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Classification of Nilsoliton metrics in dimension seven

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ABSTRACT

The aim of this paper is to classify Ricci soliton metrics on 7-dimensional nilpotent Lie groups. It can be considered as a continuation of our paper in Fernández Culma (2012). To this end, we use the classification of 7-dimensional real nilpotent Lie algebras given by Ming-Peng Gong in his Ph.D thesis and some techniques from the results of Michael Jablonski (2010, 2012) and of Yuri Nikolayevsky (2011). Of the 9 one-parameter families and 140 isolated 7-dimensional indecomposable real nilpotent Lie algebras, we have 99 nilsoliton metrics given in an explicit form and 7 one-parameter families admitting nilsoliton metrics.

Our classification is the result of a case-by-case analysis, so many illustrative examples are carefully developed to explain how to obtain the main result.

1. Introduction

The Ricci flow, introduced by Richard Hamilton in the early 1980s, is a geometric evolution equation which smoothly deforms an initial metric g_0 on a Riemannian manifold M in the direction of minus two times its Ricci tensor:

$$\begin{cases} \frac{\partial}{\partial t} g = -2\,\mathrm{ric}_g, \\ g(0) = g_0. \end{cases}$$

The "philosophy" behind this tool of geometric analysis is to try to evolve the geometry of (M, g_0) to one which looks more uniform (although in practice, it is not always possible).

A *Ricci soliton* is a complete Riemannian metric g_0 such that the solution to the Ricci flow $g(t)$ with $g(0) = g_0$ changes g_0 only by diffeomorphisms and scaling as time goes on, that is $g(t) = c(t)\varphi_t^* g_0$, where $c(t) \in \mathbb{R}_+$ and φ_t is a one-parameter group of diffeomorphisms of M; a Ricci soliton is not "improved" by the Ricci flow. These distinguished metrics are important in the study of the Ricci flow because they may be limiting cases for the Ricci flow near singularities and are a natural generalization of an *Einstein metric* [1].

In the theory of Ricci flow, homogeneous Riemannian manifolds provide a rich source of explicit examples for some concepts and behaviors. Moreover, many of the known examples of Ricci solitons correspond to *algebraic Ricci solitons*:

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these are given by a left-invariant metric g on a simply-connected Lie group G satisfying

$$Ric(g) = c\,\mathrm{Id} + D, \quad \text{for some } c \in \mathbb{R} \text{ and } D \in \mathrm{Der}(\mathfrak{g}) \tag{1.1}$$

where $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and Ric is the Ricci operator of g.

Such metrics are indeed Ricci solitons and in the particular case that G is solvable (respectively nilpotent) are called *solvsolitons* (respectively *nilsolitons*).

Our aim in this paper is to classify nilsolitons on 7-dimensional simply connected nilpotent Lie groups. In general, it is difficult to know if a nilpotent Lie algebra admits a nilsoliton (inner product) and it is very difficult to get such metric when it exists. Nilsoliton metrics have been completely classified only up to dimension 6 by Jorge Lauret and Cynthia Will in [2,3] (with a minor correction in [4]). In [5], Trayce Payne and Hülya Kadioğlu have introduced a computational method for classifying nilsoliton metrics in the family of nilpotent Lie algebras with *simple pre-Einstein derivation* and *nonsingular Gram matrix*. Their method does not rely on any preexisting classifications of nilpotent Lie algebras. They classify nilsolitons within this family, which has 33 algebras in dimension 7 and 159 algebras in dimension 8. In [6], we gave a complete classification of all seven-dimensional (complex) nilpotent Lie algebras $\mathfrak{n}$ such that a real form of $\mathfrak{n}$ admits a nilsoliton inner product. In order, to prove [6, Theorem 7], we make heavy use of results proved by Yuri Nikolayevsky in [7] (some of such results were independently proved by Michael Jablonski in [8,9] in a more general form).

In these notes, we use the classification of seven-dimensional real nilpotent Lie algebras given by Ming-Peng Gong in his Ph.D thesis [10], results from complex (and real) Geometric Invariant Theory [11,8,9,12] and calculations given in [13].

Following [10], the indecomposable 7-dimensional real Lie algebras can be seen as 9 one-parameter families of nilpotent Lie algebras plus 140 isolated nilpotent Lie algebras; i.e. 140 algebras that are not in any of such one-parameter families. We focus our attention on isolated algebras and on the family $(147E)[0 < t < 1]$ where we can give a nilsoliton metric for each t. So, we have 99 nilsoliton metrics and 7 one-parameter families admitting nilsoliton metrics. The methods explained here can also be used to study any fixed member in a one-parameter family.

Our classification is the result of a case-by-case analysis, so many illustrative examples are carefully developed to explain how to obtain the main result. Some nilsoliton metrics here were obtained in [13], we refer the reader to [6] for more details. All calculations are performed using Maple™ 14; more particularly, we use the (nice) Maple™ packages `DifferentialGeometry` and `LieAlgebras`.

2. Preliminaries

In this section, we give a brief exposition on nilsoliton metrics and real (and complex) geometric invariant theory. There is an intriguing interplay between the Ricci flow on nilpotent Lie groups and the gradient flow of the norm squared of the *moment map* associated to the natural action of $\mathrm{GL}_n(\mathbb{R})$ on $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. It is known that if $\mathfrak{n} := (\mathbb{R}^n, \mu)$ is a nilpotent Lie algebra and $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^n$, then the Ricci operator of $(\mathbb{R}^n, \mu, \langle\cdot,\cdot\rangle)$ satisfies

$$4\,Ric = m(\mu) \tag{2.1}$$

where m is the moment map. It follows that to minimize the norm of the Ricci tensor among all left-invariant metrics of $\mathfrak{n}$ with the same scalar curvature is equivalent to minimize $\|m\|^2/\|\mu\|^4$ along the $\mathrm{GL}_n(\mathbb{R})$-orbit of μ (here, the inner products on $\mathfrak{gl}_n(\mathbb{R})$ and V, which are denoted by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle$ respectively, are those induced by the canonical inner product of $\mathbb{R}^n$).

Theorem 2.1 (*Jorge Lauret*)**.** *(See for instance [14, Theorem 4.2]) The nilpotent Lie algebra $\mathfrak{n} := (\mathbb{R}^n, \mu)$ admits a nilsoliton (inner product) if and only if the $\mathrm{GL}_n(\mathbb{R})$-orbit of μ is distinguished; i.e. $\mathrm{GL}_n(\mathbb{R}) \cdot \mu$ contains a critical point of $\|m\|^2$, or equivalently, if and only if there exists a $g \in \mathrm{GL}_n(\mathbb{R})$ such that*

$$m(\widetilde{\mu}) = c\,\mathrm{Id} + D \quad \text{for some } c \in \mathbb{R} \text{ and } D \in \mathrm{Der}(\widetilde{\mathfrak{n}}) \tag{2.2}$$

where $\widetilde{\mu} := g \cdot \mu$ and $\widetilde{\mathfrak{n}} := (\mathbb{R}^n, \widetilde{\mu})$.

There is at most one nilsoliton metric on a nilpotent Lie group (up to isometry and scaling).

A point $\widetilde{\mu}$ satisfying Eq. (2.2) is called a *distinguished point* (see [9, Definition 2.6]) and the derivation D is usually called *Einstein derivation* (in connection with *Einstein solvmanifolds*) or *nilsoliton derivation*.

It is known from [15] that the eigenvalues of an Einstein derivation are all positive integers without a common divisor (up to a rational factor) and that it is, up to conjugation by an automorphism, positively proportional to the *pre-Einstein derivation*, which is defined by Yuri Nikolayevsky in [7, Definition 2].

Before stating the techniques to classify the nilsoliton metrics, let us first recall some notations.

Notation 2.2. Let $\mathfrak{a}$ denote the maximal abelian subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ contained in the vector space of symmetric matrices $\mathfrak{p}$ given by

$$\mathfrak{a} = \{\mathrm{Diag}(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$$

and set $\mathrm{A} := \exp(\mathfrak{a})$.

Let Φ be a finite subset of $\mathfrak{a}$. The convex hull of Φ will be denoted by $\mathrm{CH}(\Phi)$ while by $\mathrm{Aff}(\Phi)$ we denote the affine space generated by Φ. The notation $\mathrm{mcc}(\Phi)$ denotes the *minimal convex combination* of Φ; i.e. the unique vector closest to the origin in $\mathrm{CH}(\Phi)$. The notation $\mathrm{int}(\mathrm{CH}(\Phi))$ represents the interior of $\mathrm{CH}(\Phi)$ relative to the usual topology of $\mathrm{Aff}(\Phi)$.

Given a nilpotent Lie algebra $\mathfrak{n} := (\mathbb{R}^n, \mu)$, we denote by $\mathfrak{R}(\mu)$ the (ordered) set of weights related with μ to the action of $\mathrm{GL}_n(\mathbb{R})$ on V, i.e. if $\{C_{i,j}^k\}$ are the structural constants of $\mathfrak{n}$ in the basis $\{e_1 \dots e_n\}$ then

$$\mathfrak{R}(\mu) = \{E_{k,k} - E_{i,i} - E_{j,j} : C_{i,j}^k \neq 0\}$$

where $\{E_{i,j}\}$ is the canonical basis of $\mathfrak{gl}_n(\mathbb{R})$.

Recall that Nikolayevsky's nice basis criterion [7, Theorem 3] says that a nilpotent Lie algebra $\mathfrak{n} := (\mathbb{R}^n, \mu)$ which is written in a *nice basis* [7, Definition 3] admits a nilsoliton metric if and only if the equation

$$\boldsymbol{U}x = [1]_m \tag{2.3}$$

has at least one solution x with positive coordinates, where $m = \#(\mathfrak{R}(\mu))$ and $\boldsymbol{U}$ is the Gram matrix of $(\mathfrak{R}(\mu), \langle\cdot,\cdot\rangle)$. This result gives us an easy-to-check convex geometry condition for a nilpotent Lie algebra with a nice basis to be an Einstein nilradical.

If one carefully reads the proof of the above theorem one sees that it gives us a technique to find a nilsoliton on a nilpotent Lie algebra which is written in a nice basis and admits such distinguished metric; a nilsoliton can be found in an A-orbit. So, Nikolayevsky's nice basis criterion can be rewritten as

Theorem 2.3 (*Nikolayevsky's Nice Basis Criterion*)**.** *Let $\mathfrak{n} := (\mathbb{R}^n, \mu)$ be a nilpotent Lie algebra such that $\mathfrak{n}$ is written in a nice basis. Then $\mathfrak{n}$ admits a nilsoliton if and only if there exists $g \in \mathrm{A}$ such that*

$$m(g \cdot \mu) = \mathrm{mcc}(\mathfrak{R}(\mu)), \tag{2.4}$$

and, consequently, $\widetilde{\mu} := g \cdot \mu$ is a distinguished point in the $\mathrm{GL}_n(\mathbb{R})$-orbit of μ.

To find a nilsoliton metric for a nilpotent Lie algebra admitting such metric and which is written in a nice basis is easy in practice. We must calculate the vector $\mathrm{mcc}(\mathfrak{R}(\mu))$, which is given by

$$\frac{1}{\sum x_p}\left(\sum x_p \mathfrak{R}(\mu)_p\right)$$

where $[x_i]$ is any positive solution to Eq. (2.3) and we solve Eq. (2.4) for $g \in \mathrm{A}$. We refer the reader to [8, Corollary 3.4] or [12, Section 3] for further information on results related with Theorem 2.3.

Another application from geometric invariant theory in the study of nilsoliton metrics is the following result, which was proven independently by Michael Jablonski in [8, Theorem 6.5] and Yuri Nikolayevsky in [7, Theorem 6].

Theorem 2.4. *Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be two (real) nilpotent Lie algebras whose complexifications are isomorphic as complex nilpotent Lie algebras. If $\mathfrak{n}_1$ is an Einstein nilradical then so is $\mathfrak{n}_2$, with the same eigenvalue type.*

Consider the natural action of the complex reductive Lie group $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ and its moment map $\check{m}$ as in [11] (see [16]). Theorem 2.4 can be derived by comparing the distinguished orbits of the mentioned action with the distinguished orbits of the natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We can rephrase Theorem 2.4 as saying that

Theorem 2.5 (*[8, Theorem 4.7]*)**.** *Let $\mathfrak{n} := (\mathbb{R}^n, \mu)$ be a nilpotent Lie algebra. Then $\mathfrak{n}$ admits a nilsoliton metric if and only if the $\mathrm{GL}_n(\mathbb{C})$-orbit of μ is distinguished of the action of $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$.*

Theorem 2.5 provides us with another technique. It is easy to see that Nikolayevsky's nice basis criterion is also true in the complex case (see [12, Remark 3.2]); i.e. given a complex nilpotent Lie algebra $(\mathbb{C}^n, \mu)$ which is written in a nice basis for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ ($\check{m}(\mathrm{A} \cdot \mu) \subseteq \mathfrak{a}$), then $\mathrm{GL}_n(\mathbb{C}) \cdot \mu$ is distinguished if and only if Eq. (2.3) has at least one solution x with positive coordinates where $m = \#(\mathfrak{R}(\mu))$ and $\boldsymbol{U}$ is the Gram matrix of $(\mathfrak{R}(\mu), \langle\cdot,\cdot\rangle)$ (here, $\langle\cdot,\cdot\rangle$ is the usual Hermitian inner product on $\mathfrak{gl}_n(\mathbb{C})$).

A real nilpotent Lie algebra $\mathfrak{n} := (\mathbb{R}^n, \mu)$ could fail to admit a nice basis for the action of $\mathrm{GL}_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ (see [4, Proposition 2.1.] or [6, Section 2]). However, it may happen that $\mathfrak{n}$ admits a nice basis for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$. Suppose that $(\mathbb{C}^n, \widehat{\mu})$ is written in a nice basis with $\widehat{\mu} \in \mathrm{GL}_n(\mathbb{C}) \cdot \mu$ and $\mathrm{GL}_n(\mathbb{C}) \cdot \mu$ being a distinguished orbit; consequently, $(\mathbb{R}^n, \mu)$ admitting a nilsoliton metric. One can easily find a distinguished point $\widetilde{\mu}$ in the A-orbit of $\widehat{\mu}$ as above. By results of Linda Ness [11, Theorem 6.2], it is well known that any distinguished point in $\mathrm{GL}_n(\mathbb{C}) \cdot \mu$ is in the $\mathbb{C}^*\mathrm{U}(n)$-orbit of $\widetilde{\mu}$, hence to find a distinguished point in the $\mathrm{GL}_n(\mathbb{R})$-orbit of μ, we study the real forms in $\mathbb{C}^*\mathrm{U}(n) \cdot \widetilde{\mu}$ which are isomorphic over $\mathbb{R}$ to μ.

3. The classification

In this section, we give the classification of nilsoliton metrics on 7-dimensional (isolated) nilpotent Lie algebras. By using that any nilpotent Lie algebra of dimension less than or equal to 6 is an Einstein nilradical, one obtains that any decomposable 7-dimensional nilpotent Lie algebra is an Einstein nilradical and it is easy to give a nilsoliton metric in each case; therefore we focus on studying indecomposable algebras.

The general outline of our classification could be presented as follows:

- If the Lie algebra $(\mathbb{R}^7, \mu)$ is written in a nice basis and it admits a nilsoliton metric, then such distinguished metric can be found in the A-orbit of μ. We illustrate this technique in Example 3.2.
- If the Lie algebra $(\mathbb{R}^7, \mu)$ is not written in a nice basis but it admits a nice basis $(\mathbb{C}^7, \widehat{\mu})$ for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$. A nilsoliton metric for $(\mathbb{R}^7, \mu)$ (if such metric exists) can be found in the real forms of $\mathbb{C}^*\mathrm{U}(n)\mathrm{A} \cdot \widehat{\mu}$ which are isomorphic over $\mathbb{R}$ to μ (see Example 3.4).
- If the Lie algebra $(\mathbb{R}^7, \mu)$ is not written in a nice basis but it is possible to give a nilsoliton metric by solving the system of polynomial equations defined by the pre-Einstein derivation and the nilsoliton condition (see [6, Corollary 3]). We explain this technique in Example 3.3 (see also [6, Example 1]).

Following [10], there are 5 one-parameter families and 41 isolated nilpotent Lie algebras which are not written in a nice basis. We will use the exclamation mark to indicate such algebras (see List in Section 3.1), and where we have an extra exclamation mark, such algebra has a non-positive pre-Einstein derivation; and hence it does not admit any nilsoliton metric. The one-parameter families $(147E1)[t]$, $(1357S)[t]$ and the isolated algebras $(257J1)$, $(247E)$, $(247G)$, $(247H)$, $(247H1)$, $(247R)$, $(1357Q)$, $(1357Q1)$, $(1357R)$, $(12457L)$ can be worked out as Examples 3.3 and 3.4, while the remaining algebras were studied in [13] (to obtain [6, Theorem 7], we give in some cases nilsoliton metrics which are considered here again).

Theorem 3.1. *The classification of* 7*-dimensional nilsoliton metrics on isolated nilpotent Lie algebras (plus* $(147E)[0 < t < 1]$*) is given according to the notation in* [10] *by the list in Section* 3.1*.*

Example 3.2. Consider the one-parameter family $(1357QRS1)[t \in \mathbb{R}]$ given by $(\mathbb{R}^7, \mu_t)$ with

$$\mu_t := \begin{cases} [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_1, e_5] = e_7, \\ [e_2, e_3] = -e_6, [e_2, e_4] = e_5, [e_2, e_6] = te_7, [e_3, e_4] = (1-t)e_7. \end{cases}$$

The basis $\{e_1, \ldots, e_7\}$ is a nice basis to $(1357QRS1)[t \in \mathbb{R}]$ for any t. If $t \neq 0, 1$, the Gram matrix U is given by

$$\begin{bmatrix} 3 & 0 & 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 3 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 3 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 3 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

The general solution to the problem $Ux = [1]_8$ is given by

$$x = \frac{1}{11}(1 + 11\,a_1, 2, -1 + 11\,a_2, 5 - 11\,a_1 - 11\,a_2, 2, 4 - 11\,a_1 - 11\,a_2, 11\,a_2, 11\,a_1)^T.$$

By taking a_1 and a_2 such that $0 < 11a_1 < 3$ and $1 < 11a_2 < 4 - 11a_1$, we get a solution with positive coordinates. Hence, $(1357QRS1)[t \in \mathbb{R}]$ with $t \neq 0, 1$ admits a nilsoliton metric [7, Theorem 3].

To give a nilsoliton metric for each t in a simultaneous manner, it is incredibly difficult. But, if we fix the value of t, say $t = -1$, it is easy to give the nilsoliton metric for $(1357QRS1)[t = -1]$. To find such metric, we solve the problem

$$\begin{aligned} m(g \cdot \mu_{-1}) &= \mathrm{mcc}(\mathfrak{R}(\mu_{-1})) \\ &= \frac{1}{13}\,\mathrm{Diag}(-7, -7, -3, -3, 1, 1, 5) \end{aligned}$$

for $g \in \mathrm{A}$ (Theorem 2.3). Let $g = \mathrm{Diag}(1, 1, \frac{2\sqrt{39}}{39}, \frac{2\sqrt{13}}{13}, \frac{2\sqrt{3}}{39}, \frac{2\sqrt{3}}{39}, \frac{\sqrt{130}}{507})$, the change of basis given by g defines $\widetilde{\mu}_{-1} := g \cdot \mu_{-1} = g\mu_{-1}(g^{-1}\cdot, g^{-1}\cdot)$

$$\widetilde{\mu}_{-1} := \begin{cases} [e_1, e_2] = \dfrac{2\sqrt{39}}{39}e_3, [e_1, e_3] = \dfrac{\sqrt{13}}{13}e_5, [e_1, e_4] = \dfrac{\sqrt{39}}{39}e_6, [e_1, e_5] = \dfrac{\sqrt{390}}{78}e_7, \\ [e_2, e_3] = -\dfrac{\sqrt{13}}{13}e_6, [e_2, e_4] = \dfrac{\sqrt{39}}{39}e_5, [e_2, e_6] = -\dfrac{\sqrt{390}}{78}e_7, [e_3, e_4] = \dfrac{\sqrt{390}}{78}e_7. \end{cases}$$

Since,

$$m(\widetilde{\mu}_{-1}) = \frac{1}{13}\operatorname{Diag}(-7,-7,-3,-3,1,1,5) \\ = -\frac{11}{13}\operatorname{Id} + \underbrace{\frac{4}{13}\operatorname{Diag}(1,1,2,2,3,3,4)}_{\text{Derivation}} \tag{3.1}$$

the canonical inner product of $\mathbb{R}^7$ defines a nilsoliton metric on $(\mathbb{R}^7, \widetilde{\mu}_{-1})$. Note that $(1357QRS1)[t=-1]$ becomes $(1357R)$ over $\mathbb{C}$, therefore $(1357R)$ also admits nilsoliton metrics (Theorem 2.5).

We study $(1357QRS1)[t=1]$ $(\cong_{\mathbb{C}}(1357Q)\cong_{\mathbb{C}}(1357Q1))$ and $(1357QRS1)[t=0]$ $(\cong_{\mathbb{C}}(2357D)\cong_{\mathbb{C}}(2357D1)\cong_{\mathbb{C}}(1357)[t=1])$ in an entirely analogous way.

Example 3.3. In this example, we show how to find the nilsoliton metric for $(1357R)$ given by $(\mathbb{R}^7,\mu)$ with

$$\mu := \begin{cases} [e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_6]=e_7, \\ [e_2,e_3]=e_6,\ [e_2,e_4]=e_6,\ [e_2,e_5]=e_7,\ [e_3,e_4]=e_7. \end{cases}$$

As we explained previously, $(1357R)$ must admit such a metric. Furthermore, there must exist a $g\in \mathrm{GL}_7(\mathbb{R})$ such that $m(g\cdot\mu)$ satisfies Eq. (3.1).

It is easy to see that the pre-Einstein derivation of $(\mathbb{R}^7,\mu)$ is equal to $\frac{4}{11}\operatorname{Diag}(1,1,2,2,3,3,4)$ and as the pre-Einstein derivation of $g\cdot\mu$ must be positively proportional to the Einstein derivation, $\frac{4}{13}\operatorname{Diag}(1,1,2,2,3,3,4)$, we can try to find such g in the group

$$\mathrm{G} = \left\{ g\in \mathrm{GL}_7(\mathbb{R}) : g = \operatorname{Diag}\left(\begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2}\end{pmatrix}, \begin{pmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4}\end{pmatrix}, \begin{pmatrix} m_{5,5} & m_{5,6} \\ m_{6,5} & m_{6,6}\end{pmatrix}, m_{7,7} \right) \right\}$$

which commutates with the Einstein derivation. By solving

$$m(g\cdot\mu) = \frac{1}{13}\operatorname{Diag}(-7,-7,-3,-3,1,1,5)$$

with $g\in\mathrm{G}$, we find

$$g = \operatorname{Diag}\left(\begin{pmatrix} 1 & \frac{\sqrt{13}}{2} \\ 1 & -\frac{\sqrt{13}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{39}}{39} & \frac{\sqrt{3}}{3} \\ -\frac{2\sqrt{39}}{39} & -\frac{\sqrt{3}}{3} \end{pmatrix}, -\frac{\sqrt{130}}{39} \right)$$

which defines $\widetilde{\mu} = g\cdot\mu$

$$\widetilde{\mu} := \begin{cases} [e_1,e_2]=\frac{2\sqrt{39}}{39}e_3,\ [e_1,e_3]=\frac{\sqrt{13}}{13}e_6,\ [e_1,e_4]=\frac{\sqrt{39}}{39}e_5,\ [e_1,e_6]=\frac{\sqrt{390}}{78}e_7, \\ [e_2,e_3]=\frac{\sqrt{13}}{13}e_5,\ [e_2,e_4]=\frac{\sqrt{39}}{39}e_6,\ [e_2,e_5]=-\frac{\sqrt{390}}{78}e_7,\ [e_3,e_4]=\frac{\sqrt{390}}{78}e_7 \end{cases}$$

where the canonical inner product of $\mathbb{R}^7$ defines a nilsoliton metric on $(\mathbb{R}^7,\widetilde{\mu})$.

Example 3.4. For a final example, we consider the one-parameter family $(1357S)[t\in\mathbb{R}\setminus\{0,1\}]$ given by $(\mathbb{R}^7,\mu_t)$ with

$$\mu_t = \begin{cases} [e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_7,\ [e_2,e_3]=e_6,\ [e_2,e_4]=e_6, \\ [e_2,e_5]=e_7,\ [e_2,e_6]=te_7,\ [e_3,e_4]=e_7. \end{cases}$$

It is easily seen that $(1357S)[t]$ (with $t\neq 0,1$) is a real form of the (complex) Lie algebra $(1357QRS1)[\lambda]$ with $\lambda := \frac{t+\sqrt{t}}{t-\sqrt{t}}$. In the same manner as in Example 3.2, we can see that the $\mathrm{GL}_7(\mathbb{C})$-orbit of $(1357QRS1)[\lambda]$ is distinguished for the natural action of $\mathrm{GL}_7(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^7)^*\otimes\mathbb{C}^7$ (by using Nikolayevsky's nice basis criterion in the complex case). Consequently, the $\mathrm{GL}_7(\mathbb{R})$-orbit of $(1357S)[t]$ (with $t\neq 0,1$) is distinguished; i.e. such family admits nilsoliton metrics (Theorem 2.5).

We fix the value of t, say $t=-3$. So $(1357S)[t=-3]$ is a real form of $(1357QRS1)[\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}] := (\mathbb{C}^7,\mu)$ with

$$\mu = \begin{cases} [e_1,e_2]=e_3,\ [e_1,e_3]=e_5,\ [e_1,e_4]=e_6,\ [e_1,e_5]=e_7,\ [e_2,e_3]=-e_6,\ [e_2,e_4]=e_5, \\ [e_2,e_6]=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}\right)e_7,\ [e_3,e_4]=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}\right)e_7. \end{cases}$$

To find a nilsoliton metric for $(1357S)[t = -3]$, we can find a distinguished point $\widetilde{\mu}$ in the $\mathrm{GL}_7(\mathbb{C})$-orbit of μ and then to study the real forms in the U(7)-orbit of $\widetilde{\mu}$.

The point $\widetilde{\mu}$ can be found easily in much the same way as was done for $(1357QRS1)[t = -1]$ in Example 3.2. So we find $\widetilde{\mu}$ given by

$$\widetilde{\mu} = \begin{cases} [e_1, e_2] = \frac{\sqrt{13}}{13}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = \frac{\sqrt{26}}{26}e_6, [e_1, e_5] = \frac{\sqrt{13}}{13}e_7, [e_2, e_3] = -\frac{\sqrt{13}}{13}e_6, \\ [e_2, e_4] = \frac{\sqrt{26}}{26}e_5, [e_2, e_6] = \frac{\sqrt{13}}{13}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}\right)e_7, [e_3, e_4] = \frac{\sqrt{26}}{26}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}\right)e_7. \end{cases}$$

Since $(1357S)[t = -3]$ and $(\mathbb{C}^7, \widetilde{\mu})$ have the "same" pre-Einstein derivation, $\frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$, we can try to find a distinguished point in the $\mathrm{GL}_7(\mathbb{R})$-orbit of $(1357S)[t = -3]$ by considering the real forms in the G-orbit of $\widetilde{\mu}$, where

$$\mathrm{G} = \left\{ g \in \mathrm{U}(7) : g = \mathrm{Diag}\left(\begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}, \begin{pmatrix} z_{3,3} & z_{3,4} \\ z_{4,3} & z_{4,4} \end{pmatrix}, \begin{pmatrix} z_{5,5} & z_{5,6} \\ z_{6,5} & z_{6,6} \end{pmatrix}, z_{7,7} \right) \right\}.$$

Therefore, we find

$$\widehat{\mu} = \begin{cases} [e_1, e_2] = \frac{\sqrt{13}}{13}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = -\frac{\sqrt{26}}{26}e_5, [e_1, e_5] = \frac{\sqrt{39}}{26}e_7, [e_1, e_6] = \frac{\sqrt{13}}{26}e_7, \\ [e_2, e_3] = \frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{26}}{26}e_6, [e_2, e_5] = \frac{\sqrt{13}}{26}e_7, [e_2, e_6] = -\frac{\sqrt{39}}{26}e_7, [e_3, e_4] = \frac{\sqrt{26}}{26}e_7, \end{cases}$$

where $(1357S)[t = -3]$ is isomorphic to $(\mathbb{R}^7, \widehat{\mu})$ (over $\mathbb{R}$) and the canonical inner product of $\mathbb{R}^7$ defines a nilsoliton metric on $(\mathbb{R}^7, \widehat{\mu})$.

By a similar argument, we can to prove that $(147E1)[t > 1]$, which is a real form of the (complex) Lie algebra $(147E)[\lambda]$ with $\lambda = \left(\frac{(1-\sqrt{t^2-1}\sqrt{-1})}{t}\right)^2$, admits nilsoliton metrics.

Remark 3.5. Recently, the existence problem of nilsoliton metrics on $\mathfrak{n}_{11} = (1357S)[t = -3]$, $\mathfrak{n}_{12} = (147E1)[t = 2]$ and $\mathfrak{n}_{10} \cong_{\mathbb{R}} 1.3(i)[t = 1]$ has been studied in [17, Proposition 3.2]. In the above example, we give a nilsoliton metric for $\mathfrak{n}_{11}$, and by a similar argument we find a nilsoliton metric for $\mathfrak{n}_{12}$ given by the change of basis

$$g := \mathrm{Diag}\left(\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}, 1, -\frac{1}{4}, \begin{pmatrix} \frac{\sqrt{3}}{12} & \frac{\sqrt{3}}{6} \\ -\frac{1}{4} & 0 \end{pmatrix}, -\frac{\sqrt{3}}{48} \right)$$

which defines

$$\widetilde{\mu} = \begin{cases} [e_1, e_2] = \frac{\sqrt{3}}{6}e_4, [e_1, e_3] = -\frac{\sqrt{3}}{6}e_6, [e_1, e_5] = -\frac{\sqrt{3}}{12}e_7, [e_1, e_6] = \frac{1}{4}e_7, \\ [e_2, e_3] = -\frac{\sqrt{3}}{6}e_5, [e_2, e_5] = \frac{1}{4}e_7, [e_2, e_6] = \frac{\sqrt{3}}{12}e_7, [e_3, e_4] = -\frac{\sqrt{3}}{6}e_7, \end{cases}$$

where the canonical inner product of $\mathbb{R}^7$ is a nilsoliton metric of $(\mathbb{R}^7, \widetilde{\mu}) \cong_{\mathbb{R}} \mathfrak{n}_{12}$. The nilpotent Lie algebra $1.3(i)[t = 1]$ also admits a nilsoliton metric and it was proved in [6, Example 2]. The nilpotent Lie algebra $\mathfrak{n}_9$ in [17, Proposition 3.2], in fact, does not admit a nilsoliton metric and this can be proved by using Nikolayevsky's nice basis criterion ($\mathfrak{n}_9 = 1.1(iv)$; see [6, Table 1.]).

3.1. Classification list

The notation in the list is as follows: dim(Der) is the dimension of the algebra of derivations, rank is the dimension of a maximal torus of semisimple derivations. By ϕ we denote 4 times the Einstein derivation or the pre-Einstein derivation, depending if the respective algebra admits or does not admit a nilsoliton metric, and $\|\beta\|^2$ is the real number such that

$$4\,Ric = -\|\beta\|^2\,\mathrm{Id} + \phi.$$

Each Lie bracket $\widetilde{\mu}$ given in the list is such that the canonical inner product of $\mathbb{R}^7$ is a nilsoliton metric of scalar curvature equal to $-\frac{1}{4}$ on $\widetilde{\mathfrak{n}} := (\mathbb{R}^7, \widetilde{\mu})$.

The Betti numbers provide additional information that can be used to compare Gong's list with other classifications.

(1) (**37A**): dim(Der) $= 25$, rank $= 4$, Betti Numbers $(4, 12, 18, 18, 12, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{4.2})$.
Einstein der. $\phi = \frac{2}{3}\,\mathrm{Diag}(2, 1, 2, 2, 3, 3, 3)$, $\|\beta\|^2 = \frac{5}{3} \approx 1.667$
$[e_1, e_2] = \frac{\sqrt{6}}{6}e_5$, $[e_2, e_3] = \frac{\sqrt{6}}{6}e_6$, $[e_2, e_4] = \frac{\sqrt{6}}{6}e_7$.

(2) (**37B**): dim(Der) $= 20$, rank $= 4$, Betti Numbers $(4, 11, 16, 16, 11, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{4.1})$.
Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(5, 4, 4, 5, 9, 8, 9)$, $\|\beta\|^2 = \frac{7}{5} \approx 1.400$
$[e_1, e_2] = \frac{\sqrt{5}}{5}e_5$, $[e_2, e_3] = \frac{\sqrt{10}}{10}e_6$, $[e_3, e_4] = \frac{\sqrt{5}}{5}e_7$.

(3) (**37B1**): dim(Der) $= 20$, rank $= 4$, Betti Numbers $(4, 11, 16, 16, 11, 4, 1)$ $(\cong_{\mathbb{C}} \mathfrak{g}_{4.1})$.
Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(4, 5, 5, 4, 9, 9, 8)$, $\|\beta\|^2 = \frac{7}{5} \approx 1.400$
$[e_1, e_2] = \frac{\sqrt{10}}{10}e_5$, $[e_1, e_3] = \frac{\sqrt{10}}{10}e_6$, $[e_1, e_4] = \frac{\sqrt{10}}{10}e_7$, $[e_2, e_4] = \frac{\sqrt{10}}{10}e_6$, $[e_3, e_4] = -\frac{\sqrt{10}}{10}e_5$.

(4) (**37C**): dim(Der) $= 22$, rank $= 3$, Betti Numbers $(4, 11, 17, 17, 11, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.24})$.
Einstein der. $\phi = \frac{1}{4}\,\mathrm{Diag}(5, 3, 4, 4, 8, 7, 7)$, $\|\beta\|^2 = \frac{3}{2} \approx 1.500$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_5$, $[e_2, e_3] = \frac{\sqrt{2}}{4}e_6$, $[e_2, e_4] = \frac{\sqrt{2}}{4}e_7$, $[e_3, e_4] = \frac{\sqrt{2}}{4}e_5$.

(5) (**37D**): dim(Der) $= 19$, rank $= 3$, Betti Numbers $(4, 11, 14, 14, 11, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.12})$.
Einstein der. $\phi = \frac{5}{6}\,\mathrm{Diag}(1, 1, 1, 1, 2, 2, 2)$, $\|\beta\|^2 = \frac{4}{3} \approx 1.333$
$[e_1, e_2] = \frac{\sqrt{3}}{6}e_5$, $[e_1, e_3] = \frac{\sqrt{6}}{6}e_6$, $[e_2, e_4] = \frac{\sqrt{6}}{6}e_7$, $[e_3, e_4] = \frac{\sqrt{3}}{6}e_5$.

(6) (**37D1**): dim(Der) $= 19$, rank $= 3$, Betti Numbers $(4, 11, 14, 14, 11, 4, 1)$ $(\cong_{\mathbb{C}} \mathfrak{g}_{3.12})$.
Einstein der. $\phi = \frac{5}{6}\,\mathrm{Diag}(1, 1, 1, 1, 2, 2, 2)$, $\|\beta\|^2 = \frac{4}{3} \approx 1.333$
$[e_1, e_2] = \frac{\sqrt{3}}{6}e_5$, $[e_1, e_3] = \frac{\sqrt{3}}{6}e_6$, $[e_1, e_4] = \frac{\sqrt{3}}{6}e_7$, $[e_2, e_3] = -\frac{\sqrt{3}}{6}e_7$, $[e_2, e_4] = \frac{\sqrt{3}}{6}e_6$, $[e_3, e_4] = -\frac{\sqrt{3}}{6}e_5$.

(7) (**357A**): dim(Der) $= 18$, rank $= 3$, Betti Numbers $(3, 8, 14, 14, 8, 3, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.6})$.
Einstein der. $\phi = \frac{1}{11}\,\mathrm{Diag}(5, 7, 12, 9, 17, 16, 14)$, $\|\beta\|^2 = \frac{13}{11} \approx 1.182$
$[e_1, e_2] = \frac{\sqrt{66}}{22}e_3$, $[e_1, e_3] = \frac{\sqrt{22}}{11}e_5$, $[e_1, e_4] = \frac{\sqrt{22}}{22}e_7$, $[e_2, e_4] = \frac{\sqrt{66}}{22}e_6$.

(8) (**357B**): dim(Der) $= 17$, rank $= 3$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.23})$.
Einstein der. $\phi = \frac{1}{10}\,\mathrm{Diag}(4, 5, 9, 9, 13, 14, 13)$, $\|\beta\|^2 = \frac{11}{10} \approx 1.100$
$[e_1, e_2] = \frac{\sqrt{15}}{10}e_3$, $[e_1, e_3] = \frac{\sqrt{10}}{10}e_5$, $[e_1, e_4] = \frac{\sqrt{10}}{10}e_7$, $[e_2, e_3] = \frac{\sqrt{15}}{10}e_6$

(9) (**357C**): dim(Der) $= 16$, rank $= 2$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{2.40})$.
Einstein der. $\phi = \frac{1}{21}\,\mathrm{Diag}(9, 10, 19, 18, 28, 29, 27)$, $\|\beta\|^2 = \frac{23}{21} \approx 1.095$
$[e_1, e_2] = \frac{\sqrt{7}}{7}e_3$, $[e_1, e_3] = \frac{\sqrt{42}}{21}e_5$, $[e_1, e_4] = \frac{\sqrt{42}}{21}e_7$, $[e_2, e_3] = \frac{\sqrt{7}}{7}e_6$, $[e_2, e_4] = \frac{\sqrt{42}}{42}e_5$.

(10) (**27A**): dim(Der) $= 21$, rank $= 4$, Betti Numbers $(5, 10, 16, 16, 10, 5, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{4.3})$.
Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(4, 5, 5, 6, 5, 9, 10)$, $\|\beta\|^2 = \frac{7}{5} \approx 1.400$
$[e_1, e_2] = \frac{\sqrt{5}}{5}e_6$, $[e_1, e_4] = \frac{\sqrt{10}}{10}e_7$, $[e_3, e_5] = \frac{\sqrt{5}}{5}e_7$.

(11) (**27B**): dim(Der) $= 19$, rank $= 3$, Betti Numbers $(5, 9, 15, 15, 9, 5, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.19})$.
Einstein der. $\phi = \frac{1}{6}\,\mathrm{Diag}(5, 6, 5, 6, 6, 11, 11)$, $\|\beta\|^2 = \frac{4}{3} \approx 1.333$
$[e_1, e_2] = \frac{\sqrt{3}}{6}e_6$, $[e_1, e_5] = \frac{\sqrt{6}}{6}e_7$, $[e_2, e_3] = \frac{\sqrt{3}}{6}e_7$, $[e_3, e_4] = \frac{\sqrt{6}}{6}e_6$.

(12) (**257A**): dim(Der) $= 19$, rank $= 3$, Betti Numbers $(4, 9, 14, 14, 9, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.8})$.
Einstein der. $\phi = \frac{1}{4}\,\mathrm{Diag}(2, 3, 5, 4, 4, 7, 6)$, $\|\beta\|^2 = \frac{5}{4} \approx 1.250$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_3$, $[e_1, e_3] = \frac{\sqrt{2}}{4}e_6$, $[e_1, e_5] = \frac{\sqrt{2}}{4}e_7$, $[e_2, e_4] = \frac{\sqrt{2}}{4}e_6$.

(13) (**257B**): dim(Der) $= 18$, rank $= 3$, Betti Numbers $(4, 8, 13, 13, 8, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.11})$.
Einstein der. $\phi = \frac{1}{11}\,\mathrm{Diag}(5, 7, 12, 12, 10, 17, 17)$, $\|\beta\|^2 = \frac{13}{11} \approx 1.182$
$[e_1, e_2] = \frac{\sqrt{66}}{22}e_3$, $[e_1, e_3] = \frac{\sqrt{22}}{11}e_6$, $[e_1, e_4] = \frac{\sqrt{22}}{22}e_7$, $[e_2, e_5] = \frac{\sqrt{66}}{22}e_7$.

(14) (**257C**): dim(Der) $= 18$, rank $= 3$, Betti Numbers $(4, 9, 13, 13, 9, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.9})$.
Einstein der. $\phi = \frac{2}{11}\,\mathrm{Diag}(3, 3, 6, 6, 5, 9, 8)$, $\|\beta\|^2 = \frac{13}{11} \approx 1.182$
$[e_1, e_2] = \frac{\sqrt{66}}{22}e_3$, $[e_1, e_3] = \frac{\sqrt{22}}{11}e_6$, $[e_2, e_4] = \frac{\sqrt{22}}{22}e_6$, $[e_2, e_5] = \frac{\sqrt{66}}{22}e_7$.

(15) (**257D**): dim(Der) $= 17$, rank $= 2$, Betti Numbers $(4, 8, 12, 12, 8, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{2.45})$.
Einstein der. $\phi = \frac{1}{12}\,\mathrm{Diag}(6, 7, 13, 12, 11, 19, 18)$, $\|\beta\|^2 = \frac{7}{6} \approx 1.167$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_3$, $[e_1, e_3] = \frac{\sqrt{6}}{6}e_6$, $[e_1, e_4] = \frac{\sqrt{6}}{12}e_7$, $[e_2, e_4] = \frac{\sqrt{6}}{12}e_6$, $[e_2, e_5] = \frac{\sqrt{2}}{4}e_7$.

(16) (**257E**): dim(Der) $= 17$, rank $= 3$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.15})$.
Einstein der. $\phi = \frac{1}{10}\,\mathrm{Diag}(5, 6, 11, 7, 9, 16, 13)$, $\|\beta\|^2 = \frac{11}{10} \approx 1.100$
$[e_1, e_2] = \frac{\sqrt{15}}{10}e_3$, $[e_1, e_3] = \frac{\sqrt{15}}{10}e_6$, $[e_2, e_4] = \frac{\sqrt{10}}{10}e_7$, $[e_4, e_5] = \frac{\sqrt{10}}{10}e_6$.

(17) (**257F**): dim(Der) $= 18$, rank $= 3$, Betti Numbers $(4, 9, 12, 12, 9, 4, 1)$ $(\cong_{\mathbb{R}} \mathfrak{g}_{3.14})$.
Einstein der. $\phi = \frac{1}{11}\,\mathrm{Diag}(9, 5, 14, 9, 10, 19, 14)$, $\|\beta\|^2 = \frac{13}{11} \approx 1.182$
$[e_1, e_2] = \frac{\sqrt{22}}{11}e_3$, $[e_2, e_3] = \frac{\sqrt{66}}{22}e_6$, $[e_2, e_4] = \frac{\sqrt{22}}{22}e_7$, $[e_4, e_5] = \frac{\sqrt{66}}{22}e_6$.

(18) (**257G**): dim(Der) $= 16$, rank $= 2$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.36}$).
Einstein der. $\phi = \frac{1}{21}$ Diag$(10, 13, 23, 15, 18, 33, 28)$, $\|\beta\|^2 = \frac{23}{21} \approx 1.095$
$[e_1, e_2] = \frac{\sqrt{7}}{7}e_3, [e_1, e_3] = \frac{\sqrt{7}}{7}e_6, [e_1, e_5] = \frac{\sqrt{42}}{42}e_7, [e_2, e_4] = \frac{\sqrt{42}}{21}e_7, [e_4, e_5] = \frac{\sqrt{42}}{21}e_6$.

(19) (**257H**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.7}$).
Pre-Einstein der. $\phi = \frac{1}{3}$ Diag$(1, 2, 3, 2, 2, 4, 4)$
It does not admit nilsoliton metrics.

(20) (**257I**)!: dim(Der) $= 17$, rank $= 2$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.27}$!).
Einstein der. $\phi = \frac{1}{13}$ Diag$(6, 7, 13, 13, 14, 19, 20)$, $\|\beta\|^2 = \frac{15}{13} \approx 1.154$
$[e_1, e_2] = \frac{\sqrt{78}}{26}e_3 + \frac{\sqrt{26}}{26}e_4, [e_1, e_3] = \frac{\sqrt{26}}{26}e_6, [e_1, e_4] = \frac{\sqrt{78}}{26}e_6, [e_1, e_5] = \frac{\sqrt{26}}{26}e_7, [e_2, e_3] = \frac{\sqrt{26}}{13}e_7$.

(21) (**257J**): dim(Der) $= 16$, rank $= 2$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.38}$).
Einstein der. $\phi = \frac{1}{2}$ Diag$(1, 1, 2, 2, 2, 3, 3)$, $\|\beta\|^2 = \frac{8}{7} \approx 1.143$
$[e_1, e_2] = \frac{\sqrt{7}}{7}e_3, [e_1, e_3] = \frac{\sqrt{21}}{14}e_6, [e_1, e_5] = \frac{\sqrt{14}}{14}e_7, [e_2, e_3] = \frac{\sqrt{21}}{14}e_7, [e_2, e_4] = \frac{\sqrt{14}}{14}e_6$.

(22) (**257J1**)!: dim(Der) $= 16$, rank $= 2$, Betti Numbers $(4, 8, 11, 11, 8, 4, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{2.38}$).
Einstein der. $\phi = \frac{1}{2}$ Diag$(1, 1, 2, 2, 2, 3, 3)$, $\|\beta\|^2 = \frac{8}{7} \approx 1.143$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_3 + \frac{\sqrt{7}}{14}e_4, [e_1, e_3] = \frac{\sqrt{7}}{14}e_6, [e_1, e_4] = \frac{\sqrt{21}}{14}e_6, [e_1, e_5] = \frac{\sqrt{7}}{14}e_7, [e_2, e_3] = \frac{\sqrt{7}}{7}e_7, [e_2, e_5] = \frac{\sqrt{7}}{14}e_6$.

(23) (**257K**): dim(Der) $= 16$, rank $= 3$, Betti Numbers $(4, 6, 9, 9, 6, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.13}$).
Einstein der. $\phi = \frac{1}{22}$ Diag$(8, 11, 19, 15, 15, 27, 30)$, $\|\beta\|^2 = \frac{21}{22} \approx 0.9546$
$[e_1, e_2] = \frac{\sqrt{77}}{22}e_3, [e_1, e_3] = \frac{\sqrt{66}}{22}e_6, [e_2, e_3] = \frac{\sqrt{33}}{22}e_7, [e_4, e_5] = \frac{\sqrt{66}}{22}e_7$.

(24) (**257L**): dim(Der) $= 14$, rank $= 2$, Betti Numbers $(4, 6, 9, 9, 6, 4, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.29}$).
Pre-Einstein der. $\phi = \frac{1}{41}$ Diag$(15, 22, 37, 30, 29, 52, 59)$,
It does not admit a nilsoliton inner product.

(25) (**247A**): dim(Der) $= 19$, rank $= 3$, Betti Numbers $(3, 7, 13, 13, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.20}$).
Einstein der. $\phi = \frac{1}{4}$ Diag$(1, 4, 4, 5, 5, 6, 6)$, $\|\beta\|^2 = \frac{5}{4} \approx 1.250$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_4, [e_1, e_3] = \frac{\sqrt{2}}{4}e_5, [e_1, e_4] = \frac{\sqrt{2}}{4}e_6, [e_1, e_5] = \frac{\sqrt{2}}{4}e_7$.

(26) (**247B**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.21}$).
Einstein der. $\phi = \frac{1}{22}$ Diag$(6, 15, 11, 21, 17, 27, 28)$, $\|\beta\|^2 = \frac{21}{22} \approx 0.9546$
$[e_1, e_2] = \frac{\sqrt{66}}{22}e_4, [e_1, e_3] = \frac{\sqrt{33}}{22}e_5, [e_1, e_4] = \frac{\sqrt{66}}{22}e_6, [e_3, e_5] = \frac{\sqrt{77}}{22}e_7$.

(27) (**247C**): dim(Der) $= 16$, rank $= 2$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.43}$).
Einstein der. $\phi = \frac{1}{35}$ Diag$(11, 29, 20, 40, 31, 51, 42)$, $\|\beta\|^2 = \frac{37}{35} \approx 1.057$
$[e_1, e_2] = \frac{2\sqrt{35}}{35}e_4, [e_1, e_3] = \frac{2\sqrt{35}}{35}e_5, [e_1, e_4] = \frac{\sqrt{14}}{14}e_6, [e_1, e_5] = \frac{\sqrt{14}}{14}e_7, [e_3, e_5] = \frac{3\sqrt{70}}{70}e_6$.

(28) (**247D**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.22}$).
Einstein der. $\phi = \frac{1}{22}$ Diag$(7, 10, 12, 17, 19, 24, 29)$, $\|\beta\|^2 = \frac{10}{11} \approx 0.9091$
$[e_1, e_2] = \frac{\sqrt{55}}{22}e_4, [e_1, e_3] = \frac{\sqrt{11}}{11}e_5, [e_1, e_4] = \frac{\sqrt{11}}{11}e_6, [e_2, e_5] = \frac{\sqrt{55}}{22}e_7, [e_3, e_4] = \frac{\sqrt{11}}{11}e_7$.

(29) (**247E**)!: dim(Der) $= 14$, rank $= 2$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{2.12}$).
Einstein der. $\phi = \frac{1}{10}$ Diag$(3, 5, 5, 8, 8, 11, 13)$, $\|\beta\|^2 = \frac{9}{10} \approx 0.9000$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_4, [e_1, e_3] = \frac{\sqrt{30}}{20}e_5, [e_1, e_4] = \frac{\sqrt{10}}{10}e_6, [e_2, e_4] = \frac{\sqrt{30}}{20}e_7, [e_3, e_5] = -\frac{\sqrt{2}}{4}e_7$.

(30) (**247E1**): dim(Der) $= 14$, rank $= 2$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.12}$).
Einstein der. $\phi = \frac{1}{10}$ Diag$(3, 5, 5, 8, 8, 11, 13)$, $\|\beta\|^2 = \frac{9}{10} \approx 0.9000$
$[e_1, e_2] = \frac{\sqrt{2}}{4}e_4, [e_1, e_3] = \frac{\sqrt{30}}{20}e_5, [e_1, e_4] = \frac{\sqrt{10}}{10}e_6, [e_2, e_4] = \frac{\sqrt{30}}{20}e_7, [e_3, e_5] = \frac{\sqrt{2}}{4}e_7$.

(31) (**247F**): dim(Der) $= 13$, rank $= 3$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.4}$).
Einstein der. $\phi = \frac{1}{14}$ Diag$(6, 5, 5, 11, 11, 16, 16)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_4, [e_1, e_3] = \frac{\sqrt{21}}{14}e_5, [e_2, e_4] = \frac{\sqrt{14}}{14}e_6, [e_2, e_5] = \frac{\sqrt{14}}{14}e_7, [e_3, e_4] = \frac{\sqrt{14}}{14}e_7, [e_3, e_5] = \frac{\sqrt{14}}{14}e_6$.

(32) (**247F1**): dim(Der) $= 13$, rank $= 3$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{3.4}$).
Einstein der. $\phi = \frac{1}{14}$ Diag$(6, 5, 5, 11, 11, 16, 16)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_4, [e_1, e_3] = \frac{\sqrt{21}}{14}e_5, [e_2, e_4] = \frac{\sqrt{14}}{14}e_6, [e_2, e_5] = \frac{\sqrt{14}}{14}e_7, [e_3, e_4] = \frac{\sqrt{14}}{14}e_7, [e_3, e_5] = -\frac{\sqrt{14}}{14}e_6$.

(33) (**247G**)!: dim(Der) $= 12$, rank $= 2$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.34}$).
Einstein der. $\phi = \frac{1}{55}$ Diag$(22, 20, 21, 42, 43, 62, 64)$, $\|\beta\|^2 = \frac{47}{55} \approx 0.8546$
$[e_1, e_2] = -\frac{\sqrt{330}}{55}e_4, [e_1, e_3] = -\frac{\sqrt{10}}{10}e_5, [e_1, e_4] = \frac{\sqrt{55}}{55}e_7, [e_2, e_4] = \frac{\sqrt{66}}{22}e_6, [e_3, e_5] = -\frac{\sqrt{66}}{22}e_7$.

(34) (**247H**)!: dim(Der) $= 11$, rank $= 1$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{1.19}$).
Einstein der. $\phi = \frac{13}{34}$ Diag$(1, 1, 1, 2, 2, 3, 3)$, $\|\beta\|^2 = \frac{29}{34} \approx 0.8529$
$[e_1, e_2] = -\frac{\sqrt{119}}{34}e_4, [e_1, e_3] = -\frac{\sqrt{119}}{34}e_5, [e_1, e_4] = \frac{\sqrt{17}}{34}e_7, [e_1, e_5] = -\frac{\sqrt{17}}{34}e_6, [e_2, e_4] = \frac{3\sqrt{17}}{34}e_6, [e_3, e_5] = -\frac{3\sqrt{17}}{34}e_7$.

(35) (**247H1**)!: dim(Der) $= 11$, rank $= 1$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{1.19}$).
Einstein der. $\phi = \frac{13}{34}$ Diag$(1, 1, 1, 2, 2, 3, 3)$, $\|\beta\|^2 = \frac{29}{34} \approx 0.8529$
$[e_1, e_2] = -\frac{\sqrt{119}}{34}e_4, [e_1, e_3] = \frac{3\sqrt{1309}}{374}e_5, [e_1, e_4] = \frac{3\sqrt{187}}{187}e_6, [e_2, e_3] = \frac{\sqrt{2618}}{374}e_5, [e_2, e_4] = -\frac{7\sqrt{374}}{748}e_6, [e_2, e_5] = -\frac{\sqrt{374}}{68}e_7, [e_3, e_4] = \frac{3\sqrt{34}}{68}e_7, [e_3, e_5] = -\frac{3\sqrt{34}}{68}e_6$.

(36) (**247I**): dim(Der) $= 14$, rank $= 3$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.5}$).
Einstein der. $\phi = \frac{1}{22}$ Diag$(10, 11, 7, 21, 17, 28, 24)$, $\|\beta\|^2 = \frac{10}{11} \approx 0.9091$
$[e_1, e_2] = \frac{\sqrt{55}}{22}e_4, [e_1, e_3] = \frac{\sqrt{55}}{22}e_5, [e_2, e_5] = \frac{\sqrt{11}}{11}e_6, [e_3, e_4] = \frac{\sqrt{11}}{11}e_6, [e_3, e_5] = \frac{\sqrt{11}}{11}e_7$.

(37) (**247J**)!: dim(Der) $= 13$, rank $= 2$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.26}!$).
Einstein der. $\phi = \frac{1}{19}$ Diag$(7, 10, 7, 17, 14, 21, 24)$, $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_2] = \frac{\sqrt{114}}{38}e_4, [e_1, e_3] = \frac{\sqrt{38}}{19}e_5, [e_1, e_5] = \frac{\sqrt{114}}{38}e_6, [e_2, e_3] = \frac{\sqrt{38}}{38}e_4, [e_2, e_5] = \frac{\sqrt{114}}{38}e_7, [e_3, e_4] = \frac{\sqrt{38}}{19}e_7, [e_3, e_5] = \frac{\sqrt{38}}{38}e_6$.

(38) (**247K**): dim(Der) $= 12$, rank $= 2$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.35}$).
Einstein der. $\phi = \frac{1}{15}$ Diag$(5, 7, 6, 12, 11, 17, 18)$, $\|\beta\|^2 = \frac{13}{15} \approx 0.8667$
$[e_1, e_2] = \frac{\sqrt{330}}{55}e_4, [e_1, e_3] = \frac{\sqrt{11}}{11}e_5, [e_1, e_4] = \frac{\sqrt{15}}{15}e_6, [e_2, e_5] = \frac{\sqrt{11}}{11}e_7, [e_3, e_4] = \frac{\sqrt{330}}{66}e_7, [e_3, e_5] = \frac{\sqrt{15}}{15}e_6$.

(39) (**247L**): dim(Der) $= 17$, rank $= 2$, Betti Numbers $(3, 7, 13, 13, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.39}$).
Einstein der. $\phi = \frac{1}{14}$ Diag$(5, 11, 10, 16, 15, 21, 20)$, $\|\beta\|^2 = \frac{8}{7} \approx 1.143$
$[e_1, e_2] = \frac{\sqrt{14}}{14}e_4, [e_1, e_3] = \frac{\sqrt{21}}{14}e_5, [e_1, e_4] = \frac{\sqrt{14}}{14}e_6, [e_1, e_5] = \frac{\sqrt{7}}{7}e_7, [e_2, e_3] = \frac{\sqrt{21}}{14}e_6$.

(40) (**247M**): dim(Der) $= 14$, rank $= 2$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.42}$).
Pre-Einstein der. $\phi = \frac{1}{41}$ Diag$(11, 30, 22, 41, 33, 52, 55)$.
It does not admit a nilsoliton inner product.

(41) (**247N**): dim(Der) $= 16$, rank $= 2$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.44}$).
Einstein der. $\phi = \frac{1}{35}$ Diag$(15, 19, 23, 34, 38, 53, 42)$, $\|\beta\|^2 = \frac{37}{35} \approx 1.057$
$[e_1, e_2] = \frac{\sqrt{14}}{14}e_4, [e_1, e_3] = \frac{3\sqrt{70}}{70}e_5, [e_1, e_5] = \frac{2\sqrt{35}}{35}e_6, [e_2, e_3] = \frac{\sqrt{14}}{14}e_7, [e_2, e_4] = \frac{2\sqrt{35}}{35}e_6$.

(42) (**247O**): dim(Der) $= 15$, rank $= 1$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{1.7}$).
Einstein der. $\phi = \frac{5}{28}$ Diag$(2, 4, 3, 6, 5, 8, 7)$, $\|\beta\|^2 = \frac{29}{28} \approx 1.036$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_4, [e_1, e_3] = \frac{\sqrt{70}}{28}e_5, [e_1, e_4] = \frac{\sqrt{70}}{28}e_6, [e_1, e_5] = \frac{\sqrt{42}}{28}e_7, [e_2, e_3] = \frac{\sqrt{42}}{28}e_7, [e_3, e_5] = \frac{\sqrt{21}}{14}e_6$.

(43) (**247P**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.1(i_0)}$).
Pre-Einstein der. $\phi = \frac{1}{2}$ Diag$(1, 1, 1, 2, 2, 2, 3)$,
It does not admit a nilsoliton inner product.

(44) (**247P1**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(3, 7, 11, 11, 7, 3, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{3.1(i_0)}$).
Pre-Einstein der. $\phi = \frac{1}{2}$ Diag$(1, 1, 1, 2, 2, 2, 3)$,
It does not admit a nilsoliton inner product.

(45) (**247Q**): dim(Der) $= 14$, rank $= 2$, Betti Numbers $(3, 6, 10, 10, 6, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(v)}$).
Pre-Einstein der. $\phi = \frac{2}{19}$ Diag$(3, 5, 6, 8, 9, 11, 14)$,
It does not admit a nilsoliton inner product.

(46) (**247R**)!: dim(Der) $= 13$, rank $= 1$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{C}} \mathfrak{g}_{1.3(iv)}$).
Pre-Einstein der. $\phi = \frac{5}{17}$ Diag$(1, 2, 2, 3, 3, 4, 5)$,
It does not admit a nilsoliton inner product.

(47) (**247R1**): dim(Der) $= 13$, rank $= 1$, Betti Numbers $(3, 5, 9, 9, 5, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(iv)}$).
Pre-Einstein der. $\phi = \frac{5}{17}$ Diag$(1, 2, 2, 3, 3, 4, 5)$,
It does not admit

(48) (**2457A**): dim(Der) $= 17$, rank $= 3$, Betti Numbers $(3, 7, 10, 10, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.2}$).
Einstein der. $\phi = \frac{2}{11}$ Diag$(1, 5, 6, 7, 6, 8, 7)$, $\|\beta\|^2 = \frac{13}{11} \approx 1.182$
$[e_1, e_2] = \frac{\sqrt{66}}{22}e_3, [e_1, e_3] = \frac{\sqrt{22}}{11}e_4, [e_1, e_4] = \frac{\sqrt{66}}{22}e_6, [e_1, e_5] = \frac{\sqrt{22}}{22}e_7$.

(49) (**2457B**): dim(Der) $= 15$, rank $= 3$, Betti Numbers $(3, 7, 9, 9, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{3.3}$).
Einstein der. $\phi = \frac{1}{22}$ Diag$(5, 12, 17, 22, 15, 27, 27)$, $\|\beta\|^2 = \frac{21}{22} \approx 0.9546$
$[e_1, e_2] = \frac{\sqrt{33}}{22}e_3, [e_1, e_3] = \frac{\sqrt{77}}{22}e_4, [e_1, e_4] = \frac{\sqrt{66}}{22}e_7, [e_2, e_5] = \frac{\sqrt{66}}{22}e_6$.

(50) (**2457C**): dim(Der) $= 19$, rank $= 2$, Betti Numbers $(3, 7, 10, 10, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{2.21}$).
Einstein der. $\phi = \frac{1}{29}$ Diag$(8, 19, 27, 35, 24, 43, 32)$, $\|\beta\|^2 = \frac{31}{29} \approx 1.069$
$[e_1, e_2] = \frac{\sqrt{87}}{29}e_3, [e_1, e_3] = \frac{\sqrt{145}}{29}e_4, [e_1, e_4] = \frac{\sqrt{87}}{29}e_6, [e_1, e_5] = \frac{\sqrt{58}}{58}e_7, [e_2, e_5] = \frac{\sqrt{87}}{29}e_6$.

(51) (**2457D**)!: dim(Der) $= 15$, rank $= 1$, Betti Numbers $(3, 7, 10, 10, 7, 3, 1)$ ($\cong_{\mathbb{R}} \mathfrak{g}_{1.16}!$).
Einstein der. $\phi = \frac{11}{38}$ Diag$(1, 2, 3, 3, 4, 4, 5)$, $\|\beta\|^2 = \frac{20}{19} \approx 1.053$
$[e_1, e_2] = \frac{\sqrt{114}}{38}e_3 + \frac{\sqrt{57}}{38}e_4, [e_1, e_3] = \frac{\sqrt{190}}{38}e_6, [e_1, e_4] = \frac{\sqrt{19}}{19}e_5, [e_1, e_6] = \frac{\sqrt{114}}{38}e_7, [e_2, e_3] = \frac{\sqrt{57}}{38}e_7, [e_2, e_4] = \frac{\sqrt{114}}{38}e_7$.

(52) (**2457E**)!: dim(Der) = 14, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.11}$!).
Einstein der. $\phi = \frac{1}{38}\,\mathrm{Diag}(9, 19, 28, 28, 37, 47, 46)$, $\|\beta\|^2 = \frac{18}{19} \approx 0.9474$
$[e_1, e_2] = \frac{\sqrt{1254}}{209} e_3 + \frac{\sqrt{8778}}{418} e_4, [e_1, e_3] = \frac{2\sqrt{1463}}{209} e_5, [e_1, e_4] = \frac{3\sqrt{209}}{418} e_5, [e_1, e_5] = \frac{\sqrt{190}}{38} e_7, [e_2, e_4] = \frac{\sqrt{209}}{38} e_6$.

(53) (**2457F**): dim(Der) = 16, rank = 2, Betti Numbers (3, 7, 10, 10, 7, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.20}$).
Einstein der. $\phi = \frac{2}{35}\,\mathrm{Diag}(5, 10, 15, 20, 16, 25, 21)$, $\|\beta\|^2 = \frac{37}{35} \approx 1.057$
$[e_1, e_2] = \frac{3\sqrt{70}}{70} e_3, [e_1, e_3] = \frac{2\sqrt{35}}{35} e_4, [e_1, e_4] = \frac{\sqrt{14}}{14} e_6, [e_1, e_5] = \frac{\sqrt{14}}{14} e_7, [e_2, e_3] = \frac{2\sqrt{35}}{35} e_6$.

(54) (**2457G**): dim(Der) = 15, rank = 2, Betti Numbers (3, 6, 9, 9, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.19}$).
Pre-Einstein der. $\phi = \frac{1}{4}\,\mathrm{Diag}(1, 2, 3, 4, 4, 5, 5)$,
It does not admit a nilsoliton inner product.

(55) (**2457H**): dim(Der) = 15, rank = 2, Betti Numbers (3, 6, 10, 10, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.18}$).
Einstein der. $\phi = \frac{1}{70}\,\mathrm{Diag}(20, 31, 51, 71, 60, 82, 91)$, $\|\beta\|^2 = \frac{34}{35} \approx 0.9714$
$[e_1, e_2] = \frac{\sqrt{21}}{14} e_3, [e_1, e_3] = \frac{3\sqrt{70}}{70} e_4, [e_1, e_4] = \frac{\sqrt{21}}{14} e_7, [e_2, e_3] = \frac{\sqrt{10}}{10} e_6, [e_2, e_5] = \frac{\sqrt{70}}{35} e_7$.

(56) (**2457I**): dim(Der) = 14, rank = 2, Betti Numbers (3, 7, 9, 9, 7, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.22}$).
Einstein der. $\phi = \frac{1}{20}\,\mathrm{Diag}(5, 10, 15, 20, 14, 25, 24)$, $\|\beta\|^2 = \frac{19}{20} \approx 0.9500$
$[e_1, e_2] = \frac{\sqrt{30}}{20} e_3, [e_1, e_3] = \frac{\sqrt{15}}{10} e_4, [e_1, e_4] = \frac{\sqrt{2}}{4} e_6, [e_2, e_3] = \frac{\sqrt{10}}{20} e_6, [e_2, e_5] = \frac{\sqrt{2}}{4} e_7$.

(57) (**2457J**)!: dim(Der) = 13, rank = 1, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.18}$!).
Einstein der. $\phi = \frac{23}{94}\,\mathrm{Diag}(1, 2, 3, 3, 4, 5, 5)$, $\|\beta\|^2 = \frac{89}{94} \approx 0.9468$
$[e_1, e_2] = \frac{3\sqrt{141}}{188} e_3 + \frac{\sqrt{1551}}{188} e_4, [e_1, e_3] = \frac{3\sqrt{517}}{188} e_5, [e_1, e_4] = \frac{3\sqrt{47}}{188} e_5, [e_1, e_5] = \frac{\sqrt{282}}{47} e_7, [e_2, e_3] = \frac{\sqrt{94}}{94} e_7,$ $[e_2, e_4] = \frac{\sqrt{1222}}{94} e_6$.

(58) (**2457K**): dim(Der) = 14, rank = 1, Betti Numbers (3, 6, 9, 9, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.9}$).
Pre-Einstein der. $\phi = \frac{10}{67}\,\mathrm{Diag}(2, 3, 5, 7, 6, 8, 9)$,
It does not admit a nilsoliton inner product.

(59) (**2457L**): dim(Der) = 12, rank = 2, Betti Numbers (2, 5, 8, 8, 5, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.9}$).
Einstein der. $\phi = \frac{9}{34}\,\mathrm{Diag}(1, 1, 2, 3, 3, 4, 4)$, $\|\beta\|^2 = \frac{14}{17} \approx 0.8235$
$[e_1, e_2] = \frac{\sqrt{17}}{17} e_3, [e_1, e_3] = \frac{\sqrt{119}}{34} e_4, [e_1, e_4] = -\frac{\sqrt{17}}{17} e_6, [e_1, e_5] = -\frac{\sqrt{17}}{17} e_7, [e_2, e_3] = -\frac{\sqrt{119}}{34} e_5, [e_2, e_4] =$ $\frac{\sqrt{17}}{17} e_7, [e_2, e_5] = \frac{\sqrt{17}}{17} e_6$.

(60) (**2457L1**): dim(Der) = 12, rank = 2, Betti Numbers (2, 5, 8, 8, 5, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{2.9}$).
Einstein der. $\phi = \frac{9}{34}\,\mathrm{Diag}(1, 1, 2, 3, 3, 4, 4)$, $\|\beta\|^2 = \frac{14}{17} \approx 0.8235$
$[e_1, e_2] = \frac{\sqrt{17}}{17} e_3, [e_1, e_3] = \frac{\sqrt{119}}{34} e_4, [e_1, e_4] = -\frac{\sqrt{17}}{17} e_6, [e_1, e_5] = -\frac{\sqrt{17}}{17} e_7, [e_2, e_3] = -\frac{\sqrt{119}}{34} e_5, [e_2, e_4] =$ $\frac{\sqrt{17}}{17} e_7, [e_2, e_5] = -\frac{\sqrt{17}}{17} e_6$.

(61) (**2457M**): dim(Der) = 13, rank = 2, Betti Numbers (2, 5, 9, 9, 5, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.8}$).
Einstein der. $\phi = \frac{1}{14}\,\mathrm{Diag}(3, 5, 8, 11, 13, 16, 14)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_2] = \frac{\sqrt{14}}{14} e_3, [e_1, e_3] = \frac{\sqrt{21}}{14} e_4, [e_1, e_4] = \frac{\sqrt{14}}{14} e_7, [e_1, e_5] = \frac{\sqrt{14}}{14} e_6, [e_2, e_3] = \frac{\sqrt{21}}{14} e_5, [e_2, e_4] = \frac{\sqrt{14}}{14} e_6$

(62) (**2357A**)!: dim(Der) = 13, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.24}$!).
Einstein der. $\phi = \frac{1}{19}\,\mathrm{Diag}(5, 9, 10, 14, 19, 19, 24)$, $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_2] = -\frac{\sqrt{38}}{19} e_4, [e_1, e_4] = \frac{\sqrt{114}}{38} e_5 - \frac{\sqrt{38}}{38} e_6, [e_1, e_5] = \frac{\sqrt{38}}{19} e_7, [e_2, e_3] = \frac{\sqrt{114}}{38} e_5 + \frac{\sqrt{38}}{38} e_6, [e_3, e_4] = \frac{\sqrt{114}}{38} e_7$.

(63) (**2357B**): dim(Der) = 14, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(i_\lambda)}$).
Pre-Einstein der. $\phi = \frac{2}{19}\,\mathrm{Diag}(3, 5, 6, 8, 11, 9, 14)$,
It does not admit a nilsoliton inner product.

(64) (**2357C**): dim(Der) = 13, rank = 2, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.17}$).
Einstein der. $\phi = \frac{1}{14}\,\mathrm{Diag}(4, 5, 8, 9, 13, 14, 17)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_2] = \frac{\sqrt{21}}{14} e_4, [e_1, e_4] = \frac{\sqrt{14}}{14} e_5, [e_1, e_5] = \frac{\sqrt{21}}{14} e_7, [e_2, e_3] = \frac{\sqrt{14}}{14} e_5, [e_2, e_4] = \frac{\sqrt{14}}{14} e_6, [e_3, e_4] = -\frac{\sqrt{14}}{14} e_7$.

(65) (**2357D**): dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.2(iii)}$).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(66) (**2357D1**): dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(iii)}$).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(67) (**23457A**): dim(Der) = 13, rank = 2, Betti Numbers (2, 4, 7, 7, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.7}$).
Einstein der. $\phi = \frac{1}{20}\,\mathrm{Diag}(3, 10, 13, 16, 19, 22, 23)$, $\|\beta\|^2 = \frac{9}{10} \approx 0.9000$
$[e_1, e_2] = \frac{\sqrt{30}}{20} e_3, [e_1, e_3] = \frac{\sqrt{30}}{20} e_4, [e_1, e_4] = \frac{\sqrt{2}}{4} e_5, [e_1, e_5] = \frac{\sqrt{10}}{10} e_6, [e_2, e_3] = \frac{\sqrt{2}}{4} e_7$.

(68) (**23457B**): dim(Der) = 12, rank = 2, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.6}$).
Einstein der. $\phi = \frac{1}{70}\,\mathrm{Diag}(10, 23, 33, 43, 53, 76, 56)$, $\|\beta\|^2 = \frac{26}{35} \approx 0.7429$
$[e_1, e_2] = \frac{\sqrt{455}}{70} e_3, [e_1, e_3] = \frac{2\sqrt{35}}{35} e_4, [e_1, e_4] = \frac{\sqrt{455}}{70} e_5, [e_2, e_3] = \frac{\sqrt{35}}{35} e_7, [e_2, e_5] = \frac{\sqrt{105}}{35} e_6, [e_3, e_4] =$ $-\frac{\sqrt{105}}{35} e_6$.

(69) (**23457C**): dim(Der) $=12$, rank $=2$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.4}$).
Einstein der. $\phi=\frac{1}{10}$ Diag(1, 4, 5, 6, 7, 8, 11), $\|\beta\|^2=\frac{26}{35}\approx 0.7429$
$[e_1,e_2]=\frac{\sqrt{105}}{35}e_3, [e_1,e_3]=\frac{2\sqrt{35}}{35}e_4, [e_1,e_4]=\frac{\sqrt{455}}{70}e_5, [e_1,e_5]=\frac{\sqrt{35}}{35}e_6, [e_2,e_5]=\frac{\sqrt{105}}{35}e_7, [e_3,e_4]=-\frac{\sqrt{455}}{70}e_7$.

(70) (**23457D**): dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.5}$).
Einstein der. $\phi=\frac{5}{42}$ Diag(1, 3, 4, 5, 6, 7, 9), $\|\beta\|^2=\frac{31}{42}\approx 0.7381$
$[e_1,e_2]=\frac{\sqrt{290}}{58}e_3, [e_1,e_3]=\frac{\sqrt{21}}{14}e_4, [e_1,e_4]=\frac{5\sqrt{609}}{406}e_5, [e_1,e_5]=\frac{\sqrt{42}}{42}e_6, [e_2,e_3]=\frac{\sqrt{42}}{42}e_6, [e_2,e_5]=\frac{\sqrt{609}}{87}e_7, [e_3,e_4]=-\frac{\sqrt{290}}{58}e_7$.

(71) (**23457E**)!: dim(Der) $=12$, rank $=1$, Betti Numbers (2, 4, 7, 7, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.13}$!).
Einstein der. $\phi=\frac{13}{68}$ Diag(1, 2, 3, 4, 5, 5, 6), $\|\beta\|^2=\frac{29}{34}\approx 0.8529$
$[e_1,e_2]=\frac{3\sqrt{34}}{68}e_3, [e_1,e_3]=\frac{\sqrt{119}}{34}e_4, [e_1,e_4]=\frac{3\sqrt{187}}{187}e_5-\frac{7\sqrt{374}}{748}e_6, [e_1,e_5]=\frac{\sqrt{374}}{68}e_7, [e_2,e_3]=\frac{3\sqrt{1309}}{374}e_5+\frac{\sqrt{2618}}{374}e_6, [e_2,e_4]=\frac{3\sqrt{34}}{68}e_7$.

(72) (**23457F**)!: dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.14}$!).
Einstein der. $\phi=\frac{9}{58}$ Diag(1, 2, 3, 4, 5, 5, 7), $\|\beta\|^2=\frac{43}{58}\approx 0.7414$
$[e_1,e_2]=-\frac{\sqrt{519593}}{2378}e_3, [e_1,e_3]=\frac{\sqrt{377}}{58}e_4, [e_1,e_4]=\frac{3\sqrt{605845438}}{252068}e_5+\frac{\sqrt{126034}}{6148}e_6, [e_2,e_3]=\frac{\sqrt{777722}}{6148}e_5-\frac{3\sqrt{58406}}{6148}e_6, [e_2,e_5]=\frac{\sqrt{126034}}{1189}e_7, [e_3,e_4]=\frac{3\sqrt{13079}}{1189}e_7$.

(73) (**23457G**): dim(Der) $=10$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(iii)}$).
Einstein der. $\phi=\frac{1}{7}$ Diag(1, 2, 3, 4, 5, 6, 7), $\|\beta\|^2=\frac{5}{7}\approx 0.7143$
$[e_1,e_2]=\frac{\sqrt{14}}{14}e_3, [e_1,e_3]=\frac{\sqrt{42}}{21}e_4, [e_1,e_4]=\frac{\sqrt{14}}{14}e_5, [e_1,e_5]=\frac{\sqrt{84}}{42}e_6, [e_2,e_3]=\frac{\sqrt{84}}{42}e_5, [e_2,e_4]=\frac{\sqrt{42}}{42}e_6, [e_2,e_5]=\frac{\sqrt{14}}{14}e_7, [e_3,e_4]=-\frac{\sqrt{14}}{14}e_7$.

(74) (**17**): dim(Der) $=28$, rank $=4$, Betti Numbers (6, 14, 14, 14, 14, 6, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{4.4}$).
Einstein der. $\phi=\frac{4}{3}$ Diag(1, 1, 1, 1, 1, 1, 2), $\|\beta\|^2=\frac{5}{3}\approx 1.667$
$[e_1,e_2]=\frac{\sqrt{6}}{6}e_7, [e_3,e_4]=\frac{\sqrt{6}}{6}e_7, [e_5,e_6]=\frac{\sqrt{6}}{6}e_7$.

(75) (**157**): dim(Der) $=19$, rank $=3$, Betti Numbers (5, 10, 11, 11, 10, 5, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.18}$).
Einstein der. $\phi=\frac{2}{11}$ Diag(3, 4, 7, 6, 5, 5, 10), $\|\beta\|^2=\frac{13}{11}\approx 1.182$
$[e_1,e_2]=\frac{\sqrt{22}}{11}e_3, [e_1,e_3]=\frac{\sqrt{66}}{22}e_7, [e_2,e_4]=\frac{\sqrt{22}}{22}e_7, [e_5,e_6]=\frac{\sqrt{66}}{22}e_7$.

(76) (**147A**): dim(Der) $=15$, rank $=3$, Betti Numbers (4, 8, 9, 9, 8, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.1(iii)}$).
Pre-Einstein der. $\phi=\frac{1}{2}$ Diag(1, 1, 1, 2, 2, 2, 3),
It does not admit

(77) (**147A1**): dim(Der) $=15$, rank $=3$, Betti Numbers (4, 8, 9, 9, 8, 4, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{3.1(iii)}$).
Pre-Einstein der. $\phi=\frac{1}{2}$ Diag(1, 1, 1, 2, 2, 2, 3),
It does not admit a nilsoliton inner product.

(78) (**147B**): dim(Der) $=12$, rank $=2$, Betti Numbers (4, 8, 10, 10, 8, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.28}$).
Einstein der. $\phi=\frac{4}{35}$ Diag(4, 6, 5, 10, 9, 8, 14), $\|\beta\|^2=\frac{37}{35}\approx 1.057$
$[e_1,e_2]=\frac{2\sqrt{35}}{35}e_4, [e_1,e_3]=\frac{2\sqrt{35}}{35}e_5, [e_1,e_4]=\frac{\sqrt{14}}{14}e_7, [e_2,e_6]=\frac{\sqrt{14}}{14}e_7, [e_3,e_5]=\frac{3\sqrt{70}}{70}e_7$.

(79) (**147D**)!: dim(Der) $=15$, rank $=2$, Betti Numbers (3, 7, 9, 9, 7, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.2}$!).
Pre-Einstein der. $\phi=\frac{1}{2}$ Diag(1, 1, 1, 2, 2, 2, 3),
It does not admit a nilsoliton inner product.

(80) (**147E**) $[0<t<1]$: dim(Der) $=15$, rank $=3$, Betti Numbers (3, 7, 9, 9, 7, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.1(i_\lambda)}$).
Einstein der. $\phi=\frac{1}{2}$ Diag(1, 1, 1, 2, 2, 2, 3), $\|\beta\|^2=1$
$[e_1,e_2]=\frac{\sqrt{2}}{4}\sqrt{1-t}e_4, [e_1,e_3]=-\frac{\sqrt{2}}{4}\sqrt{t}e_6, [e_1,e_5]=-\frac{\sqrt{2}}{4}e_7, [e_2,e_3]=\frac{\sqrt{2}}{4}e_5, [e_2,e_6]=\frac{\sqrt{2}}{4}\sqrt{t}e_7, [e_3,e_4]=\frac{\sqrt{2}}{4}\sqrt{1-t}e_7$.

(81) (**147E1**) $[t>1]$!: dim(Der) $=15$, rank $=3$, Betti Numbers (3, 7, 9, 9, 7, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{3.1(i_{P(\lambda)})} \cong_{\mathbb{C}}$ (147E) $[\lambda]$ with $\lambda=\left(\frac{(1-\sqrt{t^2-1}\sqrt{-1})}{t}\right)^2$).
Einstein der. $\phi=\frac{1}{2}$ Diag(1, 1, 1, 2, 2, 2, 3), $\|\beta\|^2=1$
This family admits nilsoliton metrics.

(82) (**1457A**): dim(Der) $=16$, rank $=3$, Betti Numbers (4, 6, 9, 9, 6, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.17}$).
Einstein der. $\phi=\frac{5}{22}$ Diag(1, 3, 4, 5, 3, 3, 6), $\|\beta\|^2=\frac{21}{22}\approx 0.9546$
$[e_1,e_2]=\frac{\sqrt{66}}{22}e_3, [e_1,e_3]=\frac{\sqrt{77}}{22}e_4, [e_1,e_4]=\frac{\sqrt{33}}{22}e_7, [e_5,e_6]=\frac{\sqrt{66}}{22}e_7$.

(83) (**1457B**): dim(Der) $=15$, rank $=2$, Betti Numbers (4, 6, 8, 8, 6, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.30}$).
Einstein der. $\phi=\frac{4}{29}$ Diag(2, 4, 6, 8, 5, 5, 10), $\|\beta\|^2=\frac{27}{29}\approx 0.9310$
$[e_1,e_2]=\frac{2\sqrt{29}}{29}e_3, [e_1,e_3]=\frac{2\sqrt{29}}{29}e_4, [e_1,e_4]=\frac{\sqrt{174}}{58}e_7, [e_2,e_3]=\frac{\sqrt{174}}{58}e_7, [e_5,e_6]=\frac{\sqrt{406}}{58}e_7$.

(84) (**137A**): dim(Der) $= 14$, rank $= 3$, Betti Numbers (4, 7, 8, 8, 7, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.16}$).
Einstein der. $\phi = \frac{1}{14}\,\mathrm{Diag}(5, 8, 5, 8, 13, 13, 18)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_2] = \frac{\sqrt{7}}{7}e_5, [e_1, e_5] = \frac{\sqrt{21}}{14}e_7, [e_3, e_4] = \frac{\sqrt{7}}{7}e_6, [e_3, e_6] = \frac{\sqrt{21}}{14}e_7.$

(85) (**137A1**): dim(Der) $= 14$, rank $= 3$, Betti Numbers (4, 7, 8, 8, 7, 4, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{3.16}$).
Einstein der. $\phi = \frac{1}{14}\,\mathrm{Diag}(5, 5, 8, 8, 13, 13, 18)$, $\|\beta\|^2 = \frac{6}{7} \approx 0.8571$
$[e_1, e_3] = \frac{\sqrt{14}}{14}e_5, [e_1, e_4] = \frac{\sqrt{14}}{14}e_6, [e_1, e_5] = \frac{\sqrt{21}}{14}e_7, [e_2, e_3] = -\frac{\sqrt{14}}{14}e_6, [e_2, e_4] = \frac{\sqrt{14}}{14}e_5, [e_2, e_6] = \frac{\sqrt{21}}{14}e_7.$

(86) (**137B**): dim(Der) $= 13$, rank $= 2$, Betti Numbers (4, 7, 7, 7, 7, 4, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{2.23}$).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 2, 1, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(87) (**137B1**): dim(Der) $= 13$, rank $= 2$, Betti Numbers (4, 7, 7, 7, 7, 4, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{2.23}$).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(88) (**137C**): dim(Der) $= 15$, rank $= 3$, Betti Numbers (4, 7, 8, 8, 7, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{3.10}$).
Einstein der. $\phi = \frac{1}{22}\,\mathrm{Diag}(7, 12, 11, 16, 19, 23, 30)$, $\|\beta\|^2 = \frac{10}{11} \approx 0.9091$
$[e_1, e_2] = \frac{\sqrt{11}}{11}e_5, [e_1, e_4] = \frac{\sqrt{11}}{11}e_6, [e_1, e_6] = \frac{\sqrt{55}}{22}e_7, [e_2, e_3] = \frac{\sqrt{11}}{11}e_6, [e_3, e_5] = -\frac{\sqrt{55}}{22}e_7.$

(89) (**137D**): dim(Der) $= 14$, rank $= 2$, Betti Numbers (4, 7, 8, 8, 7, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(iv)}$).
Pre-Einstein der. $\phi = \frac{2}{19}\,\mathrm{Diag}(3, 6, 5, 8, 9, 11, 14)$,
It does not admit a nilsoliton inner product.

(90) (**1357A**): dim(Der) $= 14$, rank $= 2$, Betti Numbers (4, 7, 8, 8, 7, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(iii)}$).
Einstein der. $\phi = \frac{2}{21}\,\mathrm{Diag}(3, 5, 6, 8, 11, 9, 14)$, $\|\beta\|^2 = \frac{19}{21} \approx 0.9048$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_4, [e_1, e_4] = \frac{\sqrt{42}}{21}e_5, [e_1, e_5] = \frac{\sqrt{21}}{14}e_7, [e_2, e_3] = \frac{\sqrt{3}}{6}e_5, [e_2, e_6] = \frac{\sqrt{42}}{42}e_7, [e_3, e_4] = -\frac{\sqrt{3}}{6}e_7.$

(91) (**1357B**)!: dim(Der) $= 14$, rank $= 2$, Betti Numbers (4, 6, 7, 7, 6, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.25}$!).
Einstein der. $\phi = \frac{5}{19}\,\mathrm{Diag}(1, 2, 2, 3, 3, 4, 5)$, $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_3] = \frac{\sqrt{38}}{19}e_4, [e_1, e_4] = -\frac{\sqrt{114}}{38}e_6, [e_1, e_5] = \frac{\sqrt{38}}{38}e_6, [e_1, e_6] = \frac{\sqrt{38}}{19}e_7, [e_2, e_3] = \frac{\sqrt{114}}{38}e_6, [e_2, e_4] = \frac{\sqrt{114}}{38}e_7, [e_2, e_5] = \frac{\sqrt{38}}{38}e_7.$

(92) (**1357C**)!: dim(Der) $= 13$, rank $= 1$, Betti Numbers (4, 6, 7, 7, 6, 4, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(v)}$!).
Pre-Einstein der. $\phi = \frac{5}{17}\,\mathrm{Diag}(1, 2, 2, 3, 4, 3, 5)$,
It does not admit a nilsoliton inner product.

(93) (**1357D**): dim(Der) $= 14$, rank $= 2$, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(ii)}$).
Einstein der. $\phi = \frac{2}{21}\,\mathrm{Diag}(5, 3, 8, 6, 11, 9, 14)$, $\|\beta\|^2 = \frac{19}{21} \approx 0.9048$
$[e_1, e_2] = \frac{\sqrt{21}}{14}e_3, [e_1, e_6] = \frac{\sqrt{21}}{14}e_7, [e_2, e_3] = \frac{\sqrt{42}}{21}e_5, [e_2, e_4] = -\frac{\sqrt{3}}{6}e_6, [e_2, e_5] = -\frac{\sqrt{42}}{42}e_7, [e_3, e_4] = -\frac{\sqrt{3}}{6}e_7.$

(94) (**1357E**): dim(Der) $= 14$, rank $= 2$, Betti Numbers (3, 5, 8, 8, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.32}$).
Einstein der. $\phi = \frac{2}{29}\,\mathrm{Diag}(10, 3, 13, 8, 16, 11, 19)$, $\|\beta\|^2 = \frac{27}{29} \approx 0.9310$
$[e_1, e_2] = \frac{\sqrt{406}}{58}e_3, [e_2, e_3] = \frac{2\sqrt{29}}{29}e_5, [e_2, e_4] = \frac{\sqrt{174}}{58}e_6, [e_2, e_5] = \frac{\sqrt{174}}{58}e_7, [e_4, e_6] = \frac{2\sqrt{29}}{29}e_7.$

(95) (**1357F**): dim(Der) $= 13$, rank $= 1$, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(iii)}$).
Einstein der. $\phi = \frac{5}{19}\,\mathrm{Diag}(2, 1, 3, 2, 4, 3, 5)$, $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_2] = \frac{3\sqrt{76}}{76}e_3, [e_1, e_3] = \frac{\sqrt{380}}{76}e_7, [e_2, e_3] = \frac{\sqrt{38}}{19}e_5, [e_2, e_4] = \frac{\sqrt{380}}{76}e_6, [e_2, e_5] = \frac{\sqrt{38}}{38}e_7, [e_4, e_6] = -\frac{3\sqrt{76}}{76}e_7.$

(96) (**1357F1**): dim(Der) $= 13$, rank $= 1$, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.3(iii)}$).
Einstein der. $\phi = \frac{5}{19}\,\mathrm{Diag}(2, 1, 3, 2, 4, 3, 5)$, $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_2] = \frac{3\sqrt{19}}{38}e_3, [e_1, e_3] = \frac{\sqrt{95}}{38}e_7, [e_2, e_3] = \frac{\sqrt{38}}{19}e_5, [e_2, e_4] = \frac{\sqrt{95}}{38}e_6, [e_2, e_5] = \frac{\sqrt{38}}{38}e_7, [e_4, e_6] = \frac{3\sqrt{19}}{38}e_7.$

(97) (**1357G**): dim(Der) $= 13$, rank $= 2$, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.31}$).
Einstein der. $\phi = \frac{1}{46}\,\mathrm{Diag}(15, 14, 29, 27, 43, 42, 57)$, $\|\beta\|^2 = \frac{39}{46} \approx 0.8478$
$[e_1, e_2] = \frac{\sqrt{69}}{46}e_3, [e_1, e_4] = \frac{\sqrt{69}}{23}e_6, [e_1, e_6] = \frac{3\sqrt{23}}{46}e_7, [e_2, e_3] = \frac{\sqrt{299}}{46}e_5, [e_2, e_5] = \frac{3\sqrt{23}}{46}e_7.$

(98) (**1357H**)!: dim(Der) $= 12$, rank $= 1$, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.2(iv)}$!).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(99) (**1357I**): dim(Der) $= 12$, rank $= 2$, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.33}$).
Einstein der. $\phi = \frac{1}{41}\,\mathrm{Diag}(18, 10, 28, 15, 38, 33, 48)$, $\|\beta\|^2 = \frac{33}{41} \approx 0.8049$
$[e_1, e_2] = \frac{\sqrt{123}}{41}e_3, [e_1, e_4] = \frac{3\sqrt{82}}{82}e_6, [e_2, e_3] = \frac{\sqrt{902}}{82}e_5, [e_2, e_5] = \frac{\sqrt{123}}{41}e_7, [e_4, e_6] = \frac{3\sqrt{82}}{82}e_7.$

(100) (**1357J**): dim(Der) $= 11$, rank $= 1$, Betti Numbers (3, 5, 6, 6, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.8}$).
Pre-Einstein der. $\phi = \frac{20}{139}\,\mathrm{Diag}(4, 2, 6, 3, 8, 7, 10)$,
It does not admit a nilsoliton inner product.

(101) (**1357L**)!: dim(Der) $= 14$, rank $= 1$, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(ii)}$!).
Pre-Einstein der. $\phi = \frac{5}{17}\,\mathrm{Diag}(1, 2, 3, 2, 4, 3, 5)$,
It does not admit a nilsoliton inner product.

(102) (**1357M**) $[t \in \mathbb{R} \setminus \{0, 1\}]$: dim(Der) = 14, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(i_t)}$).
Einstein der. $\phi = \frac{2}{21}$ Diag(3, 5, 8, 6, 11, 9, 14), $\|\beta\|^2 = \frac{19}{21} \approx 0.9048$
This family admits nilsoliton metrics.

(102.1) (**1357M**) $[t = 0]$: dim(Der) = 14, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}}(2357B) \cong_{\mathbb{R}} \mathfrak{g}_{2.1(i_0)}$).
Pre-Einstein der. $\phi = \frac{2}{19}$ Diag(3, 5, 8, 6, 11, 9, 14),
It does not admit a nilsoliton inner product.

(102.2) (**1357M**) $[t = 1]$: dim(Der) = 14, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.1(i_1)}$).
Einstein der. $\phi = \frac{2}{21}$ Diag(3, 5, 8, 6, 11, 9, 14), $\|\beta\|^2 = \frac{19}{21} \approx 0.9048$
$[e_1, e_2] = \frac{\sqrt{42}}{42}e_3, [e_1, e_3] = \frac{\sqrt{42}}{21}e_5, [e_1, e_4] = \frac{\sqrt{3}}{6}e_6, [e_1, e_5] = \frac{\sqrt{21}}{14}e_7, [e_2, e_4] = \frac{\sqrt{3}}{6}e_5, [e_2, e_6] = \frac{\sqrt{21}}{14}e_7$.

(103) (**1357N**) $[t \in \mathbb{R} \setminus \{0\}]$!: dim(Der) = 13, rank = 1, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(i_\lambda)}$!).
Einstein der. $\phi = \frac{5}{19}$ Diag(1, 2, 3, 2, 4, 3, 5), $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
This family admits nilsoliton metrics.

(103.1) (**1357N**) $[t = 0]$!: dim(Der) = 13, rank = 1, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.3(i_0)}$!).
Pre-Einstein der. $\phi = \frac{5}{17}$ Diag(1, 2, 3, 2, 4, 3, 5),
It does not admit a nilsoliton inner product.

(104) (**1357O**): dim(Der) = 13, rank = 2, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.41}$).
Einstein der. $\phi = \frac{1}{8}$ Diag(3, 2, 5, 6, 8, 7, 10), $\|\beta\|^2 = \frac{7}{8} \approx 0.8750$
$[e_1, e_2] = \frac{1}{4}e_3, [e_1, e_3] = \frac{\sqrt{6}}{8}e_5, [e_1, e_6] = \frac{\sqrt{6}}{8}e_7, [e_2, e_3] = \frac{\sqrt{6}}{8}e_6, [e_2, e_4] = \frac{1}{4}e_5, [e_2, e_5] = \frac{\sqrt{6}}{8}e_7$.

(105) (**1357P**): dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.2(i_0)}$).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{\sqrt{65}}{26}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_5] = \frac{\sqrt{65}}{26}e_7, [e_2, e_3] = \frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{26}e_5, [e_2, e_6] = \frac{\sqrt{26}}{26}e_7, [e_3, e_4] = \frac{\sqrt{39}}{26}e_7$.

(106) (**1357P1**): dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(i_0)}$).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{\sqrt{65}}{26}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_5] = \frac{\sqrt{65}}{26}e_7, [e_2, e_3] = \frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{26}e_5, [e_2, e_6] = -\frac{\sqrt{26}}{26}e_7, [e_3, e_4] = \frac{\sqrt{39}}{26}e_7$.

(107) (**1357Q**)!: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.2(i_1)}$).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{\sqrt{26}}{26}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = \frac{\sqrt{39}}{26}e_6, [e_1, e_5] = \frac{\sqrt{65}}{26}e_7, [e_2, e_3] = \frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{26}e_5, [e_2, e_6] = \frac{\sqrt{65}}{26}e_7$.

(108) (**1357Q1**)!: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(i_1)}$).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{\sqrt{26}}{26}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = \frac{\sqrt{39}}{26}e_6, [e_1, e_6] = \frac{\sqrt{65}}{26}e_7, [e_2, e_3] = \frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{26}e_5, [e_2, e_5] = \frac{\sqrt{65}}{26}e_7$.

(109) (**1357R**)!: dim(Der) = 13, rank = 2, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.37}$!).
Einstein der. $\phi = \frac{4}{11}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{2\sqrt{39}}{39}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_6, [e_1, e_4] = \frac{\sqrt{39}}{39}e_5, [e_1, e_6] = \frac{\sqrt{390}}{78}e_7, [e_2, e_3] = \frac{\sqrt{13}}{13}e_5, [e_2, e_4] = \frac{\sqrt{39}}{39}e_6, [e_2, e_5] = -\frac{\sqrt{390}}{78}e_7, [e_3, e_4] = \frac{\sqrt{390}}{78}e_7$.

(110) (**1357QRS1**) $[t \in \mathbb{R} \setminus \{-1, 0, 1\}]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(i_\lambda)}$ with $\lambda = \frac{1}{t}$).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
This family admits nilsoliton metrics.

(110.1) (**1357QRS1**) $[t = -1]$: dim(Der) = 13, rank = 2, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{2.37} \cong_{\mathbb{C}}$ (1357R))
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{2\sqrt{39}}{39}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = \frac{\sqrt{39}}{39}e_6, [e_1, e_5] = \frac{\sqrt{390}}{78}e_7, [e_2, e_3] = -\frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{39}e_5, [e_2, e_6] = -\frac{\sqrt{390}}{78}e_7, [e_3, e_4] = \frac{\sqrt{390}}{78}e_7$.

(110.2) (**1357QRS1**)$[t = 1]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 8, 8, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(i_1)} \cong_{\mathbb{C}}$ (1357Q)).
Einstein der. $\phi = \frac{4}{13}$ Diag(1, 1, 2, 2, 3, 3, 4), $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
$[e_1, e_2] = \frac{\sqrt{26}}{26}e_3, [e_1, e_3] = \frac{\sqrt{13}}{13}e_5, [e_1, e_4] = \frac{\sqrt{39}}{26}e_6, [e_1, e_5] = \frac{\sqrt{65}}{26}e_7, [e_2, e_3] = -\frac{\sqrt{13}}{13}e_6, [e_2, e_4] = \frac{\sqrt{39}}{26}e_5, [e_2, e_6] = \frac{\sqrt{65}}{26}e_7$.

(110.3) (**1357QRS1**) $[t = 0]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(iii)} \cong_{\mathbb{C}}$ (2357D)).
Pre-Einstein der. $\phi = \frac{4}{11}$ Diag(1, 1, 2, 2, 3, 3, 4),
It does not admit a nilsoliton inner product.

(111) (**1357S**)! $[t \in \mathbb{R} \setminus \{0, 1\}]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(i_{P(\lambda)})} \cong_{\mathbb{C}}$ (1357QRS1) $[u]$ with $u = \frac{2\sqrt{t}+t+1}{t-1}$).
Einstein der. $\phi = \frac{4}{13}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$, $\|\beta\|^2 = \frac{11}{13} \approx 0.8462$
This family admits nilsoliton metrics.

(111.1) (**1357S**)! $[t = 0]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.2(ii)}$!).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(111.2) (**1357S**)! $[t = 1]$: dim(Der) = 12, rank = 1, Betti Numbers (3, 6, 7, 7, 6, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.2(iii)} \cong_{\mathbb{C}}$ (2357D)).
Pre-Einstein der. $\phi = \frac{4}{11}\,\mathrm{Diag}(1, 1, 2, 2, 3, 3, 4)$,
It does not admit a nilsoliton inner product.

(112) (**13457A**): dim(Der) = 14, rank = 2, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.16}$).
Einstein der. $\phi = \frac{1}{29}\,\mathrm{Diag}(5, 17, 22, 27, 32, 20, 37)$, $\|\beta\|^2 = \frac{27}{29} \approx 0.9310$
$[e_1, e_2] = \frac{\sqrt{174}}{58} e_3$, $[e_1, e_3] = \frac{2\sqrt{29}}{29} e_4$, $[e_1, e_4] = \frac{2\sqrt{29}}{29} e_5$, $[e_1, e_5] = \frac{\sqrt{174}}{58} e_7$, $[e_2, e_6] = \frac{\sqrt{406}}{58} e_7$.

(113) (**13457B**)!: dim(Der) = 13, rank = 1, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.15}$!).
Einstein der. $\phi = \frac{15}{82}\,\mathrm{Diag}(1, 3, 4, 4, 5, 6, 7)$, $\|\beta\|^2 = \frac{38}{41} \approx 0.9268$
$[e_1, e_2] = \frac{2\sqrt{4305}}{861} e_3 + \frac{\sqrt{111930}}{1722} e_4$, $[e_1, e_3] = \frac{2\sqrt{22386}}{861} e_5$, $[e_1, e_4] = \frac{5\sqrt{861}}{1722} e_5$, $[e_1, e_5] = \frac{\sqrt{902}}{82} e_6$, $[e_1, e_6] = \frac{\sqrt{82}}{41} e_7$, $[e_2, e_4] = \frac{\sqrt{861}}{82} e_7$.

(114) (**13457C**): dim(Der) = 12, rank = 2, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.10}$).
Pre-Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(1, 2, 3, 4, 5, 6, 7)$,
It does not admit a nilsoliton inner product.

(115) (**13457D**)!: dim(Der) = 12, rank = 1, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.12}$!).
Einstein der. $\phi = \frac{25}{124}\,\mathrm{Diag}(1, 2, 4, 3, 4, 5, 6)$, $\|\beta\|^2 = \frac{107}{124} \approx 0.8629$
$[e_1, e_2] = \frac{\sqrt{930}}{124} e_4$, $[e_1, e_3] = \frac{\sqrt{31}}{31} e_6$, $[e_1, e_4] = \frac{\sqrt{341}}{62} e_3 + \frac{\sqrt{62}}{62} e_5$, $[e_1, e_5] = \frac{\sqrt{682}}{124} e_6$, $[e_1, e_6] = \frac{\sqrt{341}}{62} e_7$, $[e_2, e_3] = \frac{\sqrt{1302}}{124} e_7$, $[e_2, e_4] = \frac{\sqrt{1302}}{124} e_6$.

(116) (**13457E**): dim(Der) = 11, rank = 1, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(vi)}$).
Pre-Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(1, 2, 3, 4, 5, 6, 7)$,
It does not admit a nilsoliton inner product.

(117) (**13457F**): dim(Der) = 11, rank = 1, Betti Numbers (2, 4, 7, 7, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.10}$).
Einstein der. $\phi = \frac{45}{446}\,\mathrm{Diag}(2, 3, 5, 7, 9, 8, 11)$, $\|\beta\|^2 = \frac{353}{446} \approx 0.7915$
$[e_1, e_2] = \frac{\sqrt{31220}}{892} e_3$, $[e_1, e_3] = \frac{\sqrt{15164}}{446} e_4$, $[e_1, e_4] = \frac{\sqrt{23638}}{446} e_5$, $[e_1, e_5] = \frac{3\sqrt{1338}}{446} e_7$, $[e_2, e_3] = \frac{\sqrt{84740}}{892} e_6$, $[e_2, e_6] = \frac{\sqrt{4906}}{223} e_7$.

(118) (**13457G**)!!: dim(Der) = 11, rank = 1, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.03}$!!).
Pre-Einstein der. $\phi = \frac{2}{3}\,\mathrm{Diag}(0, 1, 1, 1, 1, 2, 2)$,
It does not admit; $\phi \not> 0$

(119) (**13457I**)!!: dim(Der) = 10, rank = 0, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.7}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$

(120) (**12457A**): dim(Der) = 13, rank = 2, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.15}$).
Pre-Einstein der. $\phi = \frac{1}{6}\,\mathrm{Diag}(1, 3, 4, 5, 3, 6, 7)$, $\|\beta\|^2 = \frac{5}{6} \approx 0.8333$
$[e_1, e_2] = \frac{\sqrt{3}}{6} e_3$, $[e_1, e_3] = \frac{\sqrt{3}}{6} e_4$, $[e_1, e_4] = \frac{\sqrt{3}}{6} e_6$, $[e_1, e_6] = \frac{\sqrt{3}}{6} e_7$, $[e_2, e_5] = \frac{\sqrt{3}}{6} e_6$, $[e_3, e_5] = \frac{\sqrt{3}}{6} e_7$.

(121) (**12457B**)!!: dim(Der) = 12, rank = 1, Betti Numbers (3, 5, 7, 7, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.01(ii)}$!!).
Pre-Einstein der. $\phi = \mathrm{Diag}(0, 1, 1, 1, 0, 1, 1)$,
It does not admit; $\phi \not> 0$.

(122) (**12457C**): dim(Der) = 12, rank = 2, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.13}$).
Einstein der. $\phi = \frac{1}{86}\,\mathrm{Diag}(16, 21, 37, 53, 48, 69, 90)$, $\|\beta\|^2 = \frac{30}{43} \approx 0.6977$
$[e_1, e_2] = \frac{\sqrt{129}}{43} e_3$, $[e_1, e_3] = \frac{\sqrt{215}}{43} e_4$, $[e_1, e_4] = \frac{\sqrt{129}}{43} e_6$, $[e_2, e_5] = \frac{\sqrt{129}}{43} e_6$, $[e_2, e_6] = \frac{\sqrt{645}}{86} e_7$, $[e_3, e_4] = -\frac{\sqrt{645}}{86} e_7$.

(123) (**12457D**): dim(Der) = 11, rank = 1, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.20}$).
Pre-Einstein der. $\phi = \frac{8}{47}\,\mathrm{Diag}(2, 1, 3, 5, 6, 7, 8)$,
It does not admit a nilsoliton inner product.

(124) (**12457E**)!: dim(Der) = 11, rank = 1, Betti Numbers (3, 5, 6, 6, 5, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.11}$!).
Einstein der. $\phi = \frac{6}{31}\,\mathrm{Diag}(1, 2, 3, 3, 4, 5, 6)$, $\|\beta\|^2 = \frac{25}{31} \approx 0.8064$
$[e_1, e_2] = \frac{7\sqrt{1767}}{1767} e_3 + \frac{\sqrt{465}}{93} e_4$, $[e_1, e_3] = \frac{\sqrt{651}}{93} e_5$, $[e_1, e_4] = \frac{\sqrt{61845}}{1767} e_5$, $[e_1, e_5] = \frac{\sqrt{62}}{31} e_6$, $[e_1, e_6] = \frac{\sqrt{90706}}{1178} e_7$, $[e_2, e_4] = \frac{4\sqrt{1767}}{589} e_6$, $[e_2, e_5] = \frac{\sqrt{64790}}{1178} e_7$, $[e_3, e_4] = \frac{\sqrt{90706}}{1178} e_7$.

(125) (**12457F**)!: dim(Der) $=11$, rank $=1$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.21}$!).
Pre-Einstein der. $\phi = \frac{25}{113}\,\mathrm{Diag}(1, 2, 3, 4, 3, 5, 7)$,
It does not admit a nilsoliton inner product.

(126) (**12457G**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.8}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(127) (**12457H**): dim(Der) $=12$, rank $=2$, Betti Numbers (2, 3, 6, 6, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.5}$).
Einstein der. $\phi = \frac{1}{7}\,\mathrm{Diag}(1, 2, 3, 4, 5, 6, 7)$, $\|\beta\|^2 = \frac{5}{7} \approx 0.7143$
$[e_1, e_2] = \frac{\sqrt{14}}{14}e_3, [e_1, e_3] = \frac{\sqrt{14}}{14}e_4, [e_1, e_5] = \frac{\sqrt{14}}{14}e_6, [e_1, e_6] = \frac{\sqrt{14}}{14}e_7, [e_2, e_3] = \frac{\sqrt{14}}{14}e_5, [e_2, e_4] = \frac{\sqrt{14}}{14}e_6, [e_3, e_4] = \frac{\sqrt{14}}{14}e_7$.

(128) (**12457I**): dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 6, 6, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(iv)}$).
Pre-Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(1, 2, 3, 4, 5, 6, 7)$,
It does not admit a nilsoliton inner product.

(129) (**12457J**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 5, 5, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.6}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(130) (**12457J1**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 5, 5, 3, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{0.6}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(131) (**12457K**)!!: dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 5, 5, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.02}$!!).
Pre-Einstein der. $\phi = \frac{1}{2}\,\mathrm{Diag}(1, 0, 1, 2, 1, 2, 3)$,
It does not admit; $\phi \not> 0$.

(132) (**12457L**)!: dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.17}$!).
Einstein der. $\phi = \frac{19}{94}\,\mathrm{Diag}(1, 1, 2, 3, 3, 4, 5)$, $\|\beta\|^2 = \frac{65}{94} \approx 0.6915$
$[e_1, e_2] = \frac{\sqrt{611}}{94}e_3, [e_1, e_3] = \frac{\sqrt{235}}{94}e_4, [e_1, e_4] = -\frac{\sqrt{611}}{94}e_6, [e_1, e_6] = \frac{\sqrt{705}}{94}e_7, [e_2, e_3] = \frac{\sqrt{235}}{47}e_5, [e_2, e_5] = \frac{\sqrt{611}}{94}e_6, [e_3, e_5] = \frac{\sqrt{705}}{94}e_7$.

(133) (**12457L1**): dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.17}$!).
Einstein der. $\phi = \frac{19}{94}\,\mathrm{Diag}(1, 1, 2, 3, 3, 4, 5)$, $\|\beta\|^2 = \frac{65}{94} \approx 0.6915$
$[e_1, e_2] = \frac{\sqrt{611}}{94}e_3, [e_1, e_3] = \frac{\sqrt{235}}{94}e_4, [e_1, e_4] = -\frac{\sqrt{611}}{94}e_6, [e_1, e_6] = \frac{\sqrt{705}}{94}e_7, [e_2, e_3] = \frac{\sqrt{235}}{47}e_5, [e_2, e_5] = -\frac{\sqrt{611}}{94}e_6, [e_3, e_5] = -\frac{\sqrt{705}}{94}e_7$.

(134) (**12457N**) [**t**]!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{0.4_{P(\lambda)}}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(135) (**12457N1**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{0.4_{\lambda_0}}$!!).
Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(136) (**12457N2**)[$\mathbf{t} \geq 0$]!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{0.4(\lambda)}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(137) (**12357A**): dim(Der) $=12$, rank $=2$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.14}$).
Einstein der. $\phi = \frac{1}{7}\,\mathrm{Diag}(1, 3, 2, 4, 5, 6, 7)$, $\|\beta\|^2 = \frac{5}{7} \approx 0.7143$
$[e_1, e_2] = \frac{\sqrt{14}}{14}e_4, [e_1, e_4] = \frac{\sqrt{14}}{14}e_5, [e_1, e_5] = \frac{\sqrt{14}}{14}e_6, [e_1, e_6] = \frac{\sqrt{14}}{14}e_7, [e_2, e_3] = \frac{\sqrt{14}}{14}e_5, [e_3, e_4] = -\frac{\sqrt{14}}{14}e_6, [e_3, e_5] = -\frac{\sqrt{14}}{14}e_7$.

(138) (**12357B**)!!: dim(Der) $=11$, rank $=1$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.01(i)}$!!).
Pre-Einstein der. $\phi = \mathrm{Diag}(0, 1, 0, 1, 1, 1, 1)$,
It does not admit; $\phi \not> 0$.

(139) (**12357B1**)!!: dim(Der) $=11$, rank $=1$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{C}} \mathfrak{g}_{1.01(i)}$!!).
Pre-Einstein der. $\phi = \mathrm{Diag}(0, 1, 0, 1, 1, 1, 1)$,
It does not admit; $\phi \not> 0$.

(140) (**12357C**): dim(Der) $=10$, rank $=1$, Betti Numbers (3, 4, 4, 4, 4, 3, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(v)}$).
Pre-Einstein der. $\phi = \frac{1}{5}\,\mathrm{Diag}(1, 3, 2, 4, 5, 6, 7)$,
It does not admit a nilsoliton inner product.

(141) (**123457A**): dim(Der) $=13$, rank $=2$, Betti Numbers (2, 4, 6, 6, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{2.3}$).
Einstein der. $\phi = \frac{2}{35}\,\mathrm{Diag}(1, 16, 17, 18, 19, 20, 21)$, $\|\beta\|^2 = \frac{37}{35} \approx 1.057$
$[e_1, e_2] = \frac{\sqrt{14}}{14}e_3, [e_1, e_3] = \frac{2\sqrt{35}}{35}e_4, [e_1, e_4] = \frac{3\sqrt{70}}{70}e_5, [e_1, e_5] = \frac{2\sqrt{35}}{35}e_6, [e_1, e_6] = \frac{\sqrt{14}}{14}e_7$.

(142) (**123457B**): dim(Der) $=12$, rank $=1$, Betti Numbers (2, 4, 6, 6, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.6}$).
Einstein der. $\phi = \frac{5}{38}$ Diag(1, 4, 5, 6, 7, 8, 9), $\|\beta\|^2 = \frac{17}{19} \approx 0.8947$
$[e_1, e_2] = \frac{\sqrt{380}}{76} e_3, [e_1, e_3] = \frac{\sqrt{380}}{76} e_4, [e_1, e_4] = \frac{3\sqrt{76}}{76} e_5, [e_1, e_5] = \frac{\sqrt{38}}{19} e_6, [e_1, e_6] = \frac{\sqrt{38}}{38} e_7, [e_2, e_3] = \frac{3\sqrt{76}}{76} e_7$.

(143) (**123457C**): dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(ii)}$).
Pre-Einstein der. $\phi = \frac{1}{5}$ Diag(1, 2, 3, 4, 5, 6, 7),
It does not admit a nilsoliton inner product.

(144) (**123457D**): dim(Der) $=12$, rank $=1$, Betti Numbers (2, 4, 6, 6, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.4}$).
Einstein der. $\phi = \frac{17}{122}$ Diag(1, 3, 4, 5, 6, 7, 8), $\|\beta\|^2 = \frac{50}{61} \approx 0.8197$
$[e_1, e_2] = \frac{\sqrt{610}}{122} e_3, [e_1, e_3] = \frac{\sqrt{1281}}{122} e_4, [e_1, e_4] = \frac{3\sqrt{122}}{122} e_5, [e_1, e_5] = \frac{\sqrt{244}}{61} e_6, [e_1, e_6] = \frac{3\sqrt{122}}{122} e_7, [e_2, e_3] = \frac{\sqrt{5124}}{244} e_6, [e_2, e_4] = \frac{3\sqrt{122}}{122} e_7$.

(145) (**123457E**)!!: dim(Der) $=11$, rank $=0$, Betti Numbers (2, 4, 6, 6, 4, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.3}$!!).
Pre-Einstein der. $\phi = 0$.
It does not admit; $\phi \not> 0$.

(146) (**123457F**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.1}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(147) (**123457H**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.2}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(148) (**123457H1**)!!: dim(Der) $=10$, rank $=0$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{0.2}$!!).
Pre-Einstein der. $\phi = 0$,
It does not admit; $\phi \not> 0$.

(149) (**123457I**) [$t \in \mathbb{R} \setminus \{0, 1\}$]: dim(Der) $=10$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(i_t)}$)
Einstein der. $\phi = \frac{1}{7}$ Diag(1, 2, 3, 4, 5, 6, 7), $\|\beta\|^2 = \frac{5}{7} \approx 0.7143$
This family admits nilsoliton metrics.

(149.1) (**123457I**) [$t = 0$]: dim(Der) $=10$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(i_0)}$)
Pre-Einstein der. $\phi = \frac{1}{5}$ Diag(1, 2, 3, 4, 5, 6, 7),
It does not admit a nilsoliton inner product.

(149.2) (**123457I**) [$t = 1$]: dim(Der) $=11$, rank $=1$, Betti Numbers (2, 3, 4, 4, 3, 2, 1) ($\cong_{\mathbb{R}} \mathfrak{g}_{1.1(i_1)}$)
Pre-Einstein der. $\phi = \frac{1}{5}$ Diag(1, 2, 3, 4, 5, 6, 7),
It does not admit a nilsoliton inner product.

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