wittmann. Application of a Theorem of M. G. Krein to Singular Integrals. *Transactions of American Mathematical Society.*, 299(1987), 581–599.